Topic 13

Solution of Nonlinear Dynamic Response—Part I

Contents:

- **Basic procedure of direct integration**
- **The explicit central difference method, basic equations, details of computations performed, stability considerations, time step selection, relation of critical time step size to wave speed, modeling of problems**
- **Practical observations regarding use of the central difference method**
- **The implicit trapezoidal rule, basic equations, details of computations performed, time step selection, convergence of iterations, modeling of problems**
- **Practical observations regarding use of trapezoidal rule**
- **Combination of explicit and implicit integrations**

Textbook: Sections 9.1, 9.2.1, 9.2.4, 9.2.5, 9.4.1, 9.4.2, 9.4.3, 9.4.4, 9.5.1, 9.5.2

Examples: 9.1, 9.4, 9.5, 9.12

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SOLUTION OF DYNAMIC EQUILIBRIUM EQUATIONS

- Direct integration methods
  - Explicit
  - Implicit
- Mode superposition
- Substructuring

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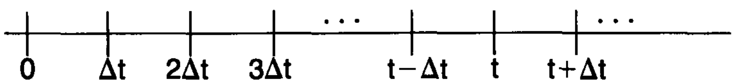
The governing equation is

$$\underbrace{\underline{F}_I(t)}_{\substack{\text{Inertia}\\ \text{forces}}} + \underbrace{\underline{F}_D(t)}_{\substack{\text{Damping}\\ \text{forces}}} + \underbrace{\underline{F}_E(t)}_{\substack{\text{``Elastic''}\\ \text{forces}}} = \underbrace{\underline{R}(t)}_{\substack{\text{Externally}\\ \text{applied loads}}}$$

"Elastic" forces ↳ nodal point forces equivalent to element stresses

This equation is to be satisfied at the discrete times



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Issues to discuss:

- What are the basic procedures for obtaining the solutions at the discrete times?
- Which procedure should be used for a given problem?

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Explicit time integration:

Central difference method

$$\underline{M}\,{}^{t}\ddot{\underline{U}} + \underline{C}\,{}^{t}\dot{\underline{U}} + {}^{t}\underline{F} = {}^{t}\underline{R}$$

$${}^{t}\dot{\underline{U}} = \frac{1}{2\Delta t}\left({}^{t+\Delta t}\underline{U} - {}^{t-\Delta t}\underline{U}\right)$$

$${}^{t}\ddot{\underline{U}} = \frac{1}{(\Delta t)^2}\left({}^{t+\Delta t}\underline{U} - 2\,{}^{t}\underline{U} + {}^{t-\Delta t}\underline{U}\right)$$

- Used mainly for wave propagation problems
- An explicit method because the equilibrium equation is used at time t to obtain the solution for time $t+\Delta t$.

Using these equations,

$$\left(\frac{1}{\Delta t^2}\underline{M} + \frac{1}{2\Delta t}\underline{C}\right) {}^{t+\Delta t}\underline{U} = {}^{t}\hat{\underline{R}}$$

where

$${}^{t}\hat{\underline{R}} = {}^{t}\underline{R} - {}^{t}\underline{F} + \frac{2}{(\Delta t)^2}\underline{M}\,{}^{t}\underline{U} - \left(\frac{1}{\Delta t^2}\underline{M} - \frac{1}{2\Delta t}\underline{C}\right) {}^{t-\Delta t}\underline{U}$$

- The method is used when $\underline{M}$ and $\underline{C}$ are diagonal:

$${}^{t+\Delta t}U_i = \left(\frac{1}{\frac{1}{\Delta t^2} m_{ii} + \frac{1}{2\Delta t} c_{ii}}\right) {}^{t}\hat{R}_i$$

and, most frequently, $c_{ii} = 0$.

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Note:

- We need $m_{ii} > 0$! (assuming $c_{ii} = 0$)
- $${}^{t}\underline{F} = \sum_{m} {}^{t}\underline{F}^{(m)}$$

  where m denotes an element.
- To start the solution, we use

  $${}^{-\Delta t}\underline{U} = {}^{0}\underline{U} - \Delta t\, {}^{0}\dot{\underline{U}} + \frac{\Delta t^2}{2}\, {}^{0}\ddot{\underline{U}}$$

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The central difference method is only conditionally stable. The condition is

$$\Delta t \leq \Delta t_{cr} = \frac{T_n}{\pi}$$

T_n ← smallest period in finite element assemblage

In nonlinear analysis, T_n changes during the time history

— becomes smaller when the system stiffens (for example, due to large displacement effects),

— becomes larger when the system softens (for example, due to material nonlinearities).

