S O L U T I O N S

Stability via Frequency Response 7

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 7.1 (P4.9)

The time-delay term has a constant magnitude of 1, and a phase of -0.01ω radians. (It is a common mistake to use units of degrees here.) Thus a pure delay is equivalent to a negative phase shift that varies linearly with ω. Applying Equations 3.46 and 3.47 from the textbook gives

$$|L(j\omega)| = \frac{a_o}{\sqrt{\omega^2 + 1}} \tag{S7.1a}$$

and

$$\sphericalangle L(j\omega) = -\tan^{-1}\omega - 0.01\omega \text{ radians} \tag{S7.1b}$$

These two expressions are used to sketch a Nyquist diagram as shown in Figure S7.1.

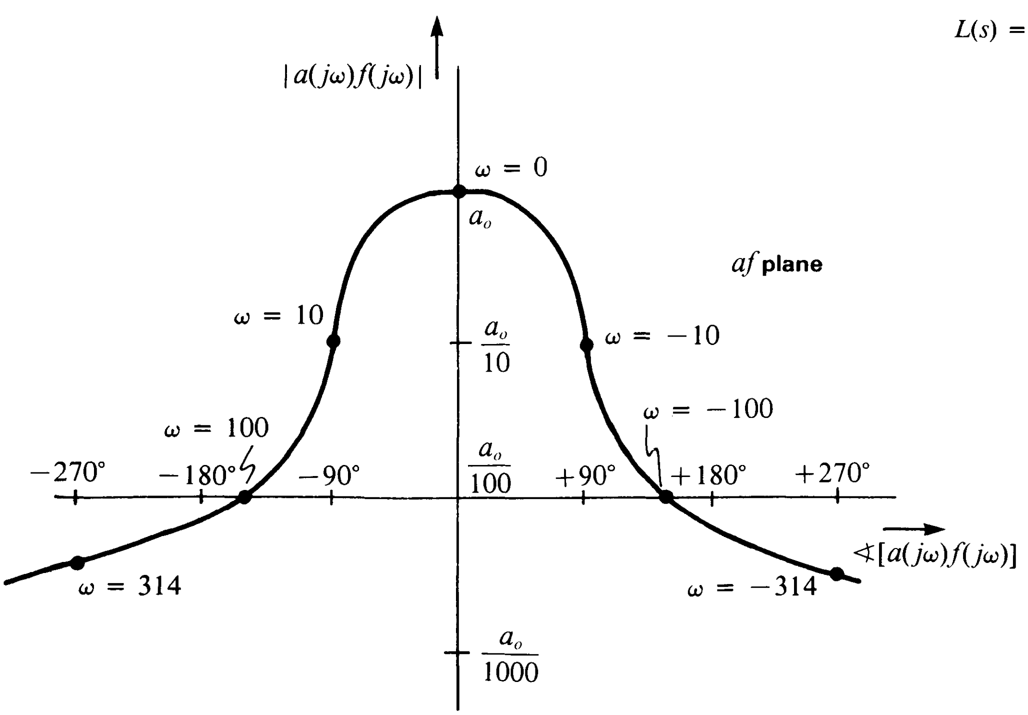


Figure S7.1 Nyquist diagram for $L(s) = -\dfrac{a_o e^{-0.01s}}{(s+1)}$.

Because we use degrees as the units for the phase axis, it is helpful to remember that there are 57.3 degrees per radian. As $\omega \rightarrow \infty$, the phase is unbounded. Thus, for a sufficiently large value of a_o, the $\pm 180°$ points will be enclosed in the contour, and the system will be unstable. The maximum value of a_o for stability is such that the $\pm 180°$ points are intersected by the af contour. Inspection of the Nyquist diagram indicates that this point will occur for $\omega > 100$. In this region, the magnitude and phase are well approximated by

$$|L(j\omega)| \simeq \frac{a_o}{\omega} \qquad \omega \gg 1 \tag{S7.2a}$$

and

$$\measuredangle L(j\omega) \simeq -\frac{\pi}{2} - 0.01\omega \qquad \omega \gg 1 \tag{S7.2b}$$

Applying Equation S7.2*b,* the frequency at which the phase is $-180°$ is:

$$-\pi = -\frac{\pi}{2} - 0.01\omega_1$$

or

$$\omega_1 = 157 \text{ rad/sec} \tag{S7.3}$$

Then, to intersect the $-180°$ point, we must have $|L(j\omega)|_{\omega=157} = 1$. Then, by Equation S7.2*a*, $a_o = 157$ is the maximum value that results in a stable system.

