Signal Processing on Databases

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Lecture 2: Group Theory

Spreadsheets, Big Tables, and the Algebra of Associative Arrays

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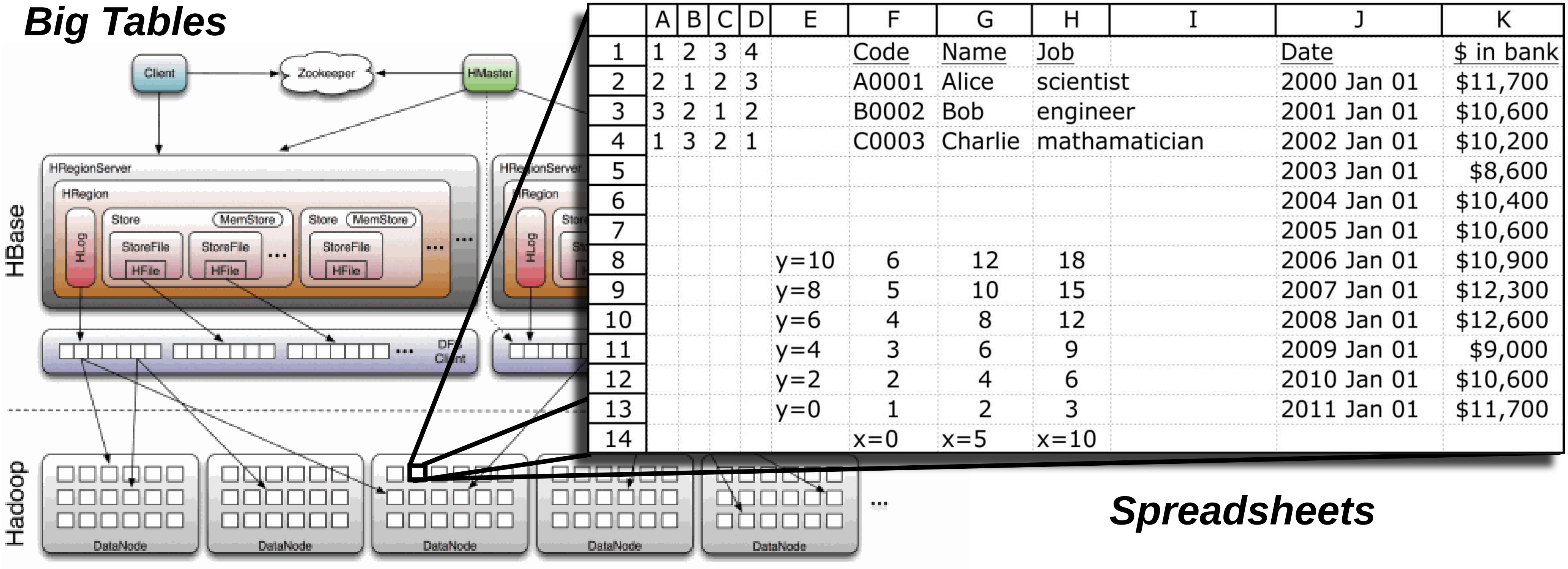
Outline

- ➡ Introduction
  - What are Spreadsheets?
  - Theoretical Goals
  - Associative Arrays
- Definitions
- Group Theory
- Vector Space
- Linear Algebra
- Summary

What are Spreadsheets and Big Tables?

Big Tables



	A	B	C	D	E	F	G	H	I	J	K
1	1	2	3	4		Code	Name	Job		Date	$ in bank
2	2	1	2	3		A0001	Alice	scientist		2000 Jan 01	$11,700
3	3	2	1	2		B0002	Bob	engineer		2001 Jan 01	$10,600
4	1	3	2	1		C0003	Charlie	mathamatician		2002 Jan 01	$10,200
5										2003 Jan 01	$8,600
6										2004 Jan 01	$10,400
7										2005 Jan 01	$10,600
8					y=10	6	12	18		2006 Jan 01	$10,900
9					y=8	5	10	15		2007 Jan 01	$12,300
10					y=6	4	8	12		2008 Jan 01	$12,600
11					y=4	3	6	9		2009 Jan 01	$9,000
12					y=2	2	4	6		2010 Jan 01	$10,600
13					y=0	1	2	3		2011 Jan 01	$11,700
14						x=0	x=5	x=10			

Spreadsheets

- ***Spreadsheets*** **are the most commonly used analytical structure on Earth (100M users/day?)**
- **Big Tables (Google, Amazon, Facebook, …) store most of the analyzed data in the world (Exabytes?)**
- **Simultaneous diverse data: strings, dates, integers, reals, …**
- **Simultaneous diverse uses: matrices, functions, hash tables, databases, …**
- **No formal mathematical basis; Zero papers in AMA or SIAM**

Goal: Signal Processing on Graphs/Strings/Spreadsheets/Tables/ …

- Create a formal basis for working with these data structures based on an Algebra of Associative Arrays
- Better Algorithms
  - Can create algorithms by applying standard mathematical tools (linear algebra and detection theory)
- Faster Implementation
  - Associative array software libraries allow these algorithms to be implemented with ~50x less effort
- Good for managers, too
  - Much simpler than Microsoft Excel; formally correct

Multi-Dimensional Associative Arrays

- **Extends associative arrays to 2D and mixed data types**

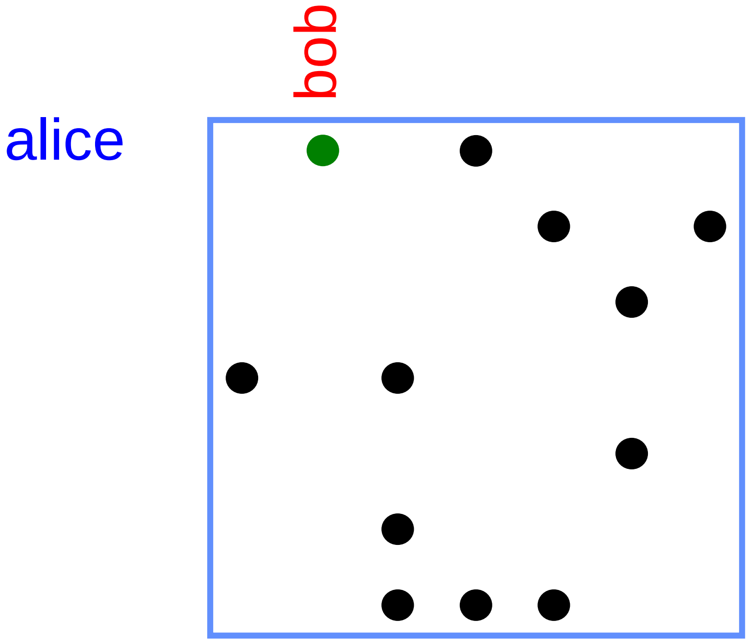
A('alice ','bob ') = 'cited '

