Topic 18

Modeling of Elasto-Plastic and Creep Response—Part II

Contents:

- **Strain formulas to model creep strains**
- **Assumption of creep strain hardening for varying stress situations**
- **Creep in multiaxial stress conditions, use of effective stress and effective creep strain**
- **Explicit and implicit integration of stress**
- **Selection of size of time step in stress integration**
- **Thermo-plasticity and creep, temperature-dependency of material constants**
- **Example analysis: Numerical uniaxial creep results**
- **Example analysis: Collapse analysis of a column with offset load**
- **Example analysis: Analysis of cylinder subjected to heat treatment**

Textbook: Section 6.4.2

References: The computations in thermo-elasto-plastic-creep analysis are described in

Snyder, M. D., and K. J. Bathe, "A Solution Procedure for Thermo-Elastic-Plastic and Creep Problems," *Nuclear Engineering and Design, 64*, 49–80, 1981.

Cesar, F., and K. J. Bathe, "A Finite Element Analysis of Quenching Processes," in *Numerical Methods for Non-Linear Problems,* (Taylor, C., et al. eds.), Pineridge Press, 1984.

References:
(continued)

The effective-stress-function algorithm is presented in

Bathe, K. J., M. Kojić, and R. Slavković, "On Large Strain Elasto-Plastic and Creep Analysis," in *Finite Element Methods for Nonlinear Problems* (Bergan, P. G., K. J. Bathe, and W. Wunderlich, eds.), Springer-Verlag, 1986.

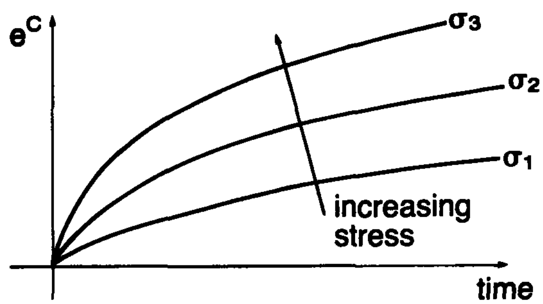
The cylinder subjected to heat treatment is considered in

Rammerstorfer, F. G., D. F. Fischer, W. Mitter, K. J. Bathe, and M. D. Snyder, "On Thermo-Elastic-Plastic Analysis of Heat-Treatment Processes Including Creep and Phase Changes," *Computers & Structures, 13*, 771–779, 1981.

Transparency 18-1

CREEP

We considered already uniaxial constant stress conditions. A typical creep law used is the power creep law $e^C = a_0 \sigma^{a_1} t^{a_2}$.



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Aside: other possible choices for the creep law are

- $e^C = a_0 \exp(a_1 \sigma) \left[1 - \exp\left(-a_2 \left(\frac{\sigma}{a_3}\right)^{a_4} t\right)\right] + a_5\, t \exp(a_6 \sigma)$
- $e^C = (a_0 (\sigma)^{a_1}) (t^{a_2} + a_3 t^{a_4} + a_5 t^{a_6}) \exp\left(\frac{-a_7}{{}^t\theta + 273.16}\right)$

  (${}^t\theta$: temperature, in degrees C)

We will not discuss these choices further.

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The creep strain formula $e^C = a_0 \sigma^{a_1} t^{a_2}$ cannot be directly applied to varying stress situations because the stress history does not enter directly into the formula.

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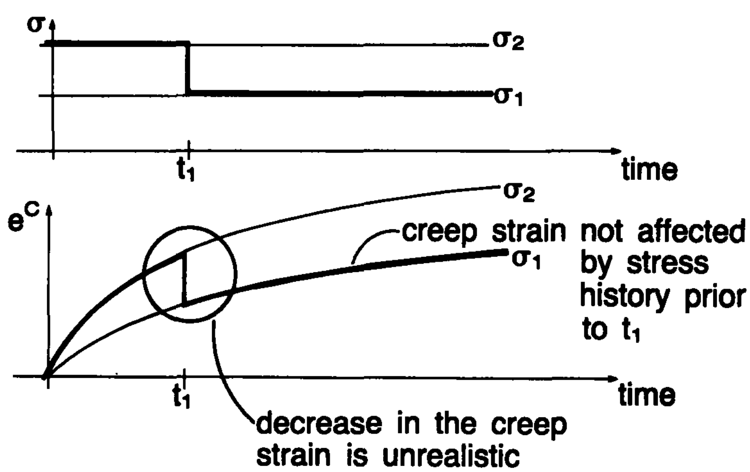
<u>Example</u>:



The assumption of strain hardening:

- The material creep behavior depends only on the current stress level and the accumulated total creep strain.
- To establish the ensuing creep strain, we solve for the "effective time" using the creep law:

$$ {}^{t}e^{C} = a_0 \, {}^{t}\sigma^{a_1} \, \underbrace{\bar{t}^{a_2}} $$

← totally unrelated to the physical time

(solve for $\bar{t}$)

Transparency 18-5

The effective time is now used in the creep strain rate formula:

$$ {}^{t}\dot{e}^{C} = a_0 \, {}^{t}\sigma^{a_1} \, a_2 \, \bar{t}^{a_2 - 1} $$

$$ = a_0^{1/a_2} \, a_2 \, ({}^{t}\sigma)^{a_1/a_2} \, ({}^{t}e^{C})^{\frac{a_2-1}{a_2}} $$

Now the creep strain rate depends on the current stress level and on the accumulated total creep strain.