Because the feedback path is frequency independent, we may apply Equation 4.88 from the textbook to solve for the value of a_o, which results in $M_p = 1.4$.

$$1.4 \simeq \frac{1}{\sin \phi_m} \tag{S7.4}$$

Thus

$$\phi_m \simeq \sin^{-1}\left(\frac{1}{1.4}\right) \simeq 45° \tag{S7.5}$$

For a 45° phase margin, Equation S7.2*b* requires crossover at a frequency such that

$$-\frac{3\pi}{4} = -\frac{\pi}{2} - 0.01\omega$$

or

$$\omega \simeq 79 \text{ rad/sec} \tag{S7.6}$$

To have crossover at $\omega = 79$ rad/sec, Equation S7.2*a* requires that $a_o = 79$.

Solution 7.2 (P4.10)

Before drawing the Nyquist plot, it is helpful to draw a Bode plot for this system. Then, the Nyquist plot may be sketched directly from the Bode plot. Figure S7.2 is a Bode plot for the transfer function of interest

$$L(s) = -\frac{a_o s^3}{(s + 1)(0.1s + 1)^2} \tag{S7.7}$$

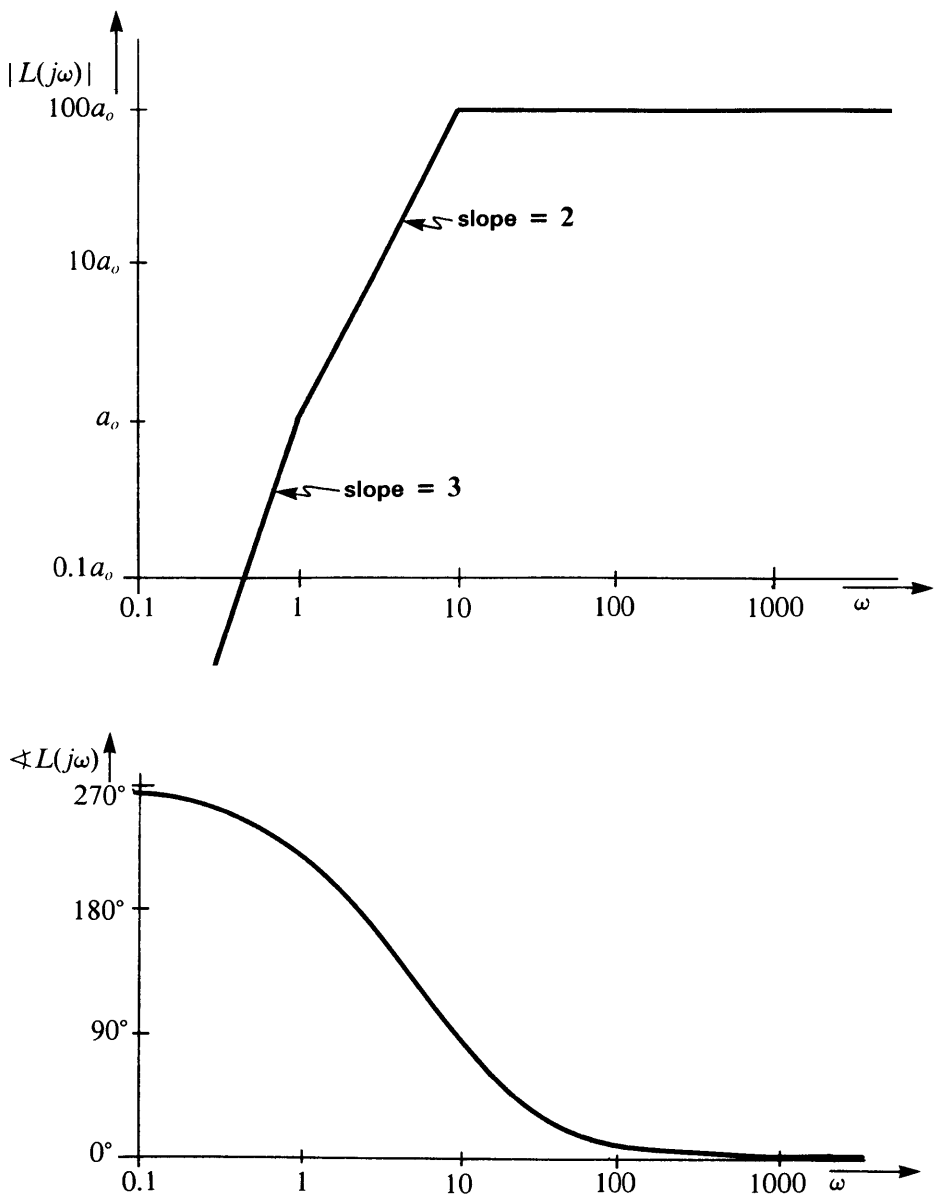
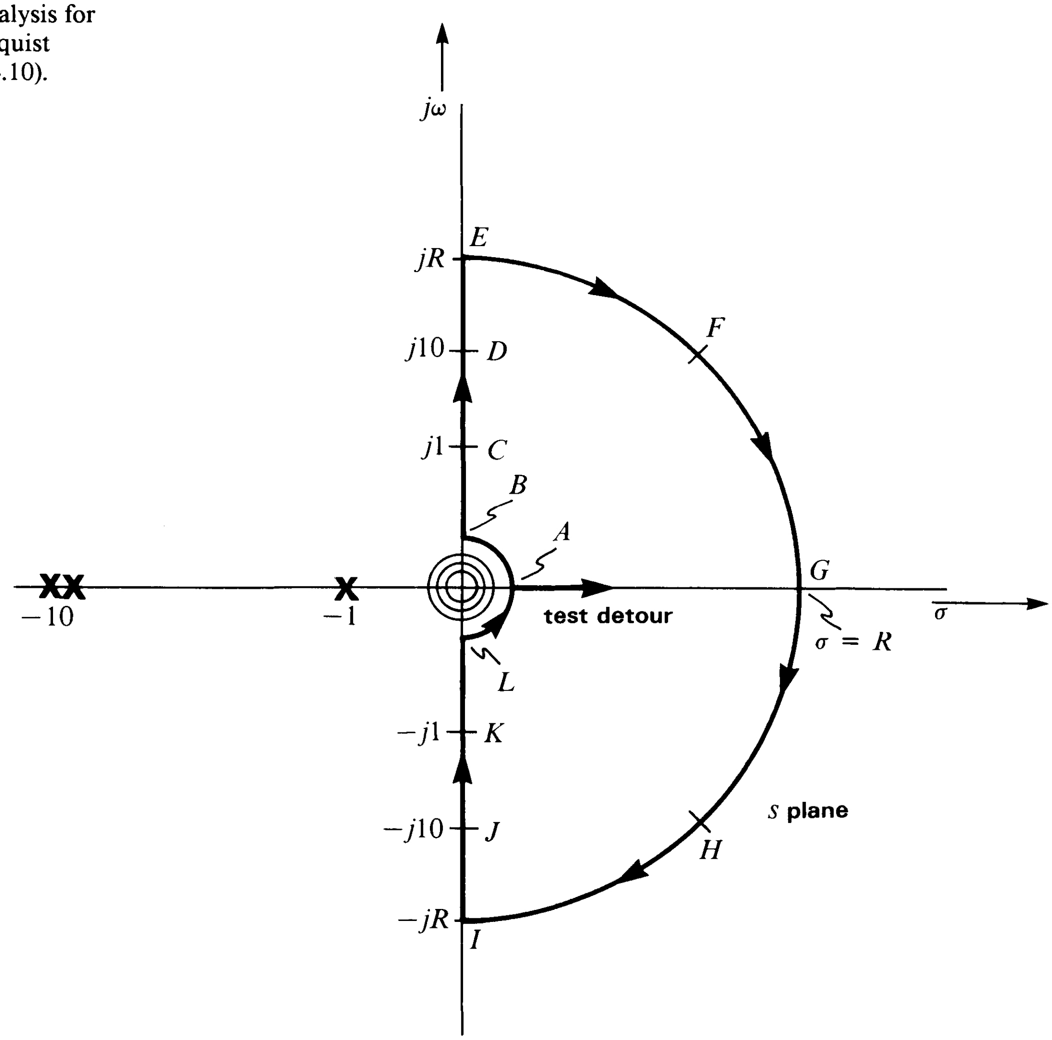