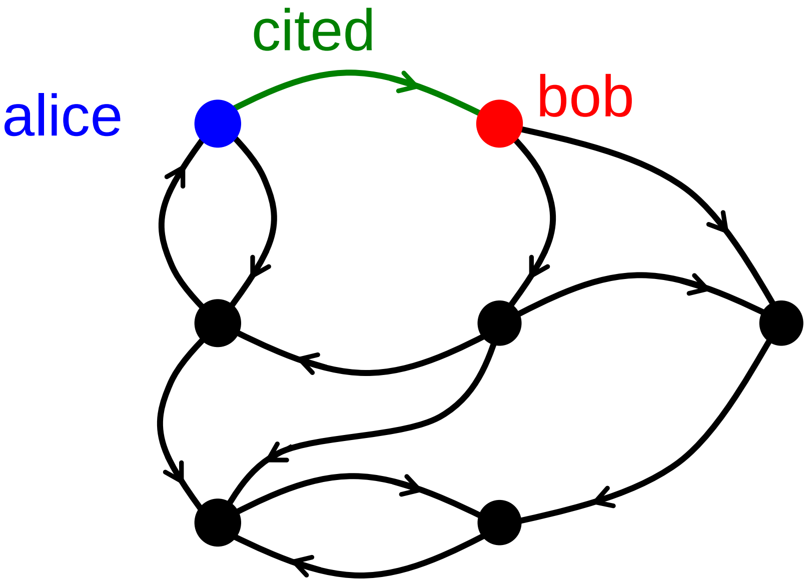
or A('alice ','bob ') = 47.0

- **Key innovation: 2D is 1-to-1 with triple store**

('alice ','bob ','cited ')

or ('alice ','bob ',47.0)





Composable Associative Arrays

- **Key innovation: mathematical closure**
  - **All associative array operations return associative arrays**
- **Enables composable mathematical operations**

A + B   A - B   A & B   A|B   A*B

- **Enables composable query operations via array indexing**

A('alice bob ',:)   A('alice ',:)   A('al* ',:)

A('alice : bob ',:)   A(1:2,:)   A == 47.0

- **Simple to implement in a library (~2000 lines) in programming environments with: 1st class support of 2D arrays, operator overloading, sparse linear algebra**

- **Complex queries with ~50x less effort than Java/SQL**
- **Naturally leads to high performance parallel implementation**

Universal “Exploded” Schema

Input Data

Time	src_ip	domain	dest_ip
2001-01-01	a		a
2001-01-02	b	b	
2001-01-03		c	c

Triple Store Table: Ttranspose

	2001-01-01	2001-01-02	2001-01-03
src_ip/a	1		
src_ip/b		1	
domain/b		1	
domain/c			1
dest_ip/a	1		
dest_ip/c			1

	src_ip/a	src_ip/b	domain/b	domain/c	dest_ip/a	dest_ip/c
2001-01-01	1				1	
2001-01-02		1	1			
2001-01-03				1		1

Triple Store Table: T

Key Innovations

- Handles all data into a *single* table representation
- Transpose pairs allows quick look up of *either* row or column

Outline

- Introduction
- Definitions
  - Values
  - Keys
  - Functions
  - Matrix multiply
- Group Theory
- Vector Space
- Linear Algebra
- Summary

Associative Array Definitions

- Keys and values are from the infinite strict totally ordered set $\mathbb{S}$

- Associative array $A(\mathbf{k}) : \mathbb{S}^d \to \mathbb{S}$, $\mathbf{k}=(k^1,\ldots,k^d)$, is a partial function from d keys (typically 2) to 1 value, where

  $A(\mathbf{k}_i) = v_i$ and $\varnothing$ otherwise

- Binary operations on associative arrays $A_3 = A_1 \oplus A_2$, where $\oplus = \cup_{f()}$ or $\cap_{f()}$, have the properties
  - If $A_1(\mathbf{k}_i) = v_1$ and $A_2(k_i) = v_2$, then $A_3(\mathbf{k}_i)$ is

    $v_1 \cup_{f()} v_2 = f(v_1,v_2)$ or $v_1 \cap_{f()} v_2 = f(v_1,v_2)$

  - If $A_1(\mathbf{k}_i) = v$ or $\varnothing$ and $A_2(\mathbf{k}_i) = \varnothing$ or v, then $A_3(\mathbf{k}_i)$ is

    $v \cup_{f()} \varnothing = v$ or $v \cap_{f()} \varnothing = \varnothing$

- **High level usage dictated by these definitions**
- **Deeper algebraic properties set by the collision function $f()$**
- **Frequent switching between "algebras" (how spreadsheets are used)**

Associative Array Values

- Value requirements
  - Diverse types: integers, reals, strings, …
  - Sortable
  - Set
- Let $\mathbb{S}$ be an infinite strict totally ordered set
  - Total order is an implementation (not theoretical) requirement
  - All values (and keys) will be drawn from this set
- Allowable operations for $v_1, v_2 \in \mathbb{S}$

  $v_1 < v_2$ $v_1 = v_2$ $v_1 > v_2$

- Special symbols: $\varnothing$, $-\infty$, $+\infty$

  $v \leq +\infty$ is always true $(+\infty \in \mathbb{S})$

  $v \geq -\infty$ is always true $(-\infty \in \mathbb{S})$

  $\varnothing$ is the empty set $(\varnothing \subset \mathbb{S})$

- **Above properties are consistent with strict totally ordered sets**

Collision Function f()

- Collision function $f(v_1,v_2)$ can have
  - two contexts ($\cup$ $\cap$)
  - three conditions ($<$ $=$ $>$)
  - $d + 5$ possible outcomes (**k** v_1 v_2 Ø $-\infty$ $+\infty$) [or sets of these]
- Combinations result in an enormous number of functions (~10^{30}) and an even greater number of associative array algebras (function pairs)
  - Impressive level of functionality given minimal assumptions
- Focus on “nice” collision functions
  - Keys are not used inside the function; results are single valued
  - No tests on special symbols

$f(v_1,v_2)$

$v_1 < v_2 : v_1 \; v_2 \; Ø \; -\infty \; +\infty$

$v_1 = v_2 : v \qquad Ø \; -\infty \; +\infty$

$v_1 > v_2 : v_1 \; v_2 \; Ø \; -\infty \; +\infty$

- **Above properties are consistent with strict totally ordered sets**
- **Note: Ø is handled by $\cup$ $\cap$; not passed into f()**

What About Concatenation?