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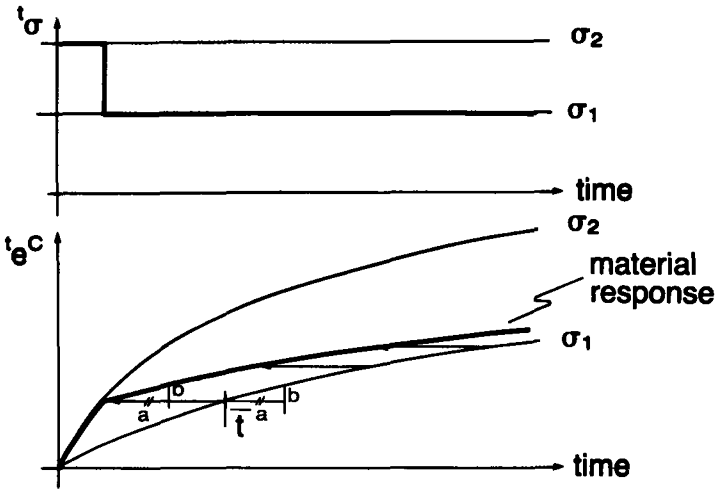
Pictorially:

- Decrease in stress



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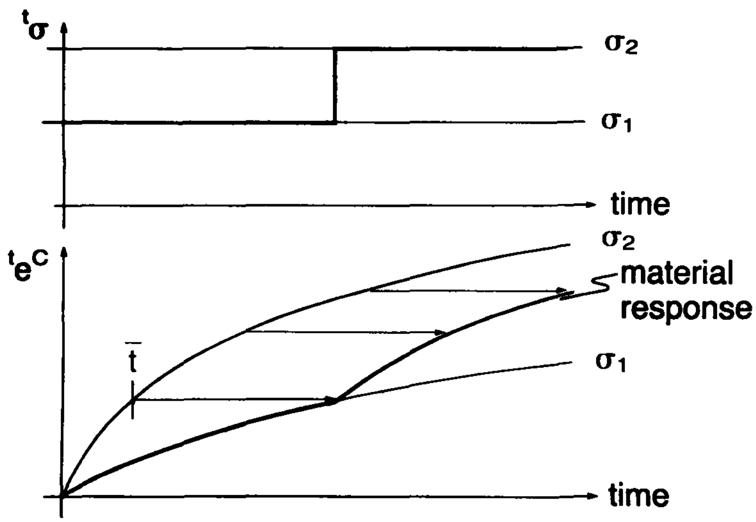
- Increase in stress



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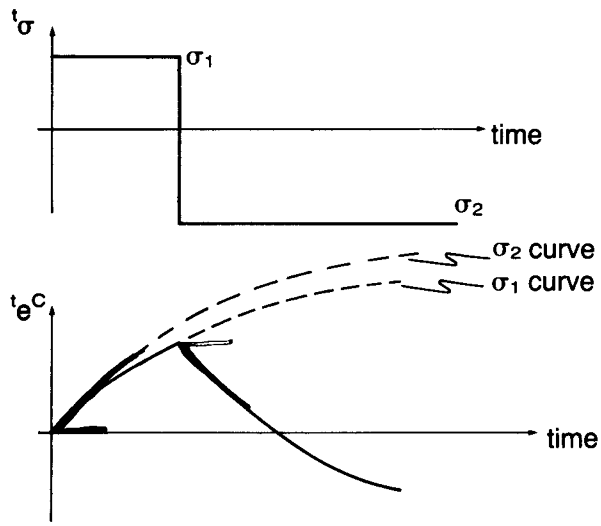
- Reverse in stress (cyclic conditions)



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MULTIAXIAL CREEP

The response is now obtained using

$${}^{t+\Delta t}\underline{\sigma} = {}^t\underline{\sigma} + \int_{{}^t\underline{e}}^{{}^{t+\Delta t}\underline{e}} \underline{C}^E \, d\,(\underline{e} - \underline{e}^C)$$

As in plasticity, the creep strains in multiaxial conditions are obtained by a generalization of the 1-D test results.

Transparency 18-11

We define

$$ {}^{t}\bar{\sigma} = \sqrt{\frac{3}{2}\,{}^{t}s_{ij}\,{}^{t}s_{ij}} \quad \text{(effective stress)} $$

$$ {}^{t}\bar{e}^{C} = \sqrt{\frac{2}{3}\,{}^{t}e_{ij}^{C}\,{}^{t}e_{ij}^{C}} \quad \text{(effective strain)} $$

and use these in the uniaxial creep law:

$$ \bar{e}^{C} = a_0\,\bar{\sigma}^{a_1}\,\bar{t}^{a_2} $$

Transparency 18-12

The assumption that the creep strain rates are proportional to the current deviatoric stresses gives

$$ {}^{t}\dot{e}_{ij}^{C} = {}^{t}\gamma\,{}^{t}s_{ij} \quad \text{(as in von Mises plasticity)} $$

${}^{t}\gamma$ is evaluated in terms of the effective stress and effective creep strain rate:

$$ {}^{t}\gamma = \frac{3}{2}\,\frac{{}^{t}\dot{\bar{e}}^{C}}{{}^{t}\bar{\sigma}} $$

$$ \left({}^{t}\dot{\bar{e}}^{C} = a_0\,a_2\,({}^{t}\bar{\sigma})^{a_1}\,(\bar{t})^{a_2-1}\right) $$

Using matrix notation,

$$d\underline{e}^C = ({}^t\gamma)\underbrace{(\underline{D}\,{}^t\underline{\sigma})}_{\text{deviatoric stresses}}dt$$

For 3-D analysis,

$$\underline{D} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & & & \\ & \frac{2}{3} & -\frac{1}{3} & & & \\ & & \frac{2}{3} & & & \\ & & & 1 & & \\ & \text{symmetric} & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Transparency 18-13

- In creep problems, the time integration is difficult due to the high exponent on the stress.
- Solution instability arises if the Euler forward integration is used and the time step Δt is too large.
  - Rule of thumb:

$$\Delta\bar{\underline{e}}^C \leq \frac{1}{10}({}^t\bar{\underline{e}}^E)$$

- Alternatively, we can use implicit integration, using the α-method:

$${}^{t+\alpha\Delta t}\underline{\sigma} = (1-\alpha)\,{}^t\underline{\sigma} + \alpha\,{}^{t+\Delta t}\underline{\sigma}$$

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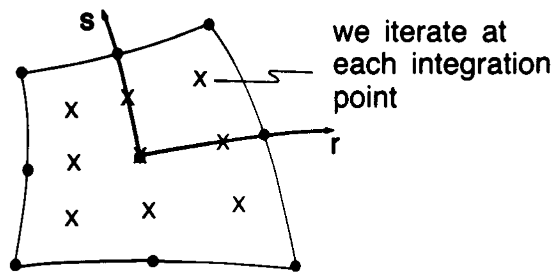
Iteration algorithm:

$$ {}^{t+\Delta t}\underline{\sigma}_{(k)}^{(i-1)} = {}^{t}\underline{\sigma} + \underline{C}^{E}\left[\underline{e}^{(i-1)} - \Delta t\ {}^{t+\alpha\Delta t}\gamma_{(k-1)}^{(i-1)}\ (\underline{D}\ {}^{t+\alpha\Delta t}\underline{\sigma}_{(k-1)}^{(i-1)})\right] $$

k = iteration counter at each integration point



Transparency 18-16

- $\alpha \geq \frac{1}{2}$ gives a stable integration algorithm. We use largely $\alpha = 1.0$.
- In practice, a form of Newton-Raphson iteration to accelerate convergence of the iteration can be used.

