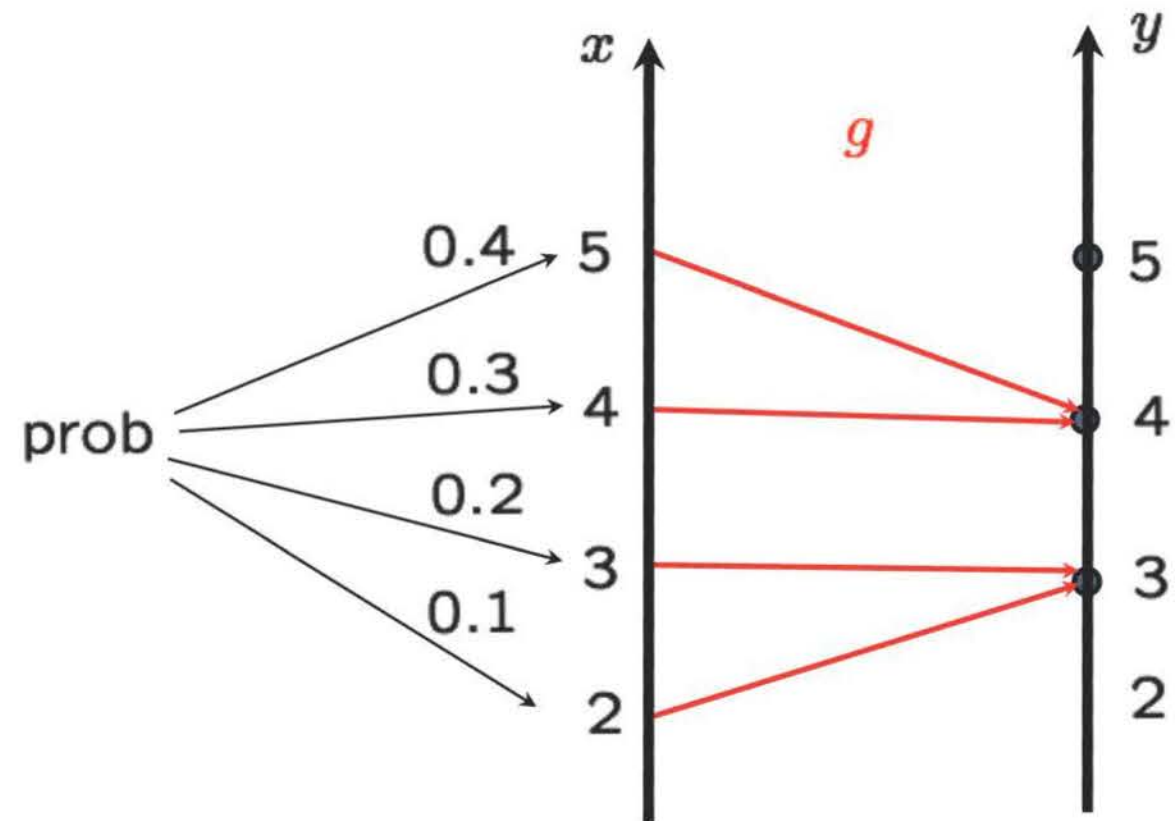


## LECTURE 11: Derived distributions

- Given the distribution of  $X$ ,  
find the distribution of  $Y = g(X)$ 
  - the discrete case
  - the continuous case
  - general approach, using CDFs
  - the linear case:  $Y = aX + b$
  - general formula when  $g$  is monotonic
- Given the (joint) distribution of  $X$  and  $Y$ ,  
find the distribution of  $Z = g(X, Y)$