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We can estimate T_n:

$$\underbrace{(\omega_n)}_{\text{frequency}}{}^2 \leq \max \{(\omega_n^{(m)})^2\} \text{ over all elements } m$$

Hence the largest frequency of all individual elements, $(\omega_n^{(m)})_{max}$, is used:

$$T_n \geq \frac{2\pi}{(\omega_n^{(m)})_{max}}$$

In nonlinear analysis $(\omega_n^{(m)})_{max}$ will in general change with the response.

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The time integration step, Δt, used can be

$$\Delta t = \frac{2}{(\omega_n^{(m)})_{max}} \leq \Delta t_{cr}$$

We may call $\frac{2}{\omega_n^{(m)}}$ the critical time step of element m.

Hence $\frac{2}{(\omega_n^{(m)})_{max}}$ is the smallest of these "element critical time steps."

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Proof that $(\omega_n)^2 \leq (\omega_n^{(m)})^2_{max}$:

Using the Rayleigh quotient (see textbook), we write

$$(\omega_n)^2 = \frac{\underline{\phi}_n^T \sum_m \underline{K}^{(m)} \underline{\phi}_n}{\underline{\phi}_n^T \sum_m \underline{M}^{(m)} \underline{\phi}_n} \quad \left(\begin{array}{l}\text{the summation is}\\ \text{taken over all}\\ \text{finite elements}\end{array}\right)$$

Let $\mathcal{U}^{(m)} = \underline{\phi}_n^T \underline{K}^{(m)} \underline{\phi}_n$, $\mathcal{J}^{(m)} = \underline{\phi}_n^T \underline{M}^{(m)} \underline{\phi}_n$,

then

$$(\omega_n)^2 = \frac{\sum_m \mathcal{U}^{(m)}}{\sum_m \mathcal{J}^{(m)}}$$

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Consider the Rayleigh quotient for a single element:

$$\rho^{(m)} = \frac{\underline{\phi}_n^T \underline{K}^{(m)} \underline{\phi}_n}{\underline{\phi}_n^T \underline{M}^{(m)} \underline{\phi}_n} = \frac{\mathcal{U}^{(m)}}{\mathcal{J}^{(m)}}$$

Using that $\rho^{(m)} \leq (\omega_n^{(m)})^2$ where $\omega_n^{(m)}$ is the largest frequency (rad/sec) of element m, we obtain

$$\mathcal{U}^{(m)} \leq (\omega_n^{(m)})^2 \mathcal{J}^{(m)}$$

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Therefore $(\omega_n)^2$ is also bounded:

$$(\omega_n)^2 \leq \frac{\sum_m (\omega_n^{(m)})^2 \mathcal{J}^{(m)}}{\sum_m \mathcal{J}^{(m)}}$$

$$\leq \frac{(\omega_n^{(m)})^2_{max} \sum_m \mathcal{J}^{(m)}}{\sum_m \mathcal{J}^{(m)}}$$

resulting in

$$(\omega_n)^2 \leq (\omega_n^{(m)})^2_{max}$$

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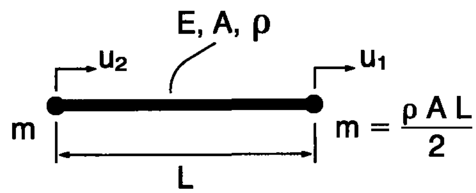
The largest frequencies of simple elements can be calculated analytically (or upper bounds can be estimated).

Example:



$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underline{\phi} = \omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \underline{\phi}$$

$$(\omega_1)^2 = 0\,,\ (\omega_2)^2 = (\omega_n)^2 = 4\,\frac{E}{\rho}\,\frac{1}{L^2} = 4\,\frac{c^2}{L^2}$$

(c — the wave speed)

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We note that hence the critical time step for this element is

$$\left(\frac{2}{\omega_n}\right) = \left(\frac{2}{\left(\frac{2c}{L}\right)}\right)$$

$$= \frac{L}{c}\,;\ L = \text{length of element!}$$

Note that $\frac{L}{c}$ is the time required for a wave front to travel through the element.

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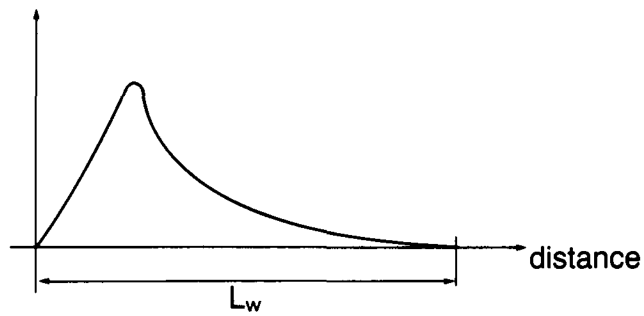
Modeling:

Let the applied wavelength be L_w



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Then $t_w = \dfrac{L_w}{c}$ ← wave speed

Choose $\Delta t = \dfrac{t_w}{n}$ ← number of time steps used to represent the wave

$L_e = c\,\Delta t$

L_e: related to element length

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Notes:

- In 1-D, $c = \sqrt{\frac{E}{\rho}}$ (E — Young's modulus, ρ — density)
- In nonlinear analysis, Δt must satisfy the stability limit throughout the analysis. Since c changes, use the largest value anticipated.
- It may also be effective to change the time step during the analysis.