Transparency 18-17

- Choice of time step Δt is now governed by need to converge in the iteration and accuracy considerations.
- Subincrementation can be employed.
- Relatively large time steps can be used with the effective-stress-function algorithm.

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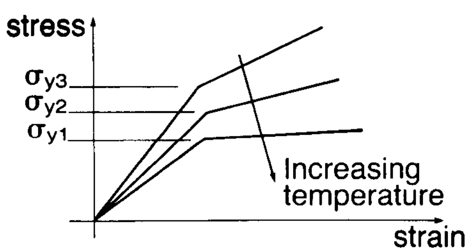
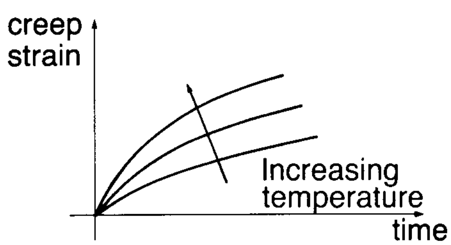
THERMO-PLASTICITY-CREEP

Plasticity:



Creep:



Transparency 18-19

Now we evaluate the stresses using

$$^{t+\Delta t}\underline{\sigma} = {}^{t}\underline{\sigma} + \int_{^{t}\underline{e}}^{^{t+\Delta t}\underline{e}} {}^{\tau}\underline{C}^{E}\, d(\underline{e} - \underline{e}^{P} - \underline{e}^{C} - \underbrace{\underline{e}^{TH}}_{\text{thermal strains}})$$

Using the α-method,

$$^{t+\Delta t}\underline{\sigma} = {}^{t+\Delta t}\underline{C}^{E}\{[\underline{e} - \underline{e}^{P} - \underline{e}^{C} - \underline{e}^{TH}] + [{}^{t}\underline{e} - {}^{t}\underline{e}^{P} - {}^{t}\underline{e}^{C} - {}^{t}\underline{e}^{TH}]\}$$

where

$$\underline{e} = {}^{t+\Delta t}\underline{e} - {}^{t}\underline{e}$$

Transparency 18-20

and

$$\underline{e}^{P} = \Delta t\, ({}^{t+\alpha\Delta t}\bar{\lambda})\, (\underline{D}\, {}^{t+\alpha\Delta t}\underline{\sigma})$$

$$\underline{e}^{C} = \Delta t\, ({}^{t+\alpha\Delta t}\gamma)\, (\underline{D}\, {}^{t+\alpha\Delta t}\underline{\sigma})$$

$$e_{ij}^{TH} = ({}^{t+\Delta t}\alpha\; {}^{t+\Delta t}\theta - {}^{t}\alpha\; {}^{t}\theta)\, \delta_{ij}$$

where

$^{t}\alpha$ = coefficient of thermal expansion at time t

$^{t}\theta$ = temperature at time t

The final iterative equation is

$$
\begin{aligned}
{}^{t+\Delta t}\underline{\sigma}_{(k)}^{(i-1)} = \underline{C}^{E}\Big|_{t+\Delta t} \Big[& {}^{t+\Delta t}\underline{e}^{(i-1)} - {}^{t}\underline{e}^{P} - {}^{t}\underline{e}^{C} - {}^{t}\underline{e}^{TH} \\
& - \Delta t \left({}^{t+\alpha\Delta t}\bar{\lambda}_{(k-1)}^{(i-1)}\right) \left(\underline{D}\ {}^{t+\alpha\Delta t}\underline{\sigma}_{(k-1)}^{(i-1)}\right) \\
& - \Delta t \left({}^{t+\alpha\Delta t}\gamma_{(k-1)}^{(i-1)}\right) \left(\underline{D}\ {}^{t+\alpha\Delta t}\underline{\sigma}_{(k-1)}^{(i-1)}\right) \\
& - \underline{e}^{TH} \Big]
\end{aligned}
$$

and subincrementation may also be used.

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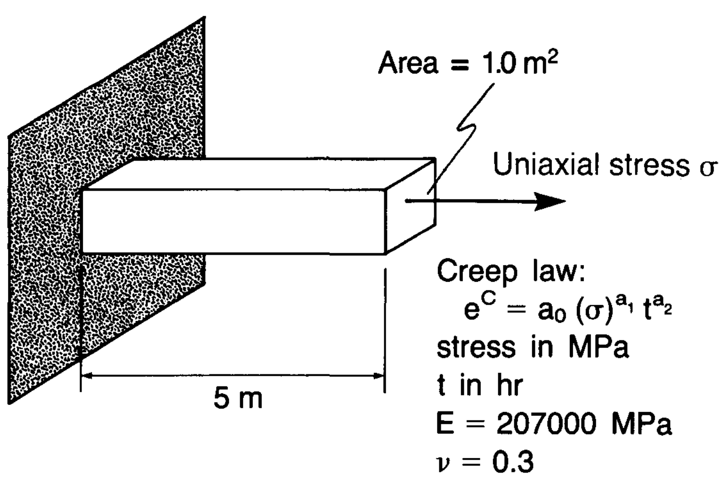
Numerical uniaxial creep results:



Creep law:

$e^{C} = a_0\,(\sigma)^{a_1}\, t^{a_2}$

stress in MPa

t in hr

E = 207000 MPa

$\nu = 0.3$

Transparency 18-22

Transparency 18-23

The results are obtained using two solution algorithms:

- $\alpha = 0$, (no subincrementation)
- $\alpha = 1$, effective-stress-function procedure