Figure S7.2 Bode plot for Problem 7.2 (P4.10).

Because there are singularities at the origin, we choose the contour shown in Figure S7.3*a*. The resulting Nyquist plot is as shown in Figure S7.3*b*. The points labeled A through L in the s plane map to the points equivalently labeled in the af plane. There are several important features to notice. For points near the origin in the s plane, the magnitude of $L(s)$ is very small. Thus, the point A in the s plane maps to the negative imaginary axis in the af plane as shown. For $|s| \gg 10$ (i.e., for points in the s plane far from the origin)

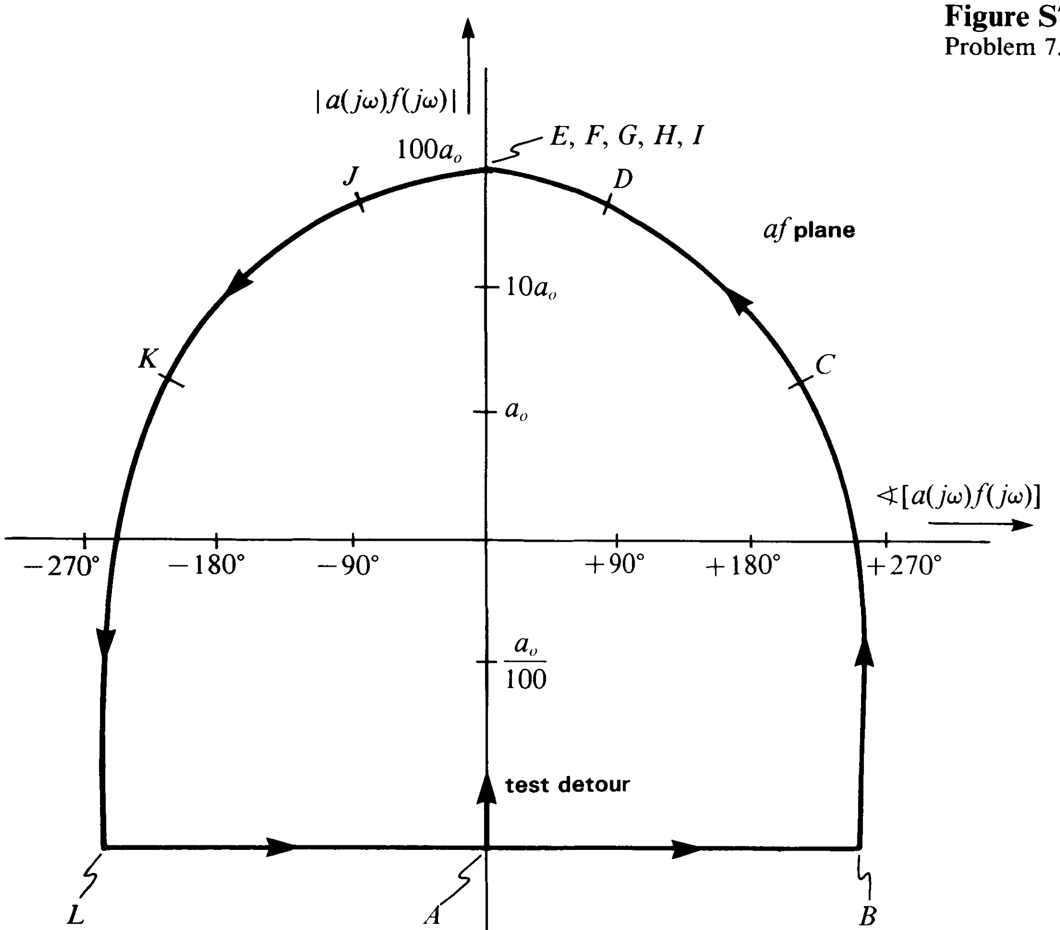
$$L(s) \simeq 100a_o \tag{S7.8}$$

Figure S7.3 Nyquist analysis for Problem 7.2 (P4.10). (*a*) Nyquist contour for Problem 7.2 (P4.10).



(*a*)

Figure S7.3 (*b*) Nyquist plot for Problem 7.2 (P4.10).



(*b*)

This is true all the way around the semicircle in the right half of the s plane. Thus, the points E, F, G, H, and I map to the point $af = 100a_o$ in the af plane. Finally, the test