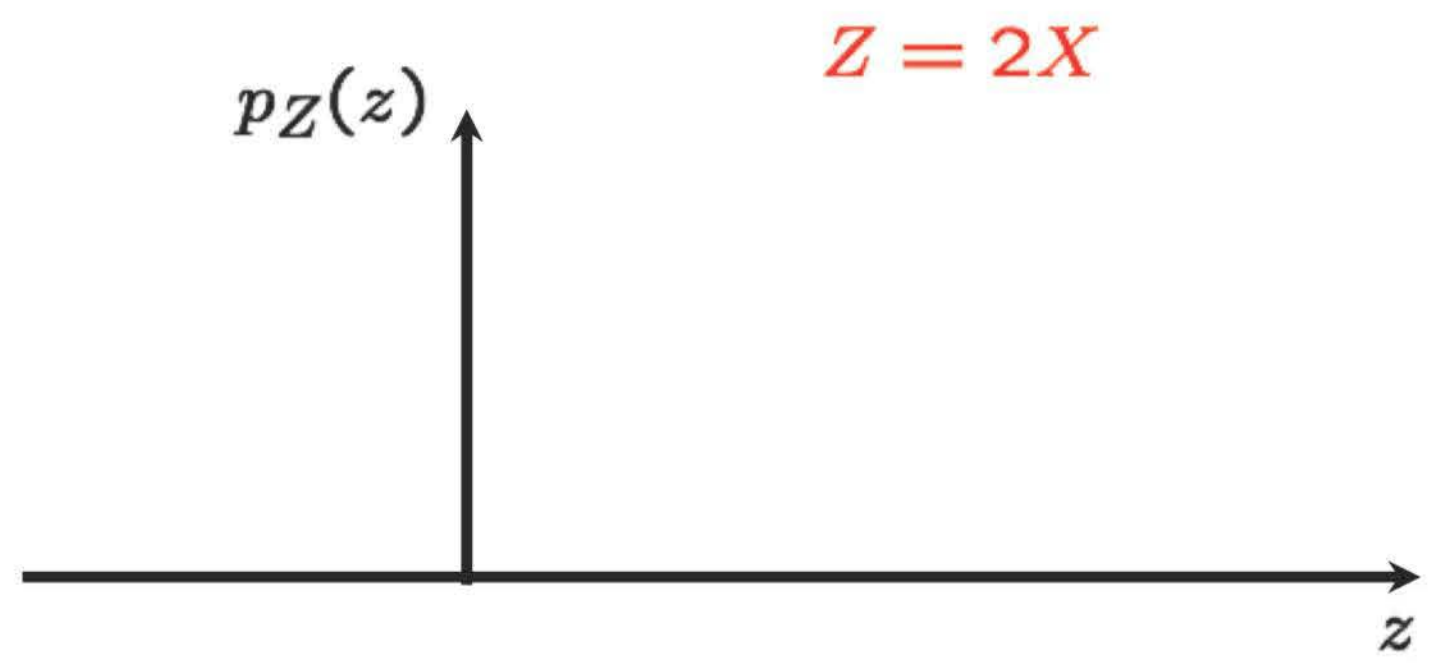
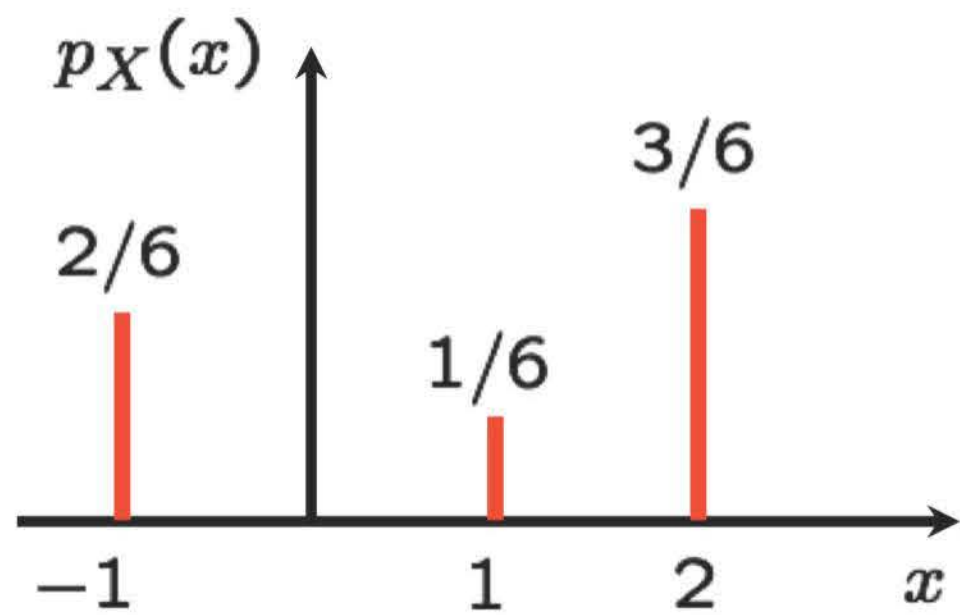
## Derived distributions — the discrete case