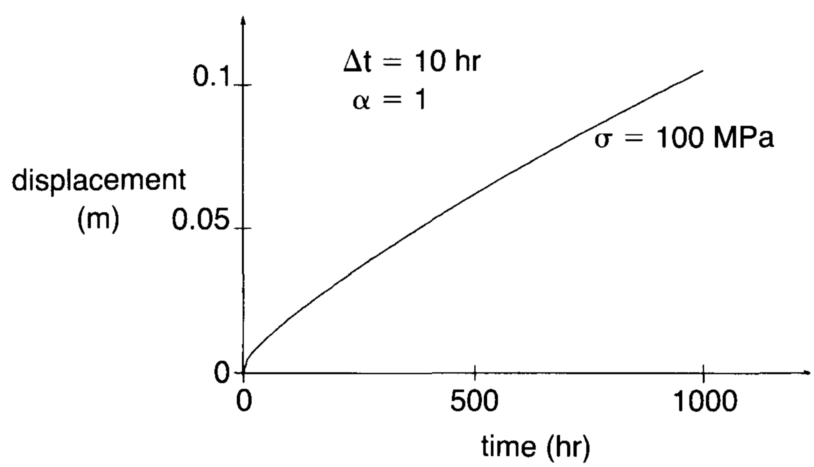
In all cases, the MNO formulation is employed. Full Newton iterations without line searches are used with

ETOL = 0.001
RTOL = 0.01
RNORM = 1.0 MN

Transparency 18-24

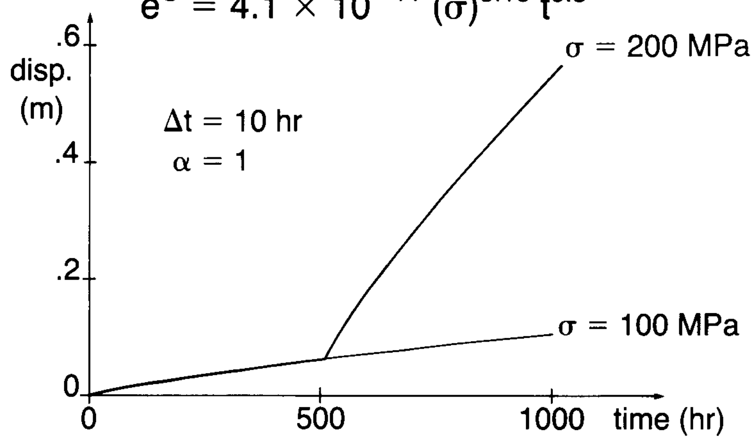
1) Constant load of 100 MPa

$$e^C = 4.1 \times 10^{-11} (\sigma)^{3.15} t^{0.8}$$



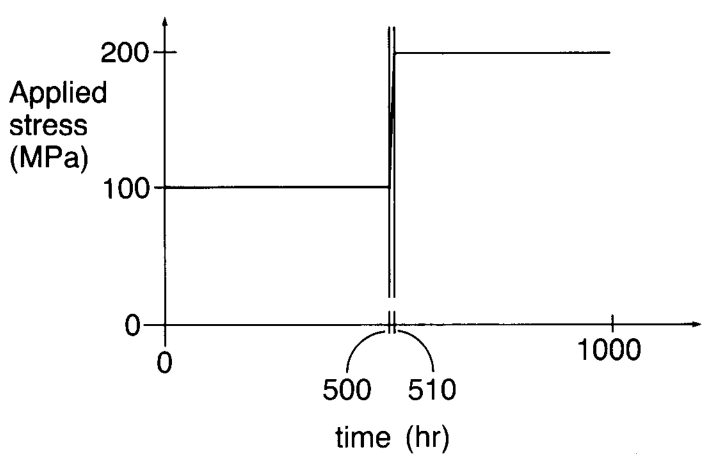
2) Stress increase from 100 MPa to 200 MPa

$$e^C = 4.1 \times 10^{-11} (\sigma)^{3.15} t^{0.8}$$



Transparency 18-25

Load function employed:

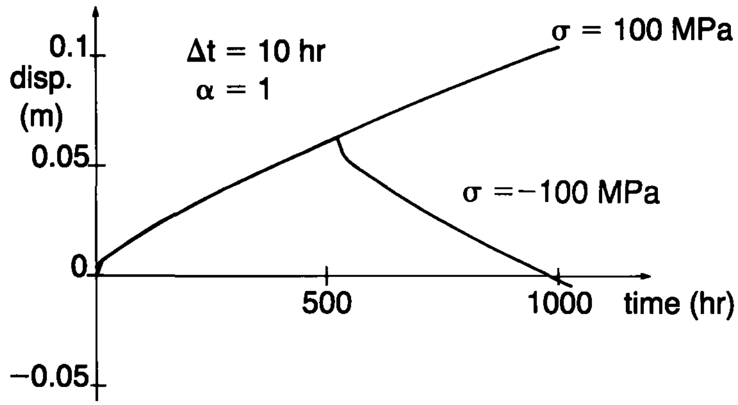


Transparency 18-26

Transparency 18-27

3) Stress reversal from 100 MPa to −100 MPa

$$e^C = 4.1 \times 10^{-11} (\sigma)^{3.15} t^{0.8}$$



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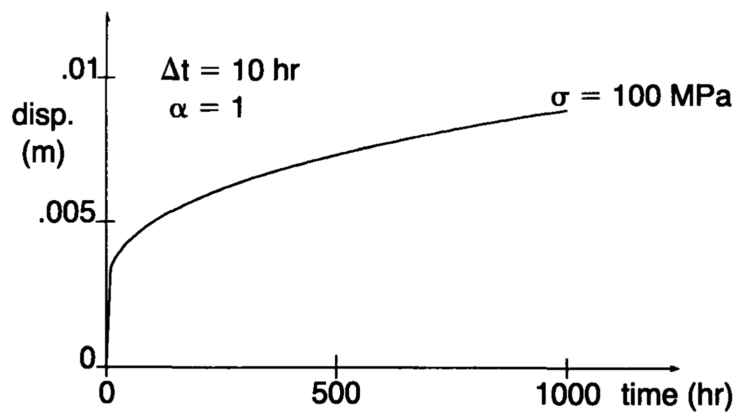
4) Constant load of 100 MPa