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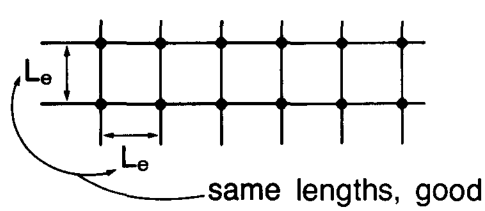
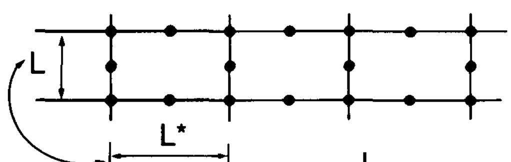
- Low-order elements:



} Usually preferable

- Higher-order elements:



different lengths, not good $L_e = \frac{L}{8}$ — conservative

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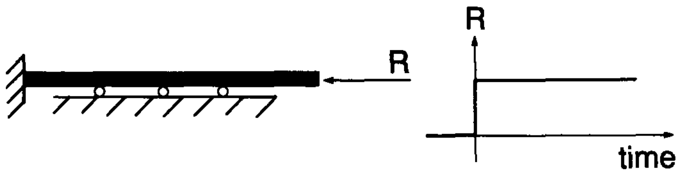
Some observations:

1) Linear elastic 1-D analysis



For this special case the exact solution is obtained for any number of elements provided $L_e = c\,\Delta t$.

Wave travels one element per time step.

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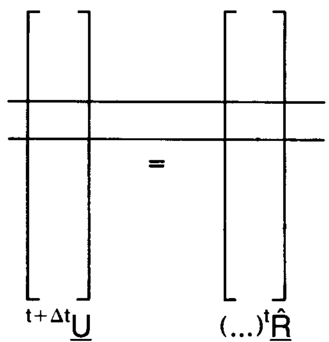
2) Uniform meshing is important, so that with the time step selected, no unduly small time step in any region of the total mesh is used.

   Different time steps for different parts of the mesh could be used, but then special coupling considerations must be enforced.

3) A system with a very large bandwidth may also be solved efficiently using the central difference method, although the problem may not be a wave propagation problem.

4) Explicit time integration lends itself to parallel processing.



Can consider a certain number of equations in parallel (by element groups)

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Implicit time integration:

Basic equation (assume modified Newton-Raphson iteration):

$$\underline{M}\ {}^{t+\Delta t}\ddot{\underline{U}}^{(k)} + \underline{C}\ {}^{t+\Delta t}\dot{\underline{U}}^{(k)} + {}^{t}\underline{K}\ \Delta\underline{U}^{(k)} =$$

$$\quad {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(k-1)}$$

$$^{t+\Delta t}\underline{U}^{(k)} = {}^{t+\Delta t}\underline{U}^{(k-1)} + \Delta\underline{U}^{(k)}$$

We use the equilibrium equation at time $t+\Delta t$ to obtain the solution for time $t+\Delta t$.

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Trapezoidal rule:

$$ {}^{t+\Delta t}\underline{U} = {}^{t}\underline{U} + \frac{\Delta t}{2}\left({}^{t}\dot{\underline{U}} + {}^{t+\Delta t}\dot{\underline{U}}\right) $$

$$ {}^{t+\Delta t}\dot{\underline{U}} = {}^{t}\dot{\underline{U}} + \frac{\Delta t}{2}\left({}^{t}\ddot{\underline{U}} + {}^{t+\Delta t}\ddot{\underline{U}}\right) $$

Hence

$$ {}^{t+\Delta t}\dot{\underline{U}} = \frac{2}{\Delta t}\left({}^{t+\Delta t}\underline{U} - {}^{t}\underline{U}\right) - {}^{t}\dot{\underline{U}} $$

$$ {}^{t+\Delta t}\ddot{\underline{U}} = \frac{4}{(\Delta t)^2}\left({}^{t+\Delta t}\underline{U} - {}^{t}\underline{U}\right) - \frac{4}{\Delta t}\,{}^{t}\dot{\underline{U}} - {}^{t}\ddot{\underline{U}} $$