$$Y = g(X)$$

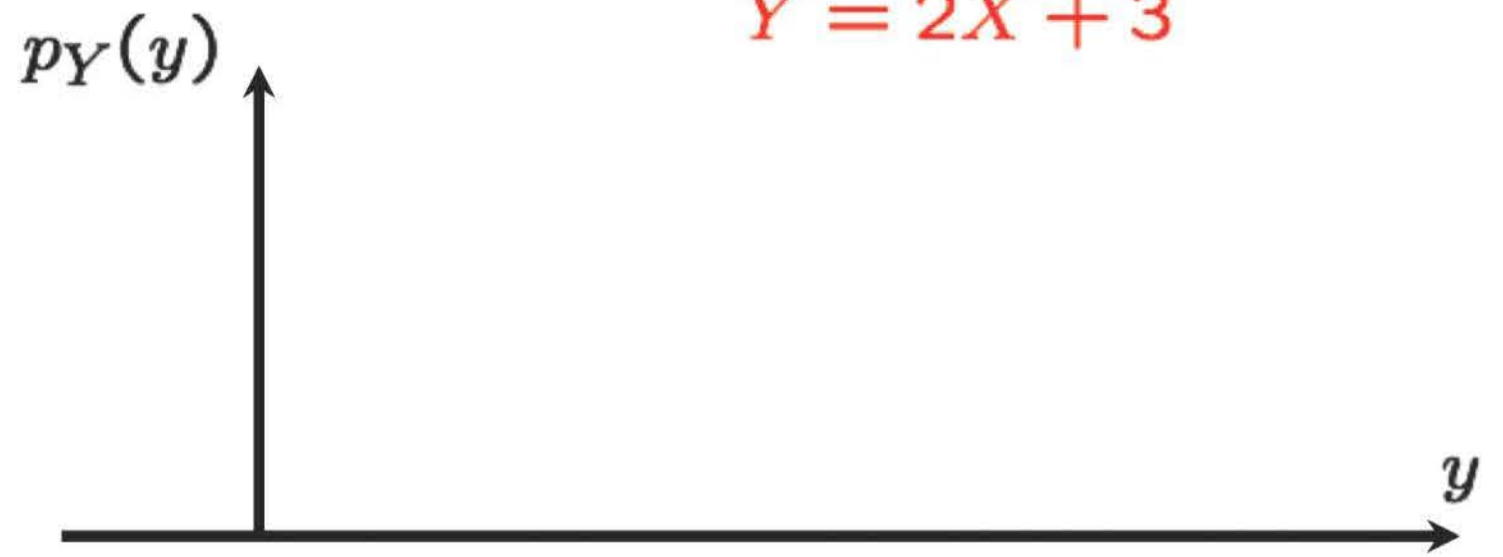


$$\begin{aligned} p_Y(y) &= \mathbf{P}(g(X) = y) \\ &= \sum_{x: g(x)=y} p_X(x) \end{aligned}$$