- Concatenation of values (or keys) can be represented by using $\cup$ or $\cap$ as collision function
  - Requires generalizing values to sets $v_1, v_2 \subset \mathbb{S}$
- Allowable operations for $v_1, v_2 \subset \mathbb{S}$

$v_1 \cup v_2$ $\qquad$ $v_1 \cap v_2$

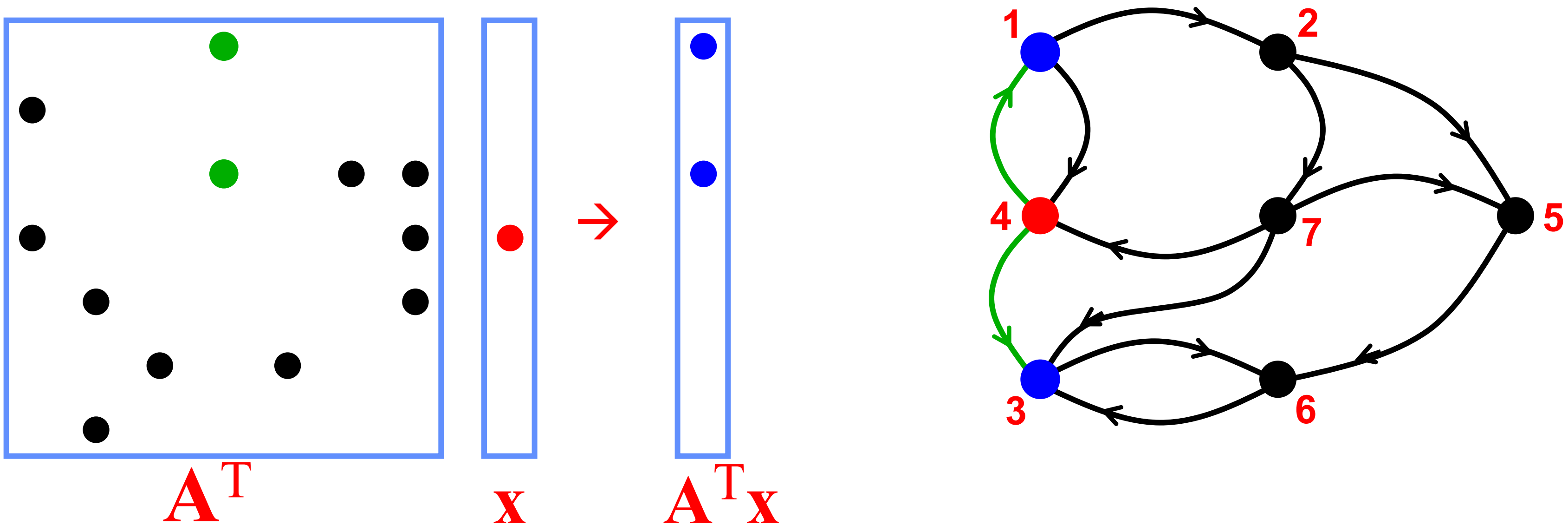
- Special symbols: $\emptyset$, $\mathbb{S}$

$v \cap \emptyset = \emptyset$	annihilator (but never reached, so identify)
$v \cup \mathbb{S} = \mathbb{S}$	annihilator
$v \cap \mathbb{S} = v$	identity
$v \cup \emptyset = v$	identity

- Possible operators: $\cup_{\cup}$, $\cap_{\cup}$, $\cup_{\cap}$, $\cap_{\cap}$

- **Concatenating collision functions are very useful**
- **Can be handled by extending values to be sets**

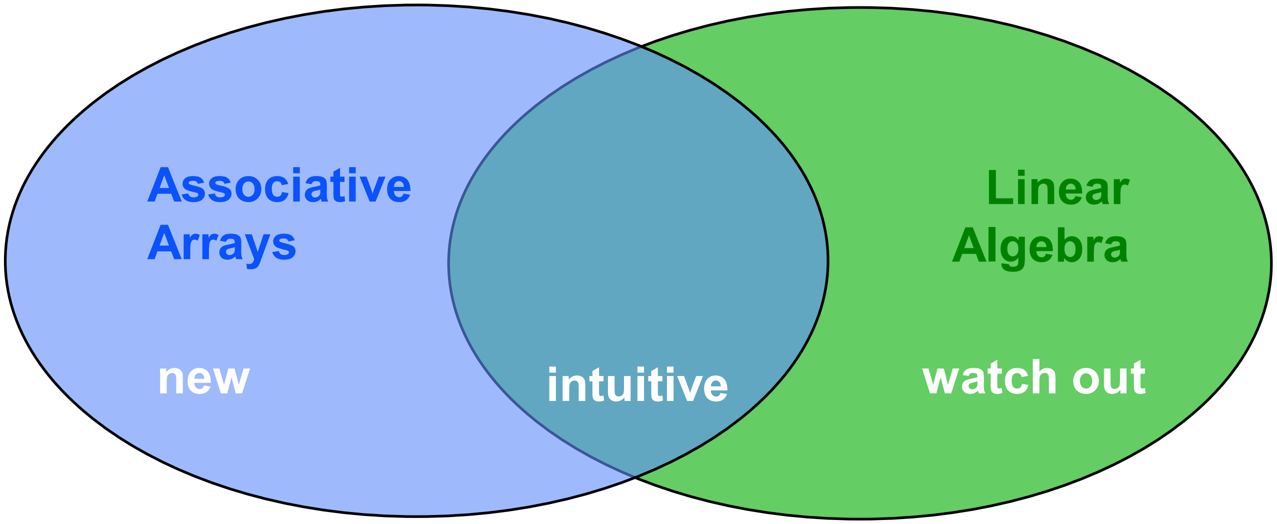
Matrix Multiply Framework



- Graphs can be represented as a sparse matrices
  - Multiply by adjacency matrix → step to neighbor vertices
  - Work-efficient implementation from sparse data structures
- Graph algorithms reduce to products on semi-rings: $A_3 = A_1 \oplus.\otimes A_2$
  - $\otimes$: associative, distributes over $\oplus$
  - $\oplus$: associative, commutative
  - Examples: +.* min.+ or.and