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In our incremental analysis, we write

$$ {}^{t+\Delta t}\dot{\underline{U}}^{(k)} = \frac{2}{\Delta t}\left({}^{t+\Delta t}\underline{U}^{(k-1)} + \Delta\underline{U}^{(k)} - {}^{t}\underline{U}\right) - {}^{t}\dot{\underline{U}} $$

$$ {}^{t+\Delta t}\ddot{\underline{U}}^{(k)} = \frac{4}{(\Delta t)^2}\left({}^{t+\Delta t}\underline{U}^{(k-1)} + \Delta\underline{U}^{(k)} - {}^{t}\underline{U}\right) - \frac{4}{\Delta t}\,{}^{t}\dot{\underline{U}} - {}^{t}\ddot{\underline{U}} $$

and the governing equilibrium equation is

$$\underbrace{\left({}^{t}\underline{K} + \frac{4}{\Delta t^2}\underline{M} + \frac{2}{\Delta t}\underline{C}\right)}_{{}^{t}\hat{\underline{K}}} \Delta\underline{U}^{(k)}$$

$$= {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(k-1)}$$

$$- \underline{M}\left[\frac{4}{\Delta t^2}({}^{t+\Delta t}\underline{U}^{(k-1)} - {}^{t}\underline{U}) - \frac{4}{\Delta t}{}^{t}\dot{\underline{U}} - {}^{t}\ddot{\underline{U}}\right]$$

$$- \underline{C}\left[\frac{2}{\Delta t}({}^{t+\Delta t}\underline{U}^{(k-1)} - {}^{t}\underline{U}) - {}^{t}\dot{\underline{U}}\right]$$

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Some observations:

1) As Δt gets smaller, entries in ${}^{t}\hat{\underline{K}}$ increase.
2) The convergence characteristics of the equilibrium iterations are better than in static analysis.
3) The trapezoidal rule is unconditionally stable in linear analysis. For nonlinear analysis,
   - select Δt for accuracy
   - select Δt for convergence of iteration

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Convergence criteria:

Energy:

$$\frac{\Delta \underline{U}^{(i)T}\left({}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} - \underline{M}\,{}^{t+\Delta t}\ddot{\underline{U}}^{(i-1)} - \underline{C}\,{}^{t+\Delta t}\dot{\underline{U}}^{(i-1)}\right)}{\Delta \underline{U}^{(1)T}\left({}^{t+\Delta t}\underline{R} - {}^{t}\underline{F} - \underline{M}\,{}^{t+\Delta t}\ddot{\underline{U}}^{(0)} - \underline{C}\,{}^{t+\Delta t}\dot{\underline{U}}^{(0)}\right)}$$

$$\leq \text{ETOL}$$

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Forces:

$$\frac{\left\| {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} - \underline{M}\,{}^{t+\Delta t}\ddot{\underline{U}}^{(i-1)} - \underline{C}\,{}^{t+\Delta t}\dot{\underline{U}}^{(i-1)} \right\|_2}{\text{RNORM}}$$

$$\leq \text{RTOL}$$

(considering only translational degrees of freedom, for rotational degrees of freedom use RMNORM).

Note: $\|\underline{a}\|_2 = \sqrt{\sum_k (a_k)^2}$

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Displacements:

$$\frac{\|\Delta \underline{U}^{(i)}\|_2}{\text{DNORM}} \leq \text{DTOL}$$

(considering only translational degrees of freedom, for rotational degrees of freedom, use DMNORM).

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Modeling:

- Identify frequencies contained in the loading.
- Choose a finite element mesh that can accurately represent the static response and all important frequencies.
- Perform direct integration with

$$\Delta t \doteq \frac{1}{20} T_{co}$$

(T_{co} is the smallest period (secs) to be integrated).

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- Method used for structural vibration problems.
- Typically it is effective to use higher-order elements.
- It can also be effective to use a consistent mass matrix.

Because a structural dynamics problem is thought of as a "static problem including inertia forces".

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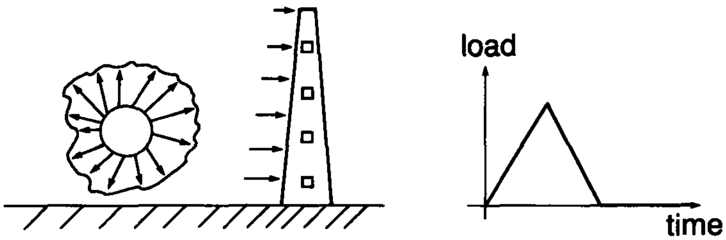
Typical problem:



Analysis of tower under blast load

- We assume that only the structural vibration is required.
- Perhaps about 100 steps are sufficient to integrate the response.

Combination of methods: explicit and implicit integration

- Use central difference method first, then switch to trapezoidal rule, for problems which show initially wave propagation, then structural vibration.

- Use central difference method for certain parts of the structure, and implicit method for other parts; for problems with "stiff" and "flexible" regions.

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