excursion shows that the interior of the contour in the s plane maps to the interior of the contour in the af plane. Clearly, for a large enough a_o, the points at $\pm 180°$ will be enclosed, and the system will be unstable.

The value of a_o required to reach the edge of instability can be solved for by finding the frequency at which $\measuredangle L(j\omega) = 180°$. Either directly from the Bode plot of Figure S7.2, or by iterating numerically on the expression for $\measuredangle L(j\omega)$

$$\measuredangle L(j\omega) = \frac{3\pi}{2} - \tan^{-1}\omega - 2\tan^{-1} 0.1\omega \qquad \textbf{(S7.9)}$$

we find that $\measuredangle L(j\omega) = 180°$ when $\omega = 2.2$ rad/sec. Then, the maximum a_o for which the system is stable is such that

$$|L(j\omega)|\Big|_{\omega=2.2} = 1 \qquad \textbf{(S7.10)}$$

Substituting in the expression for $|L(j\omega)|$ gives

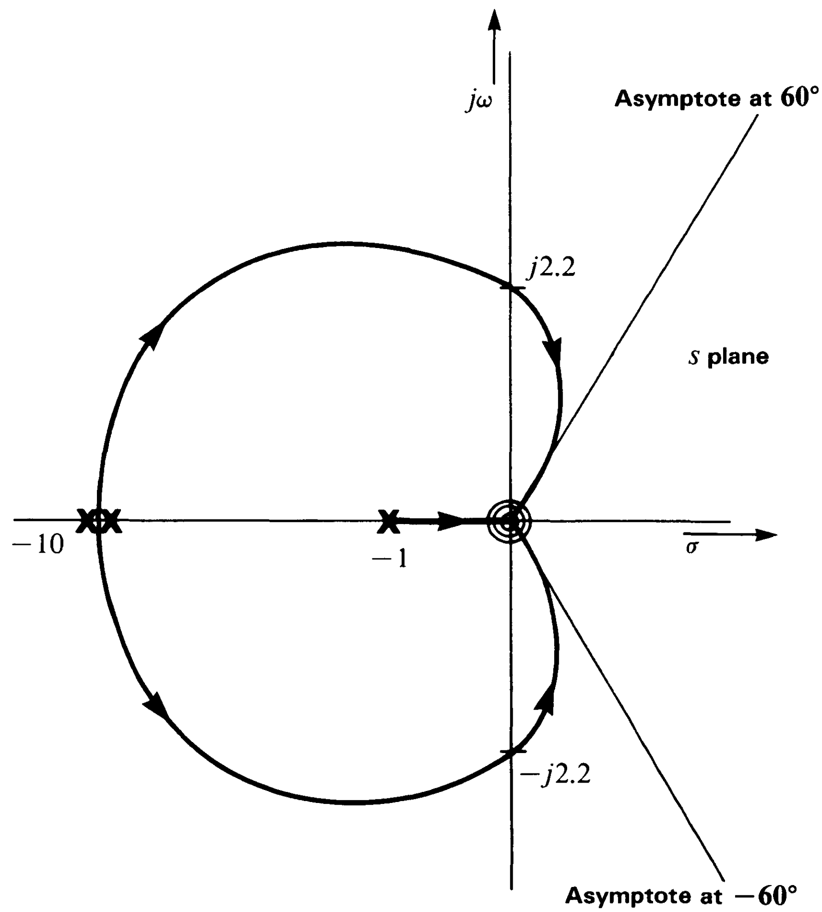
$$\frac{a_o\omega^3}{(\omega^2+1)^{1/2} \times ((0.1\omega)^2+1)}\Bigg|_{\omega=2.2} = 1 \qquad \textbf{(S7.11)}$$