Theory Questions

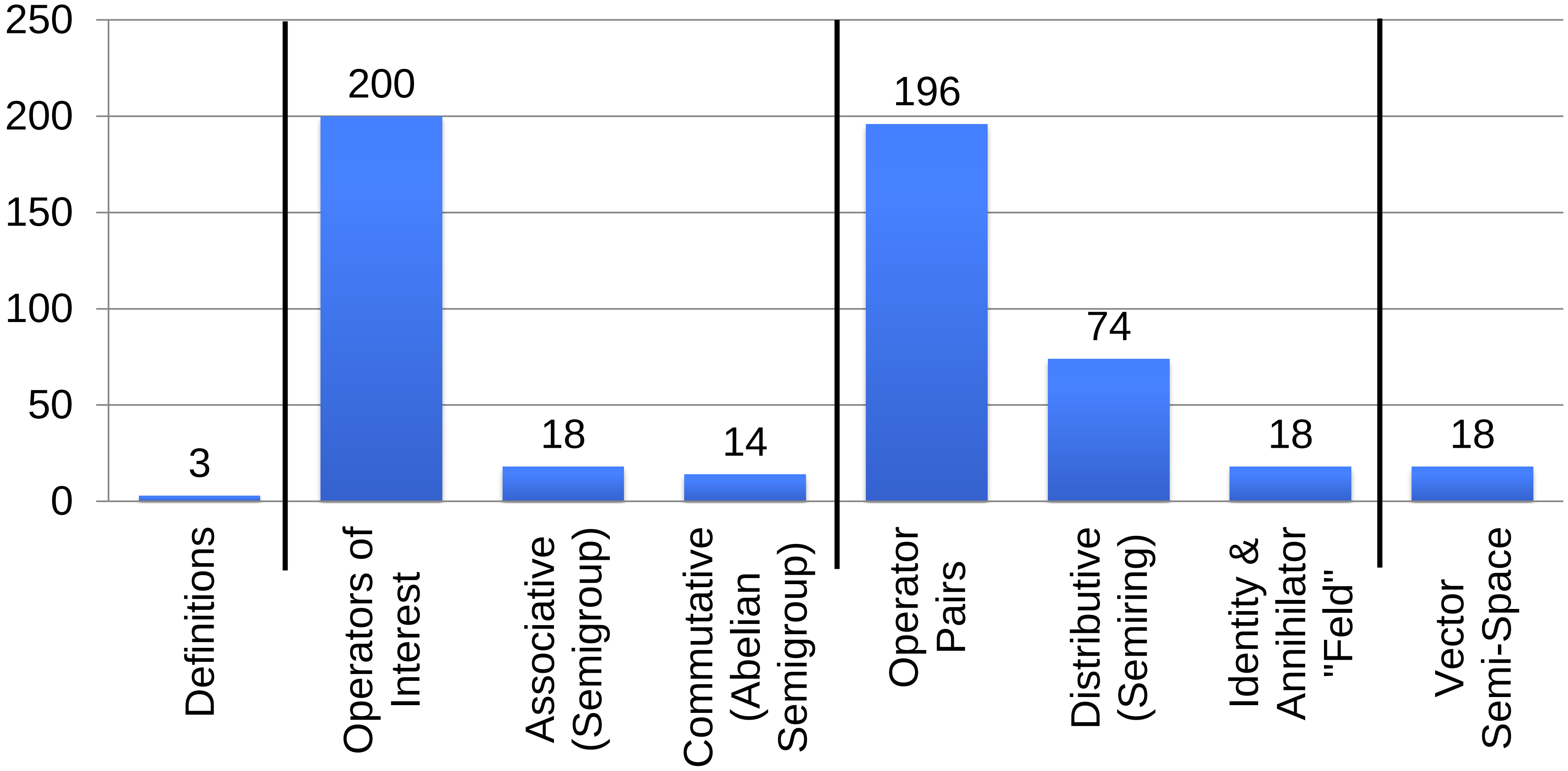
- Associative arrays can be constructed from a few definitions
- Similar to linear algebra, but applicable to a wider range of data
- Key questions
  - Which linear algebra properties do apply to associative arrays (intuitive)
  - Which linear algebra properties do not apply to associative arrays (watch out)
  - Which associative array properties do not apply to linear algebra (new)



Outline

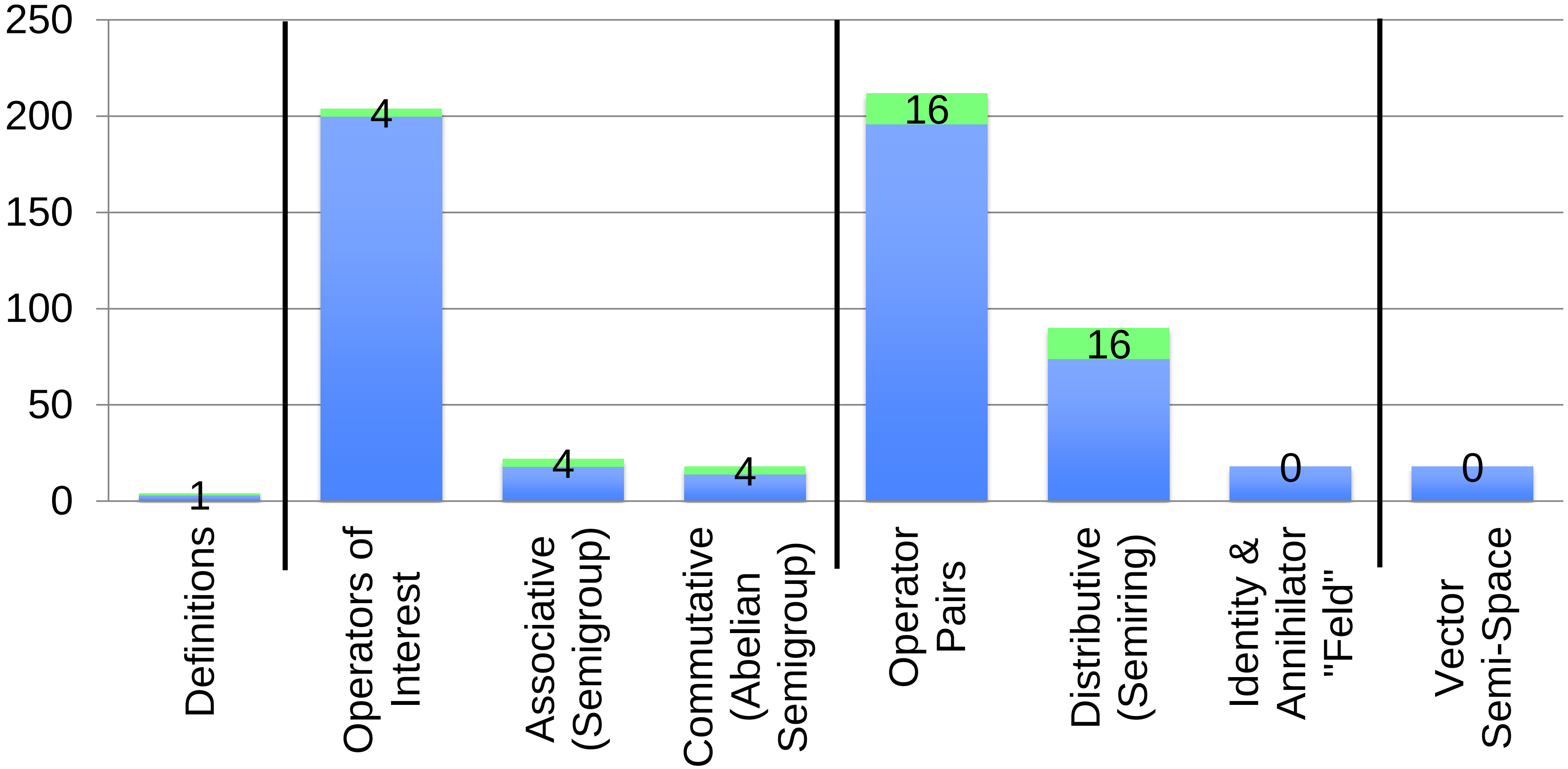
- Introduction
- Definitions
- Group Theory
  - Binary operators
  - Commutative monoids
  - Semirings
  - Feld
- Vector Space
- Linear Algebra
- Summary

Operators Roadmap



- **Begin with a few definitions**
- **Expand into many operators; reduce to well behaved**
- **Expand into many operator pairs; reduce to well behaved**

Including Concatenation



- **Including concatenation operators expands semirings**
- **Doesn’t expand vector semi-space**