## A linear function of a discrete r.v.

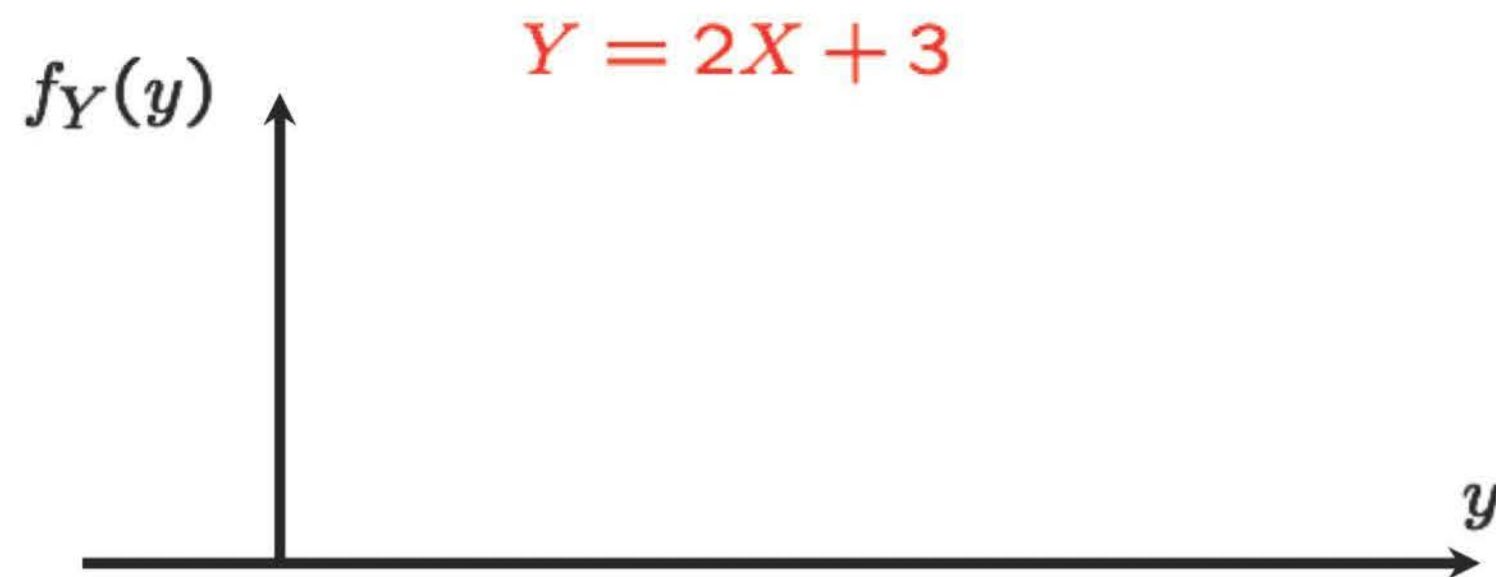
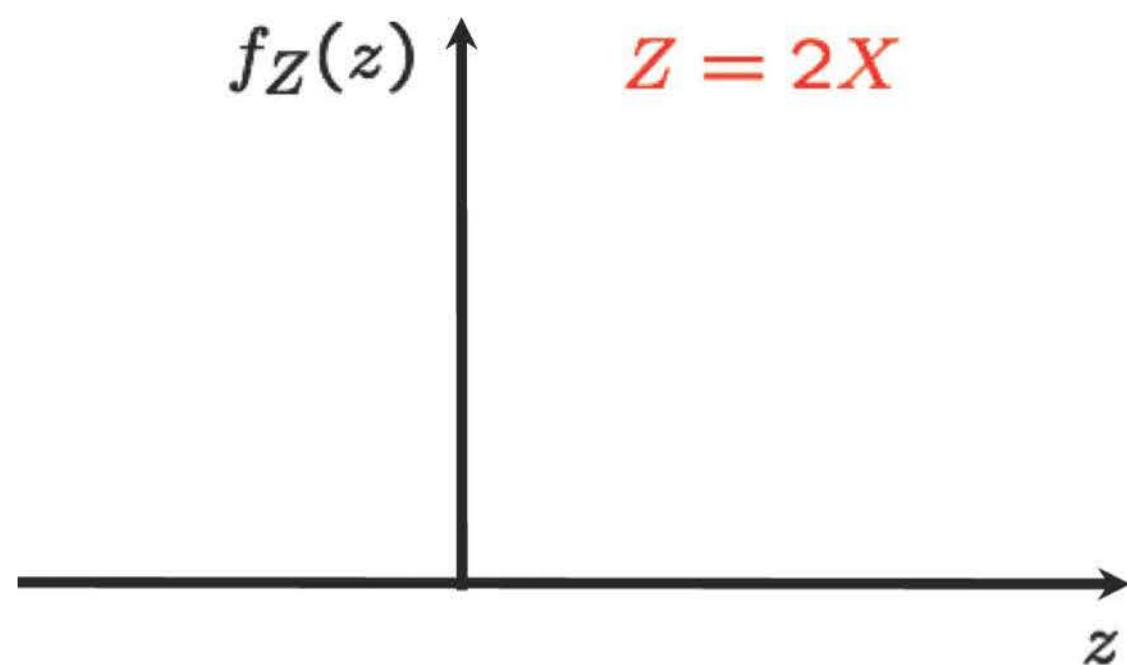
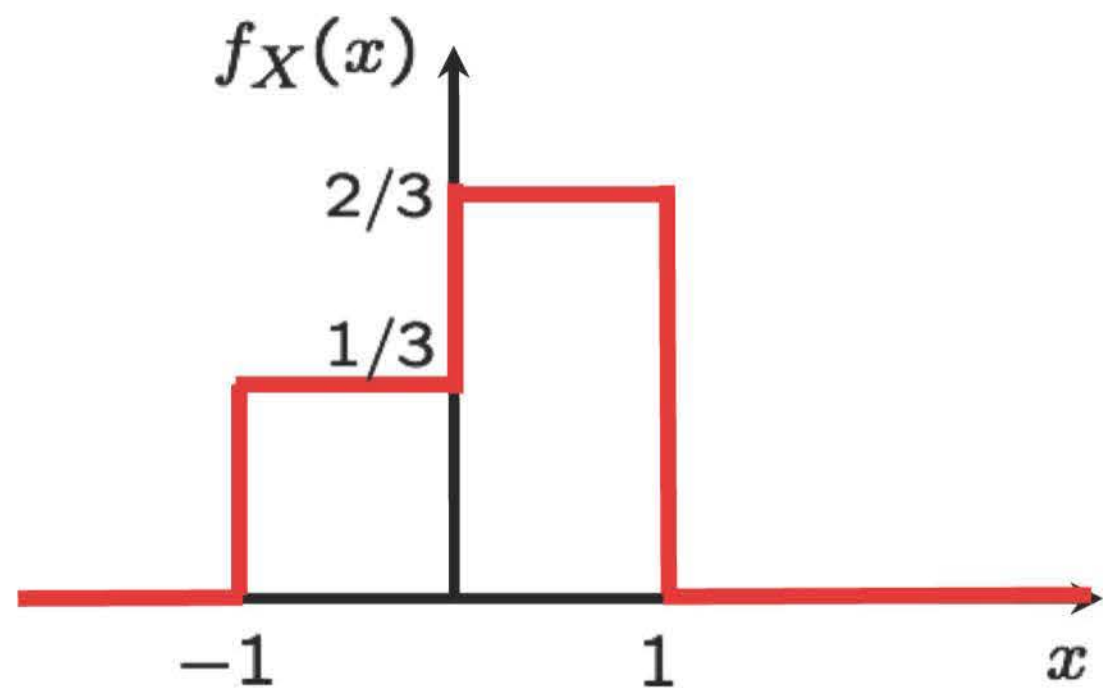


$$Y = 2X + 3$$



$$Y = aX + b : \quad p_Y(y) = p_X\left(\frac{y - b}{a}\right)$$

## A linear function of a continuous r.v.



**A linear function of a continuous r.v.**

$$Y = aX + b$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$p_Y(y) = p_X\left(\frac{y-b}{a}\right)$$