or

$$a_o = \frac{((2.2)^2+1)^{1/2}((0.22)^2+1)}{2.2^3} = 0.24 \qquad \textbf{(S7.12)}$$

Thus, the system is stable for $a_o < 0.24$.

Figure S7.4 Root locus for Problem 7.2 (P4.10).



A root-locus construction also supports the conclusion that the system is unstable for large enough values of a_o, as shown in Figure S7.4. As previously calculated, the poles cross the imaginary axis at $\omega = 2.2$ and enter the right-half plane for $a_o > 0.24$. For large a_o, the two right-half-plane poles must approach the origin along asymptotes of $\pm 60°$, while the third pole approaches along the real axis. This must be so, because as the closed-loop poles approach the origin, the angle contribution from the pole at $s = -1$, and the two poles at $s = -10$, is essentially zero. Thus, the total angle from the three zeros to the closed-loop poles must be an odd multiple of 180°, which is satisfied by the asymptotes at $\pm 60°$.

Solution 7.3 (P4.11)

Poles that have a damping ratio of less than 0.707 lie to the right of a pair of lines at $\pm 45°$ from the negative real axis, because from Figure 3.7 of the textbook, $\theta = \cos^{-1}\zeta = \cos^{-1} 0.707 = 45°$. A contour that follows these lines, and encloses all poles with damping ratios less than 0.707 is shown in Figure S7.5.