Associative and Commutative Operators

ID	Operator ⊕	$v_1 < v_2$	$v_1 = v_2$	$v_1 > v_2$
1	$\cup_{left}$	v_1	v	v_1
2	$\cap_{left}$	v_1	v	v_1
3	$\cup_{max}$	v_2	v	v_1
4	$\cap_{max}$	v_2	v	v_1
41	$\cup_{min}$	v_1	v	v_2
42	$\cap_{min}$	v_1	v	v_2
43	$\cup_{right}$	v_2	v	v_2
44	$\cap_{right}$	v_2	v	v_2
86	$\cap_{\delta}$	$\varnothing$	v	$\varnothing$
96	$\cap_{\varnothing}$	$\varnothing$	$\varnothing$	$\varnothing$
127	$\cup_{-\infty,\delta}$	$-\infty$	v	$-\infty$
128	$\cap_{-\infty,\delta}$	$-\infty$	v	$-\infty$
147	$\cup_{-\infty}$	$-\infty$	$-\infty$	$-\infty$
148	$\cap_{-\infty}$	$-\infty$	$-\infty$	$-\infty$
169	$\cup_{+\infty,\delta}$	$+\infty$	v	$+\infty$
170	$\cap_{+\infty,\delta}$	$+\infty$	v	$+\infty$
199	$\cup_{+\infty}$	$+\infty$	$+\infty$	$+\infty$
200	$\cap_{+\infty}$	$+\infty$	$+\infty$	$+\infty$

- Associative

$$(v_1 \oplus v_2) \oplus v_3 = v_1 \oplus (v_2 \oplus v_3)$$

- 18 associative operators
  - Semigroups
  - Groups w/o inverses

- Commutative

$$v_1 \oplus v_2 = v_2 \oplus v_1$$

- 14 associative & commutative operators
  - Removes left and right
  - Abelian Semigroups
  - Abelian Groups w/o inverses

Distributive Operator Pairs

- 14 x 14 = 196 Pairs of Abelian Semigroup operators
- Distributive

$$v_1 \otimes (v_2 \oplus v_3) = (v_1 \otimes v_2) \oplus (v_1 \otimes v_3)$$

- 74 distributive operator pairs
  - Semirings
  - Rings without inverses and without identity elements

- **1/3 of possible operator pairs are semirings**

Distributive Operator Pairs with Annihilators (0) and Identities (1)

- $\oplus$ identity: $v_1 \oplus 0 = v_1$ $0 = \varnothing, -\infty, +\infty$
- $\otimes$ identity: $v_1 \otimes 1 = v_1$ $1 = \varnothing, -\infty, +\infty$
- $\otimes$ annihilator: $v_1 \otimes 0 = 0$ $0 = \varnothing, -\infty, +\infty$

- 12 Semirings with appropriate 0 1 set (4 with two)
- 16 total over six operators: $\cup_{max}$, $\cap_{max}$, $\cup_{min}$, $\cap_{min}$, $\cup_{-\infty}$, $\cup_{+\infty}$
  - Felds? (Fields w/o inverses)

- $\oplus = \cup_{f()}$ in 10/16 ($\cup$ feels more like plus)
- $\otimes = \cap_{f()}$ in 10/16 ($\cap$ feels more like multiply)
- $\oplus = \cup_{f()}$ and $\otimes = \cap_{f()}$ in 8/16
- $0 = \varnothing$ in 6/8 ($\varnothing$ feels more like zero, $0 > 1$ might be a problem)

- **1/5 of semirings are Felds (Fields w/o inverses)**

Operator Pairs

$\otimes$

$\oplus$

0 1	$\cup_{max}$	$\cap_{max}$	$\cup_{min}$	$\cap_{min}$	$\cap_{\delta}$	$\cap_{\emptyset}$	$\cup_{-\infty,\delta}$	$\cap_{-\infty,\delta}$	$\cup_{-\infty}$	$\cap_{-\infty}$	$\cup_{+\infty,\delta}$	$\cap_{+\infty,\delta}$	$\cup_{+\infty}$	$\cap_{+\infty}$
$\cup_{max}$	D	$\emptyset$ $-\infty$		$-\infty$ $+\infty$ $\emptyset$ $+\infty$		D				D			D	D
$\cap_{max}$		D	$-\infty$ $+\infty$ $-\infty$ $\emptyset$	$-\infty$ $+\infty$		D			$-\infty$ $\emptyset$	D				D
$\cup_{min}$	$+\infty$ $-\infty$ $+\infty$ $\emptyset$	$+\infty$ $-\infty$ $\emptyset$ $-\infty$	D	$\emptyset$ $+\infty$		D			D	D				D
$\cap_{min}$		$+\infty$ $-\infty$		D		D				D			$+\infty$ $\emptyset$	D
$\cap_{\delta}$					D	D								
$\cap_{\emptyset}$		D		D	D	D		D		D		D		D
$\cup_{-\infty,\delta}$						D	D	D	D	D				D
$\cap_{-\infty,\delta}$						D		D		D				D
$\cup_{-\infty}$				$\emptyset$ $+\infty$		D		D		D				
$\cap_{-\infty}$				D		D		D		D				
$\cup_{+\infty,\delta}$						D				D	D	D	D	D
$\cap_{+\infty,\delta}$						D				D		D		D
$\cup_{+\infty}$		$\emptyset$ $-\infty$				D						D		D
$\cap_{+\infty}$		D				D						D		D

D=distributes; 0=Plus Identity/Multiply Annihilator; 1=Multiply Identity

Concatenate Operators

ID	Operator $\oplus$	$f(v_1,v_2)$
201	$\cup_\cup$	$v_1 \cup v_2$
202	$\cap_\cup$	$v_1 \cup v_2$
203	$\cup_\cap$	$v_1 \cap v_2$
204	$\cap_\cap$	$v_1 \cap v_2$

- Recall v_1 and v_2 are sets
- All operators are associative and commutative
  - – 4 Abelian Semigroups

$\otimes$

$\oplus$

0 1	$\cup_\cup$	$\cap_\cup$	$\cup_\cap$	$\cap_\cap$
$\cup_\cup$	D	∅ -∞	D	-∞ +∞ ∅ +∞
$\cap_\cup$	D	D	-∞ +∞ -∞ ∅	-∞ +∞
$\cup_\cap$	+∞ -∞ +∞ ∅	+∞ -∞ ∅ -∞	D	∅ +∞
$\cap_\cap$	D	+∞ -∞	D	D

- All operator pairs distribute
  - – 16 Semirings

Outline

- Introduction
- Definitions
- Group Theory
- ➡ Vector Space
  - Vector Semispace
  - Uniqueness
- Linear Algebra
- Summary

Vector Space over a Feld

- Associative Array Vector $\oplus$
  - All associative arrays are conformant (unlike matrices)
- Associative Array Scalar $\otimes$
  - Scalar is a value applied directly to values; similar to constant function; or a function that takes on keys of non-scalar argument

- Vector Space $\oplus$ requirements
  - Commutes [Yes]; Associative [Yes]; 0 Identity element [Yes]
  - Inverse [No]
- Vector Space scalar $\otimes$ requirements
  - Commutes [Yes]; Associative [Yes]; Distributes over addition [Yes]; 1 Identity element [Yes]

- **All associative array operator pairs that yield Felds also result in Vector Spaces wo/inverses (Vector Semispace?)**