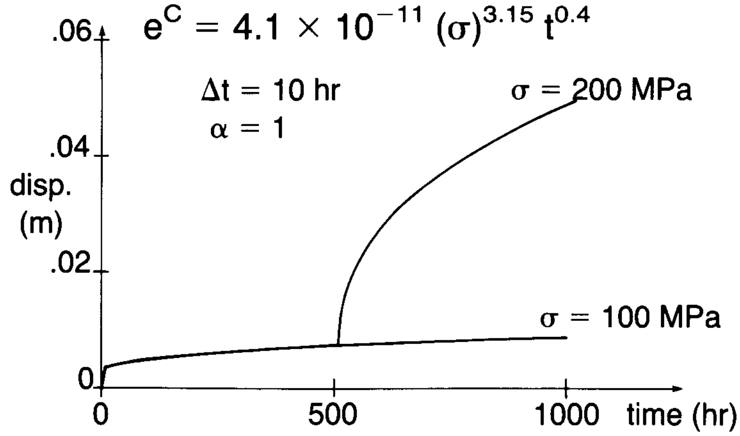
$$e^C = 4.1 \times 10^{-11} (\sigma)^{3.15} t^{0.4}$$



Transparency 18-29

5) Stress increase from 100 MPa to 200 MPa

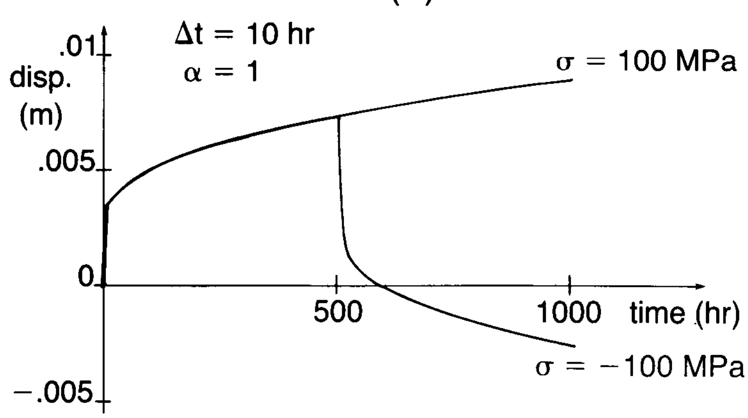
$$e^C = 4.1 \times 10^{-11} (\sigma)^{3.15} t^{0.4}$$



Transparency 18-30

6) Stress reversal from 100 MPa to −100 MPa

$$e^C = 4.1 \times 10^{-11} (\sigma)^{3.15} t^{0.4}$$



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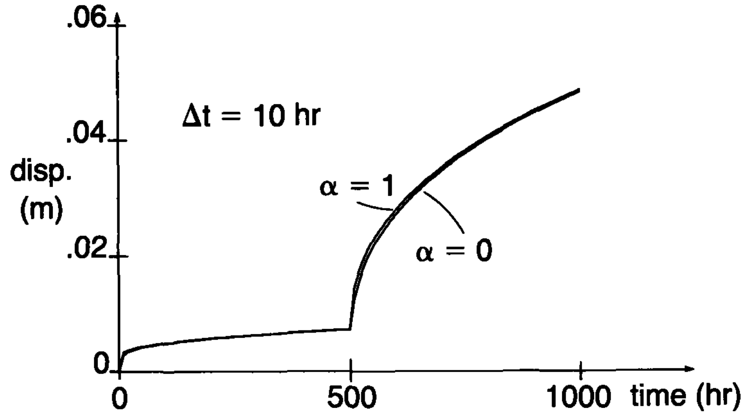
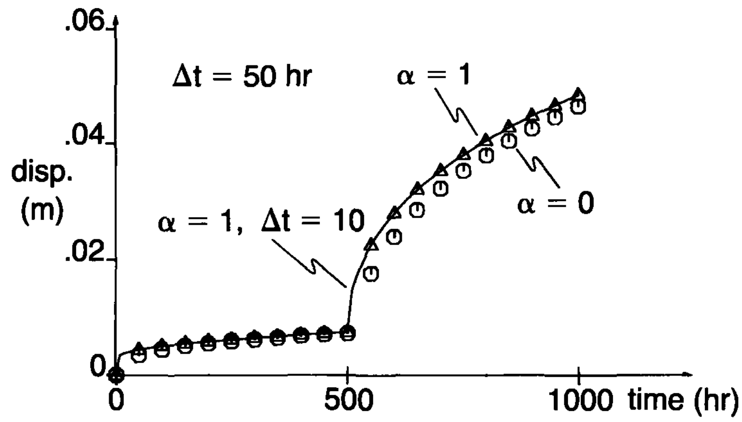
Consider the use of α = 0 for the "stress increase from 100 MPa to 200 MPa" problem solved earlier (case #5):

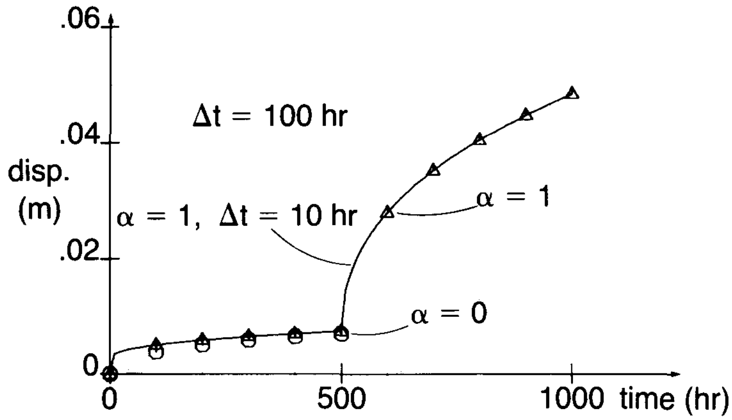


Transparency 18-32

Using Δt = 50 hr, both algorithms converge, although the solution becomes less accurate for α = 0.



Using Δt = 100 hr, α = 0 does not converge at t = 600 hr. α = 1 still gives good results.