## A linear function of a normal r.v. is normal

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

$$Y = aX + b, \quad a \neq 0$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

## A general function $g(X)$ of a continuous r.v.

- **Two-step procedure:**
  - Find the CDF of  $Y$ :  $F_Y(y) = \mathbf{P}(Y \leq y)$
  - Differentiate:  $f_Y(y) = \frac{dF_Y}{dy}(y)$

**Example:**  $Y = X^3$ ;  $X$  uniform on  $[0, 2]$

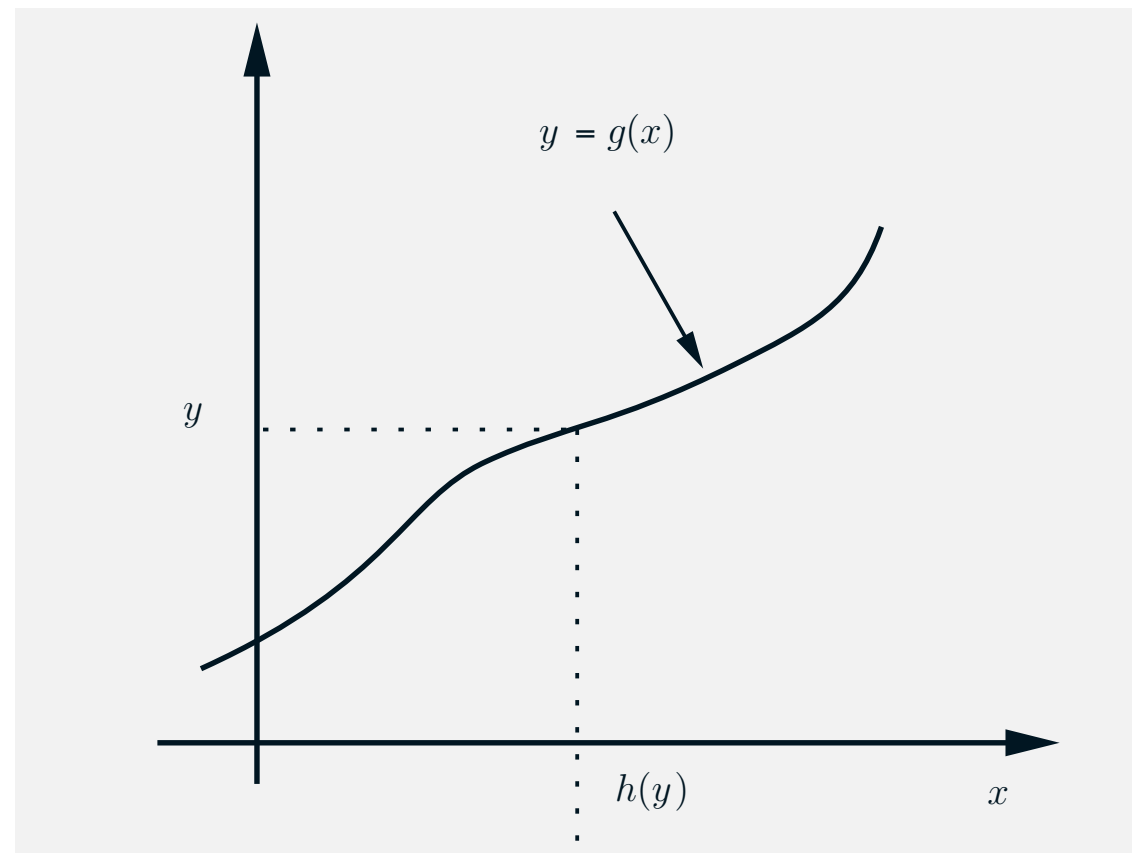


**Example:**  $Y = a/X$

- You go to the gym and set the speed  $X$  of the treadmill to a number between 5 and 10 km/hr (with a uniform distribution). Find the PDF of the time it takes to run 10km.