Vector Semispace Properties

- Scalar $\oplus$ identity annihilates under $\otimes$ [Yes]
- Subspace [Yes]
  - Any linear combination of vectors taken from the subspace is in the subspace and obeys the properties of a vector space
  - Theorem: Intersection of any subspaces is a subspace?
- Span [Yes+]
  - Given a set of vectors A_j, their span is all linear combinations of those vectors (includes vectors of different lengths)

$$\oplus_j (a_j \otimes A_j)$$

- Span = Subspace [Yes?]
  - Given an arbitrary set of vectors, their span is a vector space?
- Linear dependence [No]
  - There is a non-trivial linear combination of vectors equal to the $\oplus$ identity; can't do this without additive inverse
  - Need to redefine linear independence or all vectors are linearly independent; use minimum vectors in a subspace definition?
  - Likewise need to redefine basis as it depends upon linear dependence

- **Key question: under what conditions does the result of a linear combination of associative arrays uniquely determine the coefficients**

Unique Coefficient Conditions

- Consider a linear combinations of two associative array vectors

$$A_3 = (a_1 \otimes A_1) \;\oplus\; (a_2 \otimes A_2)$$

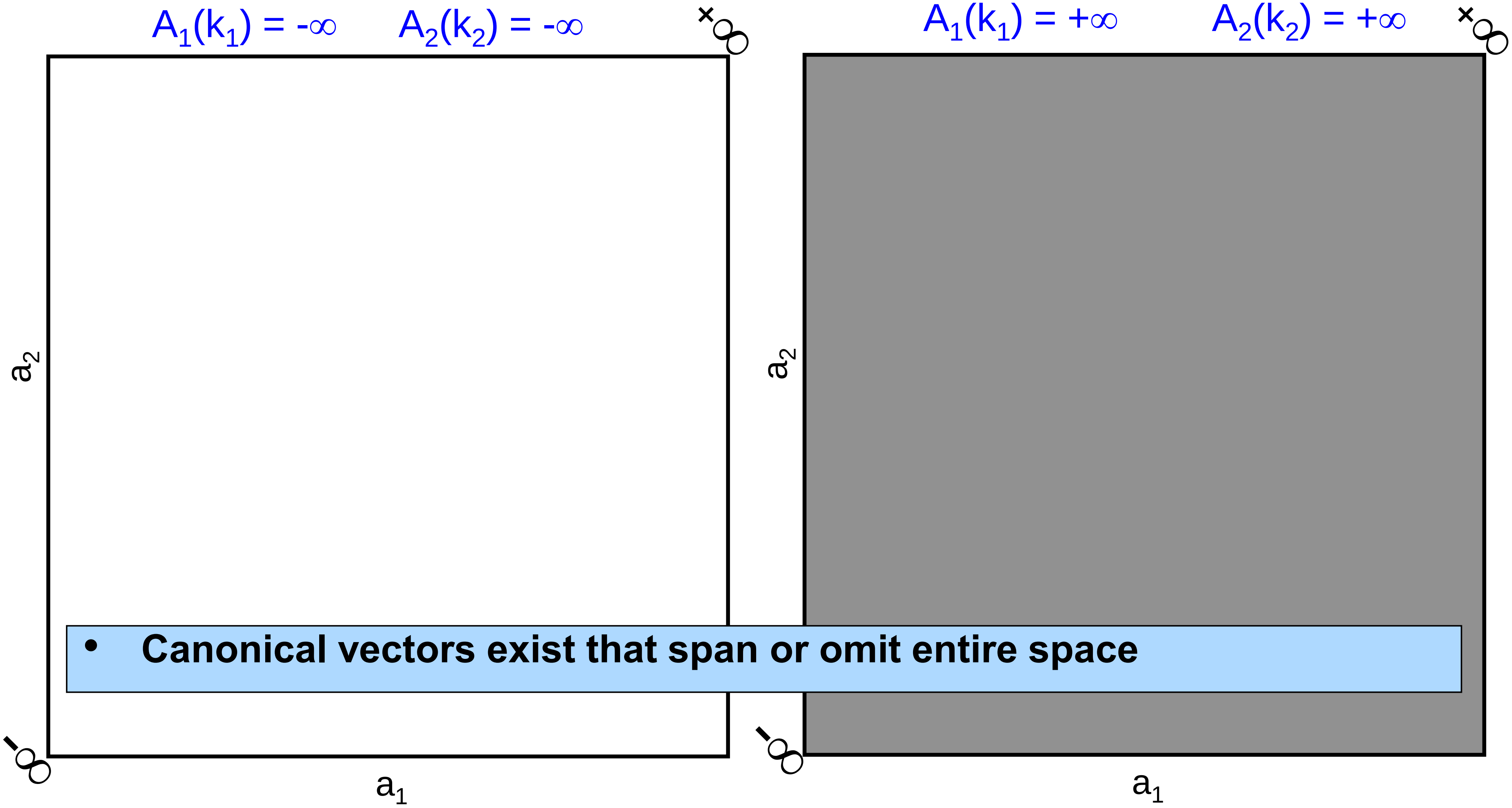
- Let $\oplus = \cup_{\text{min}}$, $\otimes = \cap_{\text{max}}$, $0 = \emptyset$, and $1 = -\infty$

- When are a_1 and a_2 uniquely determined by A_1, A_2 and A_3 ?

Canonical Vectors	Single valued	Multi-valued
$A_1(k_1) = -\infty$ $A_2(k_2) = -\infty$		$A_1(k_1\ k_2) = (v_1\ v_2)$ $A_2 = A_1$ $v_1 < v_2$
$A_1(k_1) = +\infty$ $A_2(k_2) = +\infty$	$A_1(k_1\ k_2) = (v\ v)$ $A_2 = A_1$	$A_1(k_1\ k_2) = (v_1\ v_2)$ $A_2(k_1\ k_2) = (v_2\ v_1)$ $v_1 < v_2$