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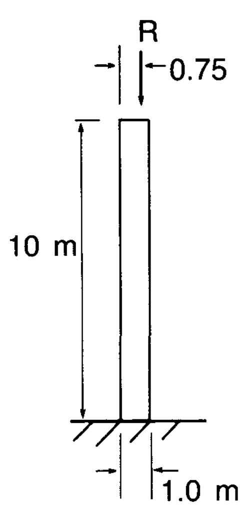
<u>Example</u>: Column with offset load



$E = 2 \times 10^6$ KPa
$\nu = 0.0$
plane stress
thickness = 1.0 m

Euler buckling load = 4100 KN

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Transparency 18-35

Goal: Determine the collapse response for different material assumptions:

— Elastic

— Elasto-plastic

— Creep

The total Lagrangian formulation is employed for all analyses.

Transparency 18-36

Solution procedure:

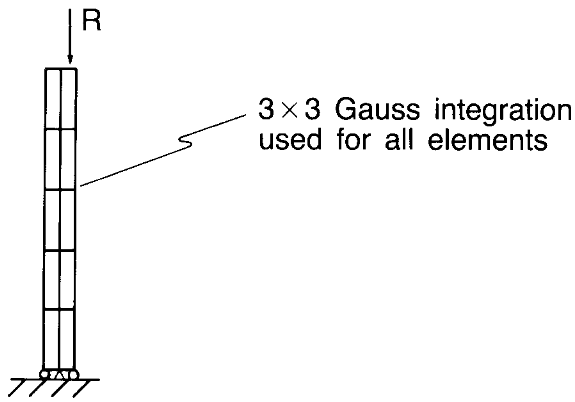
- The full Newton method without line searches is employed with

  ETOL = 0.001

  RTOL = 0.01

  RNORM = 1000 KN

Mesh used: Ten 8-node quadrilateral elements



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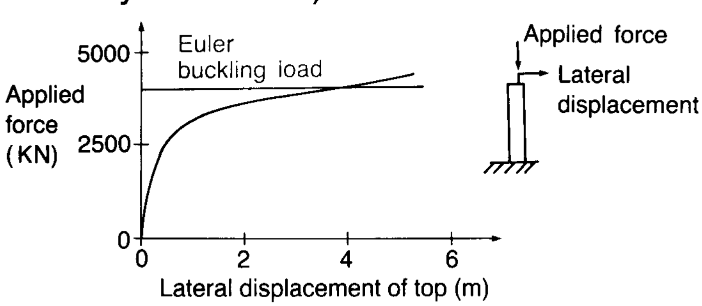
Elastic response: We assume that the material law is approximated by

$${}_0^tS_{ij} = {}_0^tC_{ijrs}\ {}_0^t\varepsilon_{rs}$$

where the components ${}_0^tC_{ijrs}$ are constants determined by E and ν (as previously described).



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Elasto-plastic response: Here we use

$E_T = 0$

$\sigma_y = 3000$ KPa (von Mises yield criterion)

and

$$ {}^{t+\Delta t}_{0}\underline{S} = {}^{t}_{0}\underline{S} + \int_{{}^{t}_{0}\underline{\varepsilon}}^{{}^{t+\Delta t}_{0}\underline{\varepsilon}} {}_{0}\underline{C}^{EP}\, d_{0}\underline{\varepsilon} $$

where ${}_{0}\underline{C}^{EP}$ is the incremental elasto-plastic constitutive matrix.

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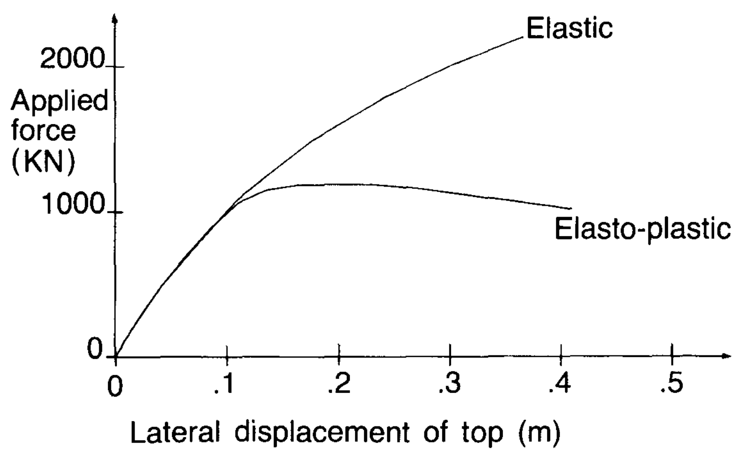
Plastic buckling is observed.



Creep response:

- Creep law: $\bar{e}^C = 10^{-16}(\bar{\sigma})^3 t$ (t in hours)
  No plasticity effects are included.
- We apply a constant load of 2000 KN and determine the time history of the column.
- For the purposes of this problem, the column is considered to have collapsed when a lateral displacement of 2 meters is reached. This corresponds to a total strain of about 2 percent at the base of the column.

Transparency 18-41

We investigate the effect of different time integration procedures on the obtained solution:

- Vary Δt ($\Delta t = .5, 1, 2, 5$ hr.)
- Vary α ($\alpha = 0, 0.5, 1$)

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Transparency 18-43

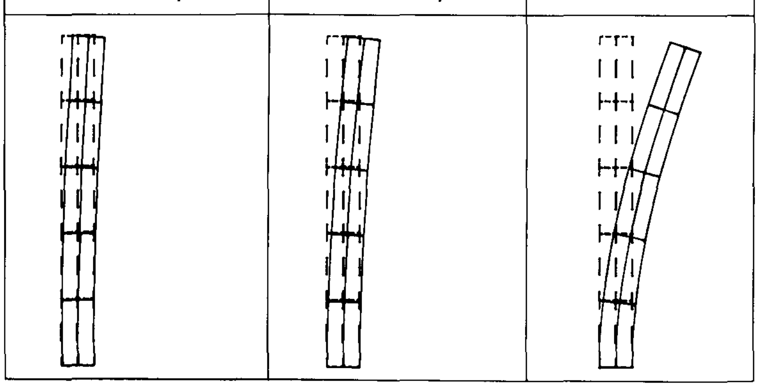
Collapse times: The table below lists the first time (in hours) for which the lateral displacement of the column exceeds 2 meters.