## A general formula for the PDF of $Y = g(X)$ when $g$ is monotonic

Assume  $g$  strictly increasing  
and differentiable

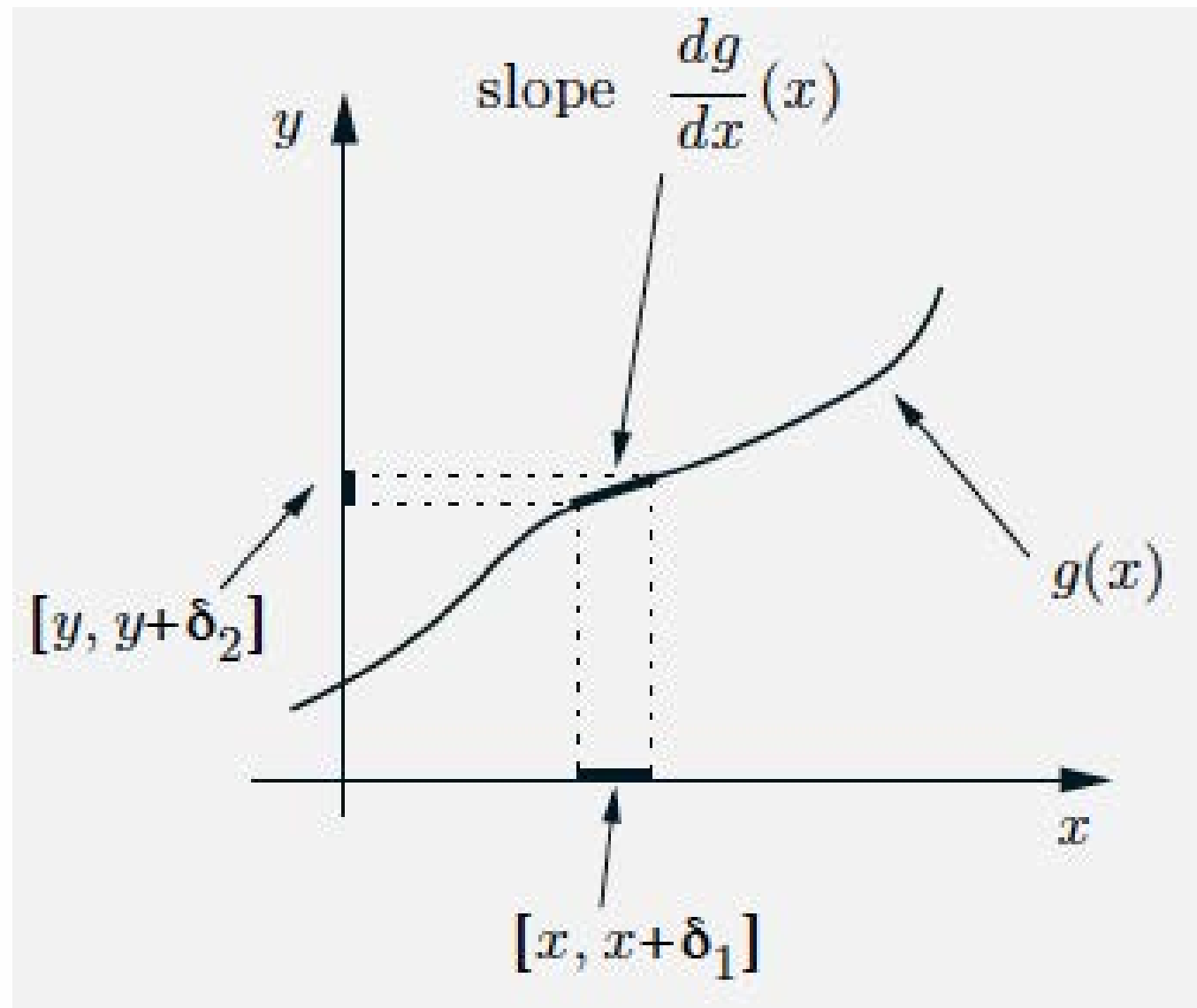


inverse function  $h$

$$f_Y(y) = f_X(h(y)) \left| \frac{dh}{dy}(y) \right|$$

**Example:**  $Y = X^2$ ;  $X$  uniform on  $[0, 1]$

## An intuitive explanation for the monotonic case



## A nonmonotonic example: $Y = X^2$

- The discrete case:

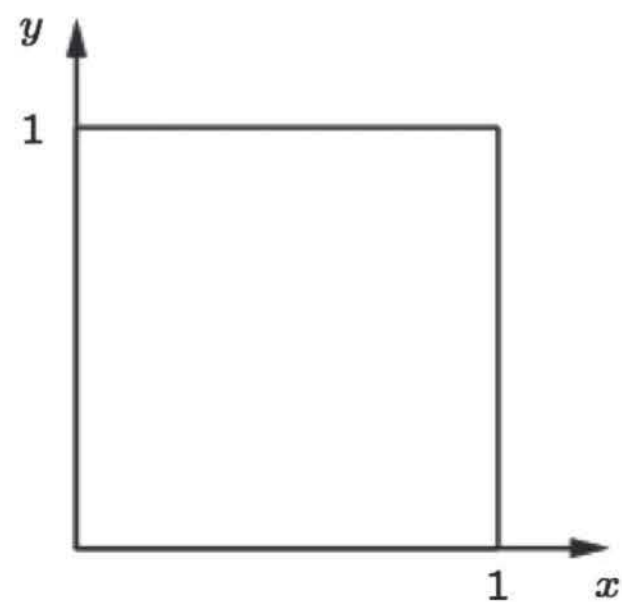
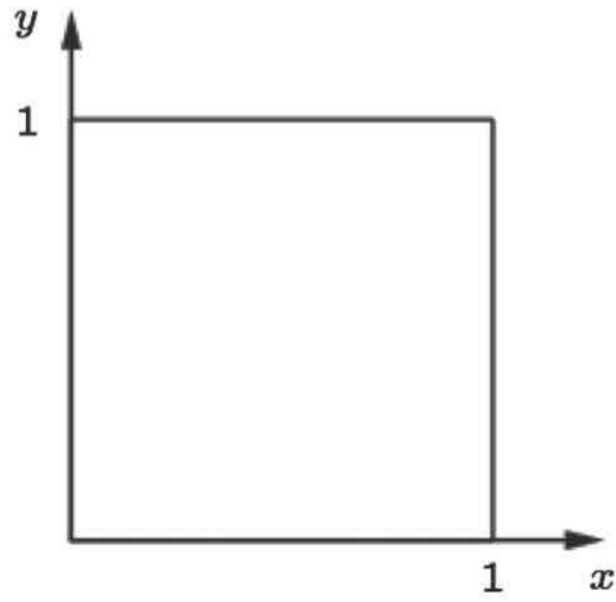
$$p_Y(9) =$$

$$p_Y(y) =$$

- The continuous case:

## A function of multiple r.v.'s: $Z = g(X, Y)$

- Same methodology: find CDF of  $Z$
- Let  $Z = Y/X$ ;  $X, Y$  independent, uniform on  $[0, 1]$



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