- **Consider specific cases to show existence of uniqueness**

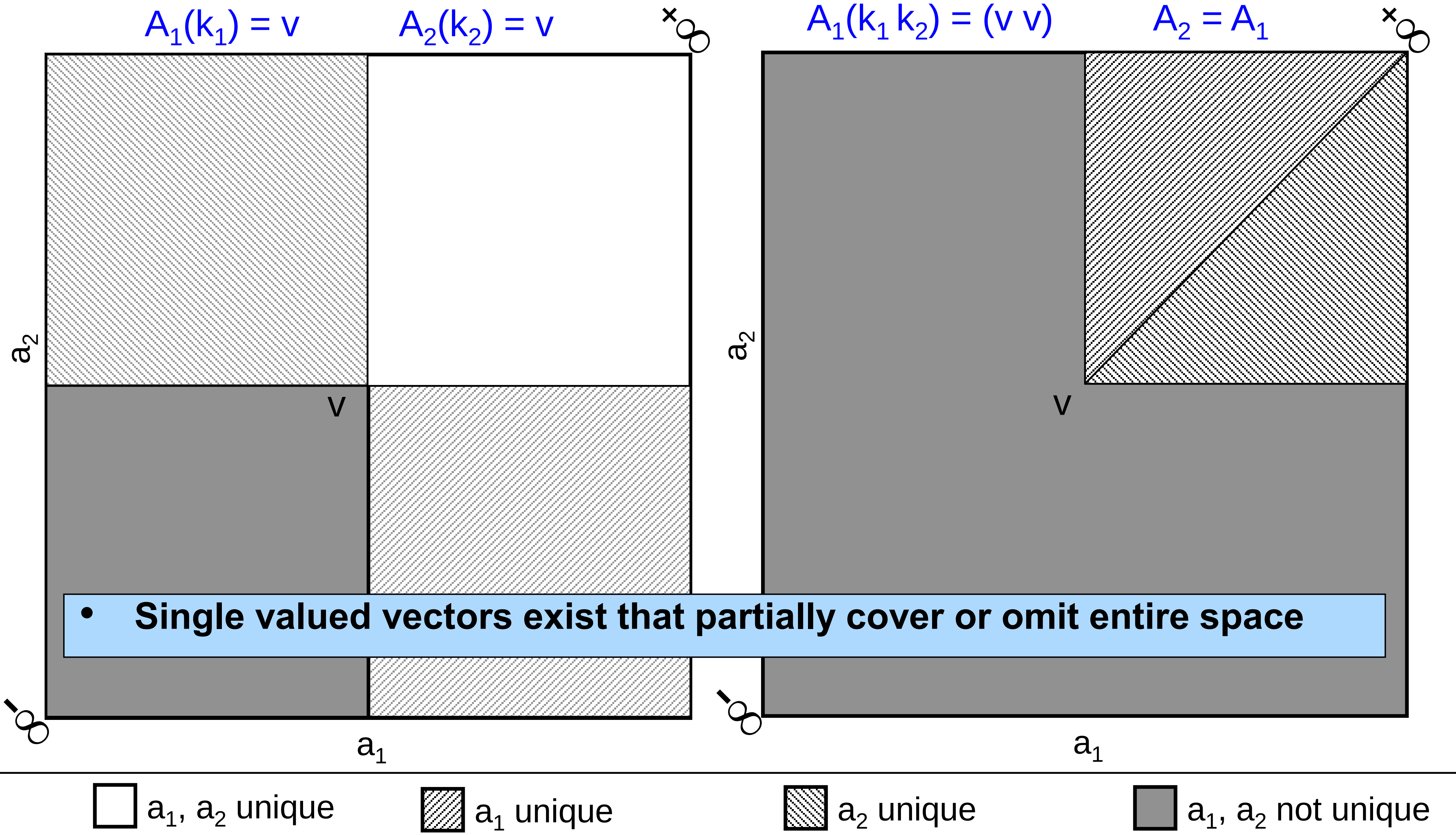
Canonical Vectors



- **Canonical vectors exist that span or omit entire space**

□ a_1, a_2 unique ▨ a_1 unique ▧ a_2 unique ■ a_1, a_2 not unique

Single Valued Vectors

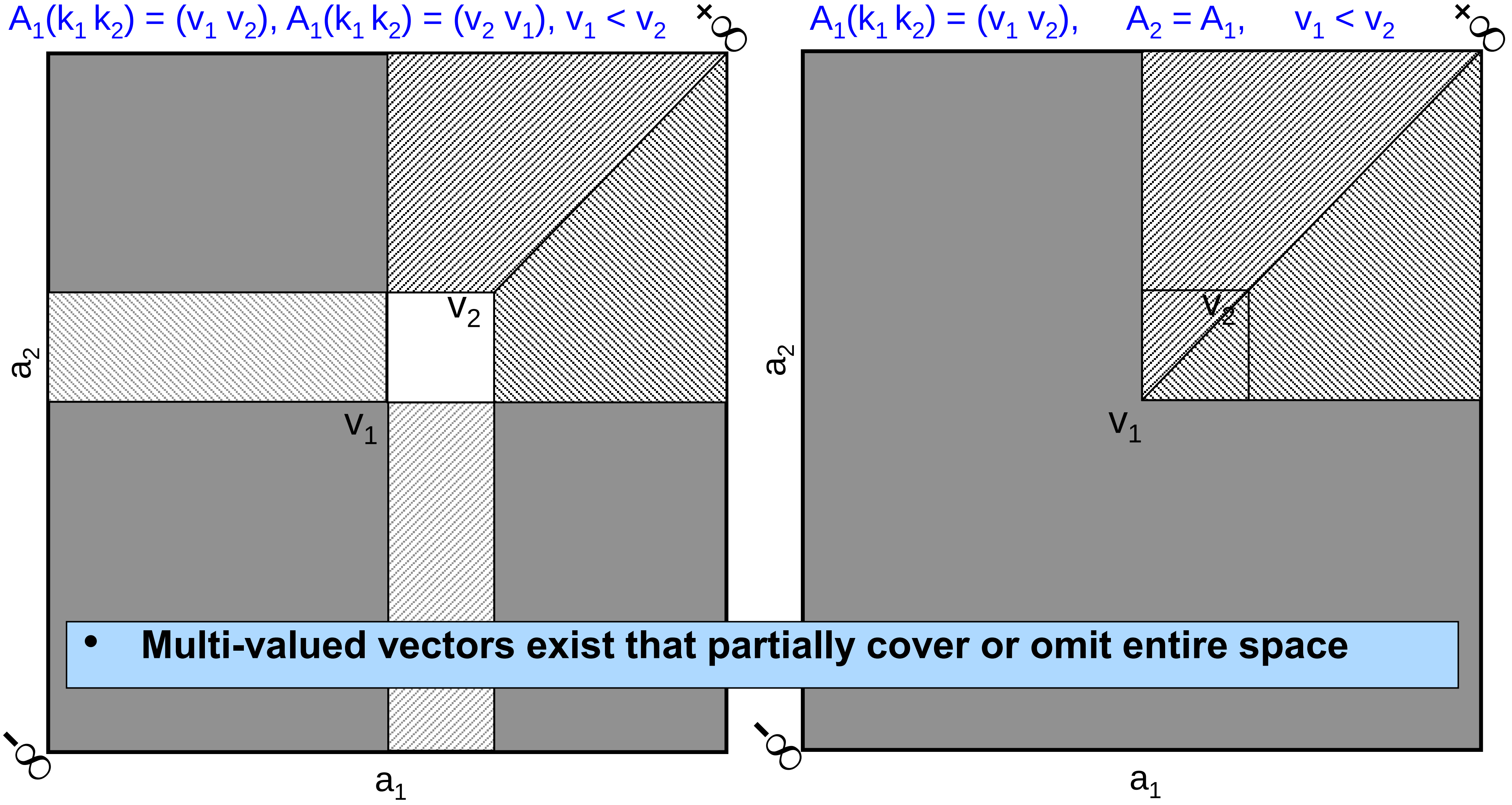


- **Single valued vectors exist that partially cover or omit entire space**

Multi-Valued Vectors

$A_1(k_1\,k_2) = (v_1\,v_2)$, $A_1(k_1\,k_2) = (v_2\,v_1)$, $v_1 < v_2$

$A_1(k_1\,k_2) = (v_1\,v_2)$, $A_2 = A_1$, $v_1 < v_2$



- **Multi-valued vectors exist that partially cover or omit entire space**

□ a_1, a_2 unique ▨ a_1 unique ▧ a_2 unique ■ a_1, a_2 not unique

Outline

- Introduction
- Definitions
- Group Theory
- Vector Space
- Linear Algebra
  - Transpose
  - Special Matrices
  - Matrix Multiply
  - Identity
  - Inverses
  - Eigenvectors
- Summary