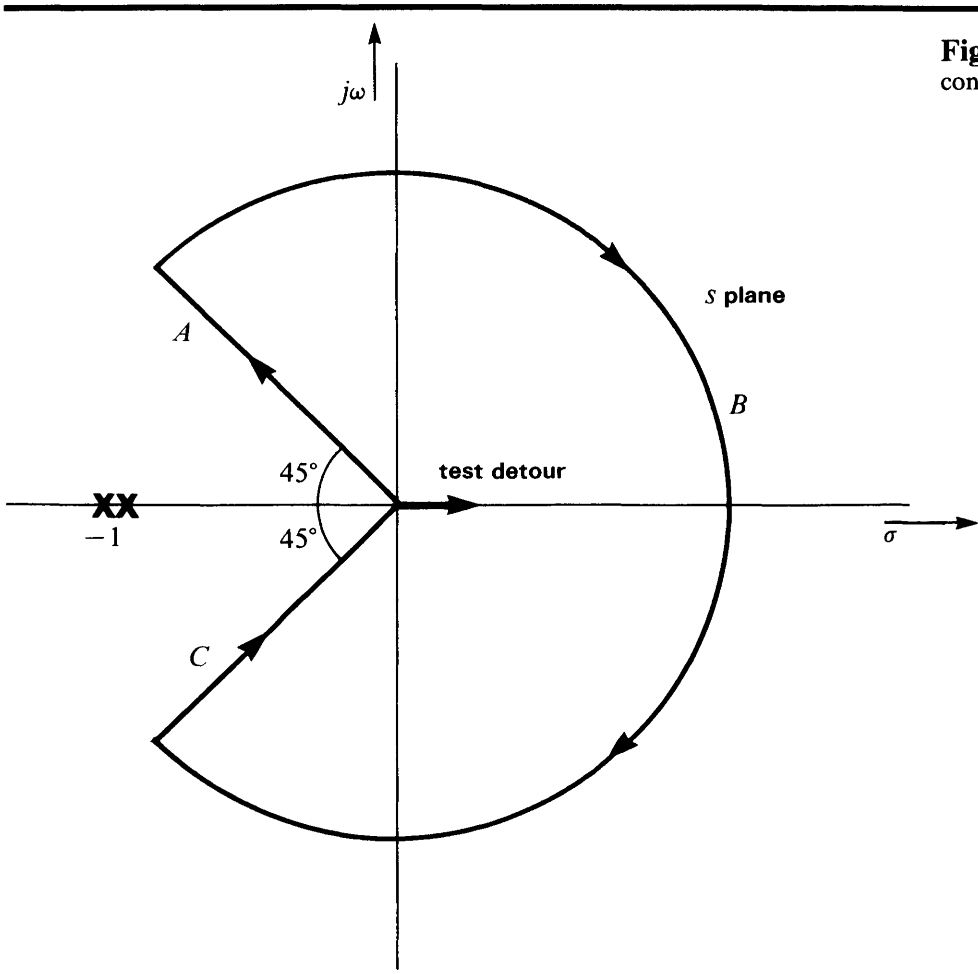


Figure S7.5 Modified Nyquist contour.

To make the modified Nyquist test we are interested in whether there are any poles within the contour of Figure S7.5, that is, whether there are any solutions of the characteristic equation $1 + a(s)f(s) = 0$ that occur for s within the contour of Figure S7.5. If there are such solutions, then the system has closed-loop poles with damping ratios less than 0.707. Thus the test in the af plane is unmodified. We look for points such that $a(s)f(s) = -1$ in exactly the same manner as the Nyquist test. Only the contour in the s plane needs to be changed to that shown in Figure S7.5. As in the Nyquist test, a test detour is used to determine where the interior of the contour in the s plane maps to in the af plane. The poles indicated at $s = -1$ are the poles of the transfer function

$$a(s)f(s) = \frac{a_o}{(s+1)^2} \tag{S7.13}$$

which we wish to evaluate using the modified Nyquist test. This test, then, is made by picking points in the s plane along the contour ABC, then plotting the value of $a(s)f(s)$ in the af plane for each of these points.

Applying this to the transfer function of Equation S7.13, at $s = 0$, $|a(s)f(s)| = a_o$, and $\sphericalangle a(s)f(s) = 0°$. As $|s| \to \infty$ along contour A, $|a(s)f(s)| \to 0$ and $\sphericalangle a(s)f(s) \to -270°$. Along the contour B, the magnitude of $a(s)f(s)$ remains small, and the angle changes from $-270°$ to $+270°$. The values of $a(s)f(s)$ resulting as the contour C is traversed are identical in magnitude and opposite in phase from the values generated along contour A. This preliminary analysis gives a general indication of the characteristics of the plot in the af plane. A more detailed analysis requires solving numerically. Along the contour A, $s = -\omega + j\omega$, thus

$$\begin{aligned} |a(s)f(s)|\Big|_{s=-\omega+j\omega} &= \frac{a_o}{|(-\omega + j\omega + 1)^2|} \\ &= \frac{a_o}{2\omega^2 - 2\omega + 1} \end{aligned} \tag{S7.14}$$

and

$$\begin{aligned} \sphericalangle a(s)f(s)\Big|_{s=-\omega+j\omega} &= \sphericalangle \frac{a_o}{(-\omega + j\omega + 1)^2} \\ &= -2\tan^{-1}\frac{\omega}{1-\omega} \end{aligned} \tag{S7.15}$$