	$\alpha=0$	$\alpha=.5$	$\alpha=1$
$\Delta t=.5$	100.0	100.0	98.5
$\Delta t=1$	101	101	98
$\Delta t=2$	102	102	96
$\Delta t=5$	105	105	90

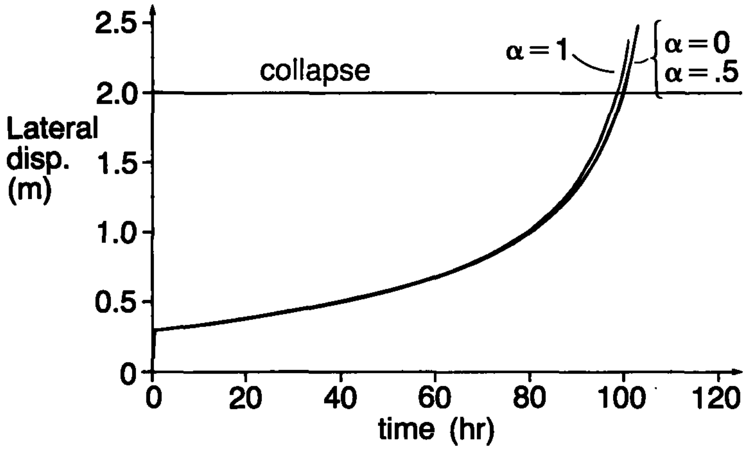
Transparency 18-44

Pictorially, using $\Delta t=0.5$ hr., $\alpha=0.5$, we have

Time = 1 hr (negligible creep effects)	Time = 50 hr (some creep effects)	Time = 100 hr (collapse)



Choose $\Delta t = 0.5$ hr.
— All solution points are connected with straight lines.



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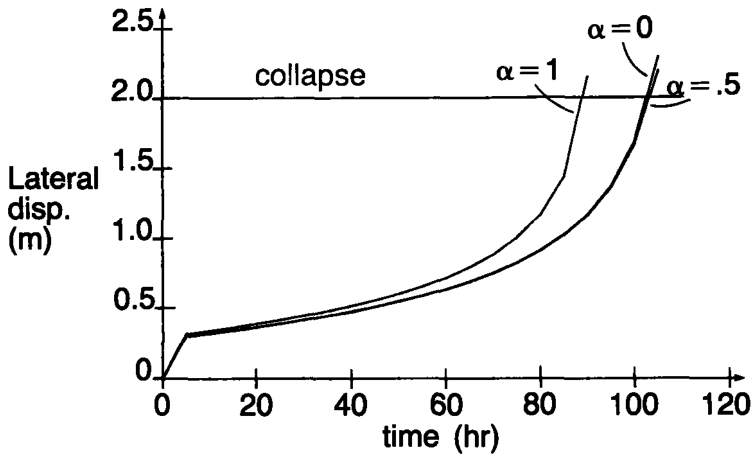
Effect of α: Choose $\Delta t = 5$ hr.
— All solution points are connected with straight lines.



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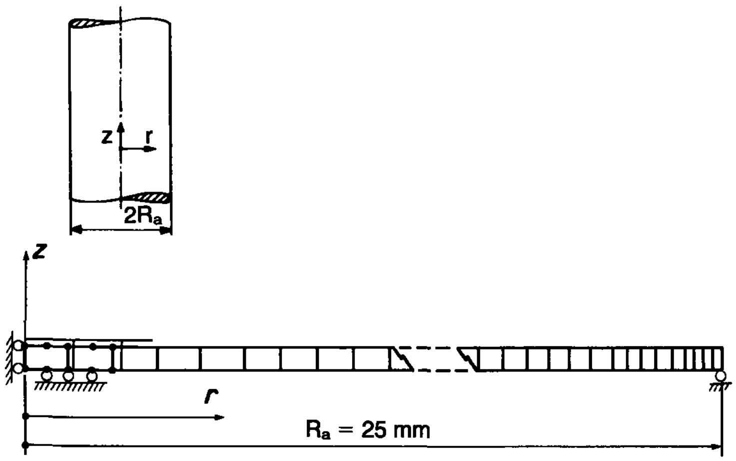
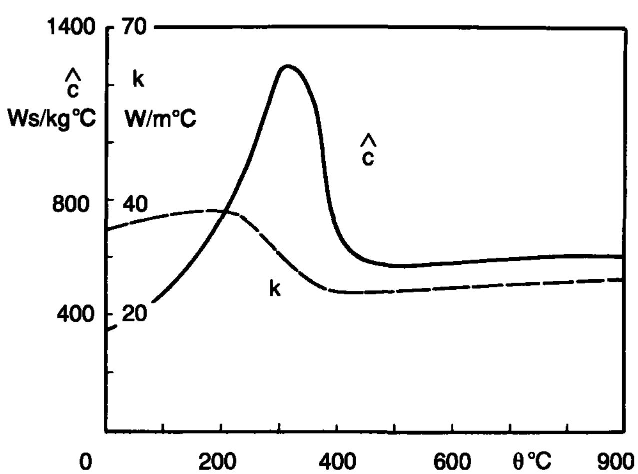
We conclude for this problem:

- As the time step is reduced, the collapse times given by $\alpha = 0$, $\alpha = .5$, $\alpha = 1$ become closer. For $\Delta t = .5$, the difference in collapse times is less than 2 hours.
- For a reasonable choice of time step, solution instability is not a problem.



Analysis of a cylinder subjected to heat treatment

Slide 18-1



Temperature-dependence of the specific heat, $\hat{c}$, and the heat conduction coefficient, k.