Matrix Transpose

- Swap keys (rows and columns)

$$A(r,c)^{T} = A(c,r)$$

- No change with even number of transposes

- Transpose distributes across $\oplus$ and scalar $\otimes$

$$((a_1 \otimes A_1) \;\oplus\; (a_2 \otimes A_1))^{T} = (a_1 \otimes A_1^{T}) \;\oplus\; (a_2 \otimes A_1^{T})$$

- **Similar to linear algebra**

Special Matrices

- Submatrices [Yes]
- Zero matrix [Yes?] (empty set)
- Square matrix [Yes]
- Diagonal matrix [Yes]
- Upper/lower triangular [Yes]
- Skew symmetric [No] (no ⊕ inverse)
- Hermitian [No] (no ⊕ inverse)
- Elementary row/column operations [Yes?]
  - Swap both keys or values? No ⊗ inverse.
  - If both key and value swap, then equivalent to matrix multiply
- Row/column equivalence [Yes?]
  - If limit to swaps

- **Similar and different from linear algebra**
- **Possible to construct these forms, but may not be applicable to associative arrays that have fixed keys (i.e., functions over a keys)**

Matrix Multiply

- Matrix multiply

$$A_3 = A_1\, A_2 = A_1 \oplus.\otimes A_2$$

- Always conformant (can multiply any sizes)

- Inner product formulation (computation)

$$A_3(r_i,c_j) = \oplus_k \,(\, A_1(r_i,k) \otimes A_2(k,c_j)\,)$$

- Outer product formulation (theory)

$$A_k(r_i,c_j) = A_1(r_i,k) \otimes A_2(k,c_j)$$

$$A_3 = \oplus_k\, A_k$$

- **Different from linear algebra**
- **Associative arrays have no conformance requirements**

Matrix Multiply Examples

- 1x2 Row matrix: $A_1(r,k_1\,k_2) = v_1$
- 2x1 Column matrix: $A_1(k_2\,k_3,c) = v_2$

- Example 1: 1x1 Matrix: $A_3(r,c) = A_1\,A_2 =$ [See Table]
- Example 2: 2x2 Matrix (r≠c): $A_3(k_1\,k_2,\,k_2\,k_3) = A_2\,A_1 =$ [See Table]
- Example 3: 2x2 Matrix (r=c): $A_3(k_1\,k_2,\,k_2\,k_3) = A_2\,A_1 = f(v_1,v_2)$

- Value of A_3 depends upon specifics of $\oplus$ and $\otimes$

Example 1	$\otimes = \cup_{f()}$	$\otimes = \cap_{f()}$
$\oplus = \cup_{g()}$	$g(g(v_1,f(v_1,v_2),v_2)$	$f(v_1,v_2)$
$\oplus = \cap_{g()}$	$g(g(v_1,f(v_1,v_2),v_2)$	Ø

Example 2	$\otimes = \cup_{f()}$	$\otimes = \cap_{f()}$
$\oplus = \cup_{g()}$	$g(v_1,v_2)$	Ø
$\oplus = \cap_{g()}$	$g(v_1,v_2)$	Ø

- **Wide range of behaviors possible given specific operator choices**

Identity

- Left Identity: $I_{left} = \text{diag}(\text{Row}(A)) = 1$
- When does? $I_{left}\, A = A$

- Right Identity: $I_{right} = \text{diag}(\text{Col}(A)) = 1$
- When does? $A\, I_{right} = A$

- Generally possible when

$$\oplus = \cup_{g()} \qquad \otimes = \cap_{f()}$$

- In some circumstances

$$I = I_{left} \oplus I_{right} \qquad \text{and} \qquad A\, I = A = I\, A$$

- **Similar to linear algebra for a limited set of $\oplus$ and $\otimes$**

Inverses

- Left Inverse: $A\ A^{-1} = I_{\text{left}}$
- Right Inverse: $A^{-1}\ A = I_{\text{right}}$
- Is it possible to construct matrix inverses with no $\oplus$ inverse and no $\otimes$ inverse
- Generally, no. Exception
  - A is a column/row vector
  - $\oplus = \cup_{g()}$, $\otimes = \cap_{f()}$
  - $I_{\text{right/left}}$ is 1x1 equal to “local” 1 (i.e., 1 wrt to A)

- **Different from linear algebra**
- **Inverses generally do not appear in associative arrays**

Eigenvectors (simple case)

- Let $\oplus = \cup_g$, $\otimes = \cap_f$

- Let A, A_e, A_λ be NxN and have 1 element per row and column

$$A(r_i,r_i) = v_i \qquad A_e(r_i,c_i) = e_i \qquad A_\lambda(c_i,c_i) = v_i$$

- Eigenvector equation

$$A\ A_e \ = \ A_e\ A_\lambda \qquad = A_{e\lambda}$$

- where: $A_{e\lambda}(r_i,c_i) = f(v_i,e_i)$

- Eigenvector equation satisfied in a simple case
- Row and column keys must match

Pseudoinverse (simple case)

- Let $\oplus = \cup_g$, $\otimes = \cap_f$

- Let A, A^+ be NxN (or N_rxN_c?) and have 1 element per row and column

$$A(r_i,c_i) = v_i \qquad A^+(c_i,r_i) = v_i^+$$

- Pseudoinverse requires

$$A = A\,A^+\,A$$

$$A = A^+\,A\;\;A^+$$

$$(A\,A^+)^T = A\,A^+$$

$$(A\,A^+)^T = A\,A^+$$

- where: $f(v_i,v_i^+) = v_i$

- **Pseudoinverse equation satisfied in a simple case**
- **Row and column keys can be different**

Future Work: Got Theorems?

- Spanning theorems: when is a span a vector space?
- Linear dependence: adding a vector doesn’t change span?
- Identity Array: when do left/right identity exist?
- Inverse: why doesn’t it exist?
- Determinant: existance?
- Pseudoinverse: existence? How to compute?
- Linear transforms: existance?
- Norms or inner product space
- Compressive sensing requirements
- Eigenvectors
- Convolution (with next operator)
- Complementary matrices

- For which $\oplus$, $\otimes$, 0/1 do these apply

Summary

- Algebra of Associative Arrays provides the mathematics for representing and operating on Spreadsheets and Big Tables

- Small number of assumptions yields a rich mathematical environment

- Much of linear algebra is available without $\oplus$ inverse and $\otimes$ inverse

Example Code & Assignment

- Example Code
  - d4m_api/examples/1Intro/3GroupTheory
- Assignment 2
  - Define, in words, a list of operations that make “sense” for your associative arrays in Assignment 1
  - Explain your reasoning

Relational Model High Level Comparison

	Relational Database	Associative Arrays
Fill	Dense	Sparse
Columns	Static	Dynamic
Data	Typed	Untyped
#Rows	Unlimited	Unlimited
#Columns	Small	Unlimited
Dimensions	2 different	N same
Main Operation	Join	Linear Algebra

- Relational algebra (Codd 1970) is the de facto theory of databases
- The design goal of relational algebra and associative arrays algebra are fundamentally different
- Result in a fundamental differences in the theory