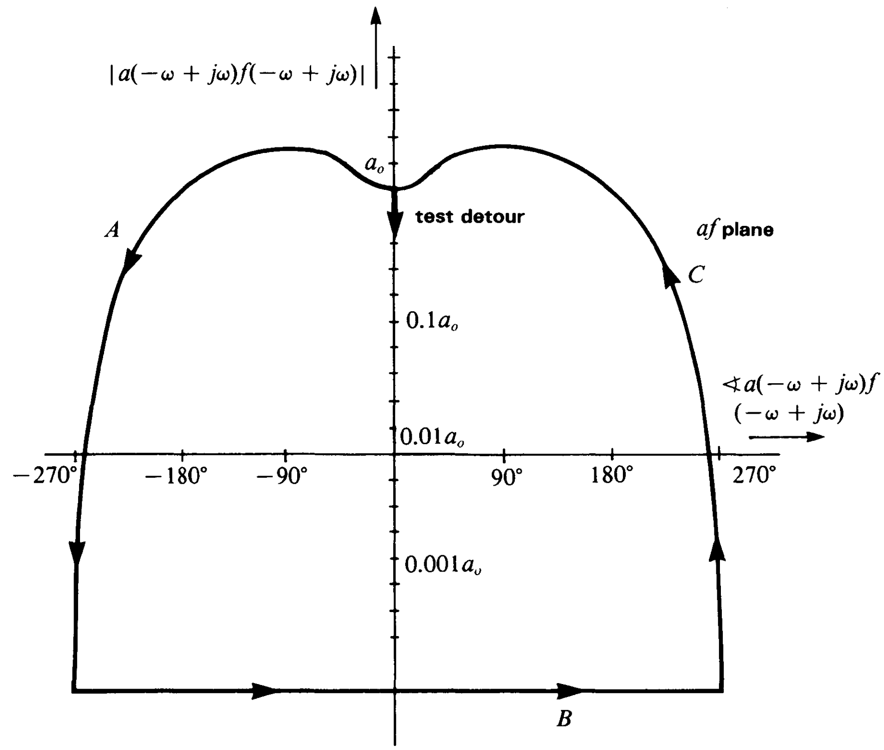
The magnitude and angle of $a(s)f(s)$ can then be solved numerically as s takes on the values $s = -\omega + j\omega$, and ω is allowed to vary. (A programmable calculator is quite helpful here, as it is throughout the subject.) When solving for the angle, be careful because the arc tangent is not a single-valued function. The earlier preliminary analysis serves as a check on the numerical results. Some values are summarized in Table S7.1.

ω	$\lvert a(-\omega + j\omega)f(-\omega + j\omega)\rvert$	$\measuredangle a(-\omega + j\omega)f(-\omega + j\omega)$
0	$1.00a_o$	0°
0.01	$1.02a_o$	−1.1°
0.05	$1.10a_o$	−6.0°
0.10	$1.22a_o$	−12.7°
0.25	$1.60a_o$	−36.9°
0.50	$2.00a_o$	−90.0°
0.75	$1.60a_o$	−143°
1.00	$1.00a_o$	−180°
1.25	$0.62a_o$	−203°
1.50	$0.40a_o$	−217°
1.75	$0.28a_o$	−226°
2.50	$0.12a_o$	−242°
5.00	$0.02a_o$	−257°
10.00	$0.006a_o$	−264°
$\to \infty$	$\to 0$	$\to -270°$

Table S7.1 Magnitude and angle of $\frac{a_o}{(s+1)^2}$, evaluated along the contour $s = -\omega + j\omega$.

Using these values, the contour of Figure S7.6 is then drawn in the *af* plane. The test detour indicates that the interior of the contour in the s plane maps to the interior of the contour in the *af* plane. Then, there are closed-loop poles with a damping ratio of less than 0.707 when the *af* plot of Figure S7.6 encloses the points at unity magnitude and an angle of $\pm 180°$. There is a pair of poles with a damping ratio of exactly 0.707 when the *af* contour intersects the -1 point. From the numerical values of Table S7.1, or by examining Figure S7.6, this occurs for $a_o = 1$.

Figure S7.6 Modified Nyquist diagram for Problem 7.3 (P4.11).



We check this result by factoring the characteristic equation for $a_o = 1$. With $a_o = 1$, the characteristic equation is

$$1 + \frac{1}{(s + 1)^2} = 0 \tag{S7.16}$$

After clearing fractions, we have

$$s^2 + 2s + 2 = 0 \tag{S7.17}$$

which has roots at

$$s = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm j \tag{S7.18}$$

These roots lie on lines at $\pm 45°$ from the negative real axis. Thus, as predicted, they have a damping ratio of 0.707.