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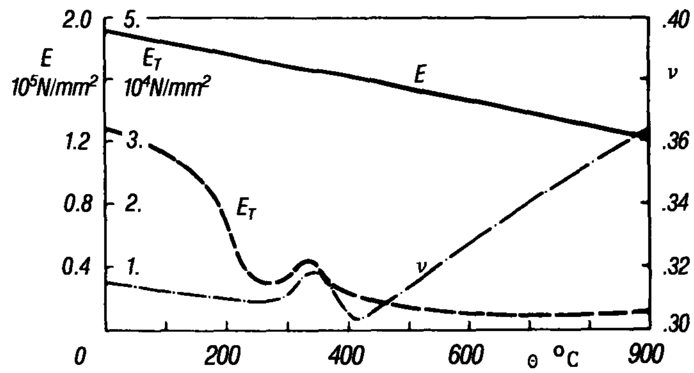
Slide 18-3



Temperature-dependence of the Young's modulus, E, Poisson's ratio, ν, and hardening modulus, E_T

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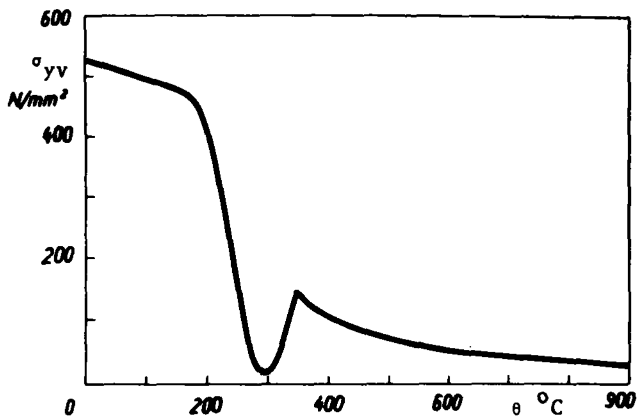
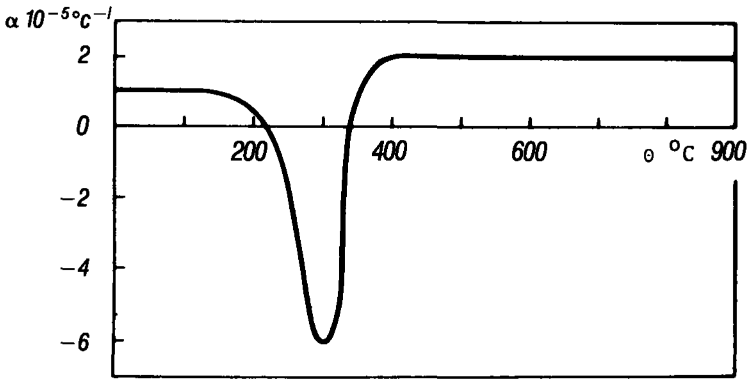


Temperature-dependence of the material yield stress



Temperature-dependence of the instantaneous coefficient of thermal expansion (including volume change due to phase transformation), α

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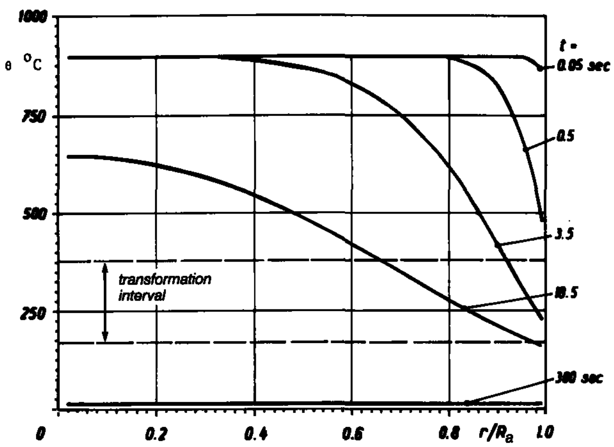

The calculated transient temperature field

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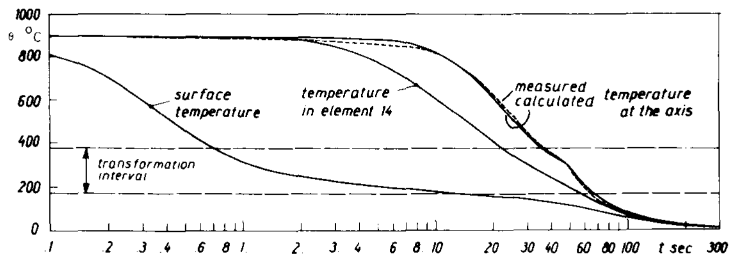
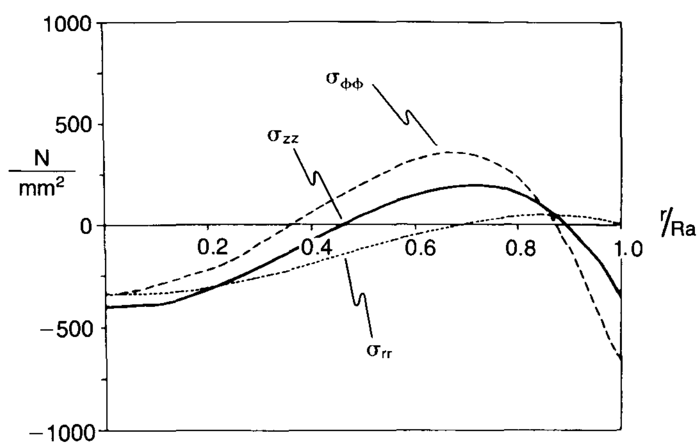
Slide 18-7

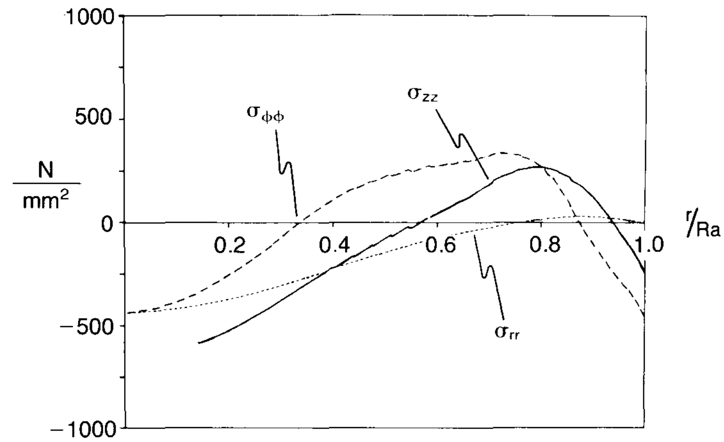


Surface and core temperature; comparison between measured and calculated results

Slide 18-8



Measured residual stress field



Calculated residual stress field

Slide 18-9