MODE SUPERPOSITION ANALYSIS; TIME HISTORY

LECTURE 11

48 MINUTES

LECTURE 11 Solution of dynamic response by mode superposition

The basic idea of mode superposition

Derivation of decoupled equations

Solution with and without damping

Caughey and Rayleigh damping

Calculation of damping matrix for given damping ratios

Selection of number of modal coordinates

Errors and use of static correction

Practical considerations

TEXTBOOK: Sections: 9.3.1, 9.3.2, 9.3.3

Examples: 9.6, 9.7, 9.8, 9.9, 9.10, 9.11

Mode Superposition Analysis

Basic idea is:

transform dynamic equilibrium equations into a more effective form for solution, using

$$\underset{n\times 1}{\underline{U}} = \underset{n\times n}{\underline{P}} \; \underset{n\times 1}{\underline{X}(t)}$$

$\underline{P}$ = transformation matrix

$\underline{X}(t)$ = generalized displacements

Using

$$\underline{U}(t) = \underline{P}\,\underline{X}(t) \qquad (9.30)$$

on

$$\underline{M}\,\ddot{\underline{U}} + \underline{C}\,\dot{\underline{U}} + \underline{K}\,\underline{U} = \underline{R} \qquad (9.1)$$

we obtain

$$\tilde{\underline{M}}\,\ddot{\underline{X}}(t) + \tilde{\underline{C}}\,\dot{\underline{X}}(t) + \tilde{\underline{K}}\,\underline{X}(t) = \tilde{\underline{R}}(t) \qquad (9.31)$$

where

$$\tilde{\underline{M}} = \underline{P}^T\,\underline{M}\,\underline{P} \;\; ; \;\; \tilde{\underline{C}} = \underline{P}^T\,\underline{C}\,\underline{P} \; ;$$

$$\tilde{\underline{K}} = \underline{P}^T\,\underline{K}\,\underline{P} \;\; ; \;\; \tilde{\underline{R}} = \underline{P}^T\,\underline{R} \qquad (9.32)$$

An effective transformation matrix $\underline{P}$ is established using the displacement solutions of the free vibration equilibrium equations with damping neglected,

$$\underline{M}\,\ddot{\underline{U}} + \underline{K}\,\underline{U} = \underline{0} \qquad (9.34)$$

Using

$$\underline{U} = \underline{\phi}\,\sin\omega(t - t_0) \qquad (9.35)$$

we obtain the generalized eigenproblem,

$$\underline{K}\,\underline{\phi} = \omega^2\,\underline{M}\,\underline{\phi} \qquad (9.36)$$

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with the n eigensolutions $(\omega_1^2, \underline{\phi}_1)$, $(\omega_2^2, \underline{\phi}_2), \ldots, (\omega_n^2, \underline{\phi}_n)$, and

$$\underline{\phi}_i^T\,\underline{M}\,\underline{\phi}_j \begin{cases} = 1 & ; \quad i = j \\ = 0 & ; \quad i \neq j \end{cases} \qquad (9.37)$$

$$0 \leq \omega_1^2 \leq \omega_2^2 \leq \omega_3^2 \cdots \leq \omega_n^2 \qquad (9.38)$$

Defining

$$\underline{\Phi} = [\underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_n] \; ; \quad \underline{\Omega}^2 = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{bmatrix} \tag{9.39}$$

we can write

$$\underline{K}\,\underline{\Phi} = \underline{M}\,\underline{\Phi}\,\underline{\Omega}^2 \tag{9.40}$$

and have

$$\underline{\Phi}^T \underline{K}\,\underline{\Phi} = \underline{\Omega}^2 \quad ; \quad \underline{\Phi}^T \underline{M}\,\underline{\Phi} = \underline{I} \tag{9.41}$$

Now using

$$\underline{U}(t) = \underline{\Phi}\,\underline{X}(t) \tag{9.42}$$

we obtain equilibrium equations that correspond to the modal generalized displacements

$$\ddot{\underline{X}}(t) + \underline{\Phi}^T \underline{C}\,\underline{\Phi}\,\dot{\underline{X}}(t) + \underline{\Omega}^2 \underline{X}(t) = \underline{\Phi}^T \underline{R}(t) \tag{9.43}$$

The initial conditions on $\underline{X}(t)$ are obtained using (9.42) and the $\underline{M}$ - orthonormality of $\underline{\Phi}$; i.e., at time 0 we have

$${}^0\underline{X} = \underline{\Phi}^T \underline{M}\,{}^0\underline{U} \; ; \quad {}^0\dot{\underline{X}} = \underline{\Phi}^T \underline{M}\,{}^0\dot{\underline{U}} \tag{9.44}$$

Analysis with Damping Neglected

$$\ddot{\underline{X}}(t) + \underline{\Omega}^2 \underline{X}(t) = \underline{\Phi}^T \underline{R}(t) \qquad (9.45)$$

i.e., n individual equations of the form

$$\left.\begin{aligned} &\ddot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t) \\ &\text{where} \\ &\qquad r_i(t) = \underline{\phi}_i^T \underline{R}(t) \end{aligned}\right\} i = 1,2,\ldots,n \qquad (9.46)$$

with

$$\begin{aligned} x_i\Big|_{t=0} &= \underline{\phi}_i^T \underline{M}\,{}^0\underline{U} \\ \dot{x}_i\Big|_{t=0} &= \underline{\phi}_i^T \underline{M}\,{}^0\dot{\underline{U}} \end{aligned} \qquad (9.47)$$

Using the Duhamel integral we have

$$x_i(t) = \frac{1}{\omega_i} \int_0^t r_i(\tau) \sin \omega_i (t - \tau) d\tau \qquad (9.48)$$

$$+ \alpha_i \sin \omega_i t + \beta_i \cos \omega_i t$$

where α_i and β_i are determined from the initial conditions in (9.47). And then

$$\underline{U}(t) = \sum_{i=1}^{n} \underline{\phi}_i x_i(t) \qquad (9.49)$$

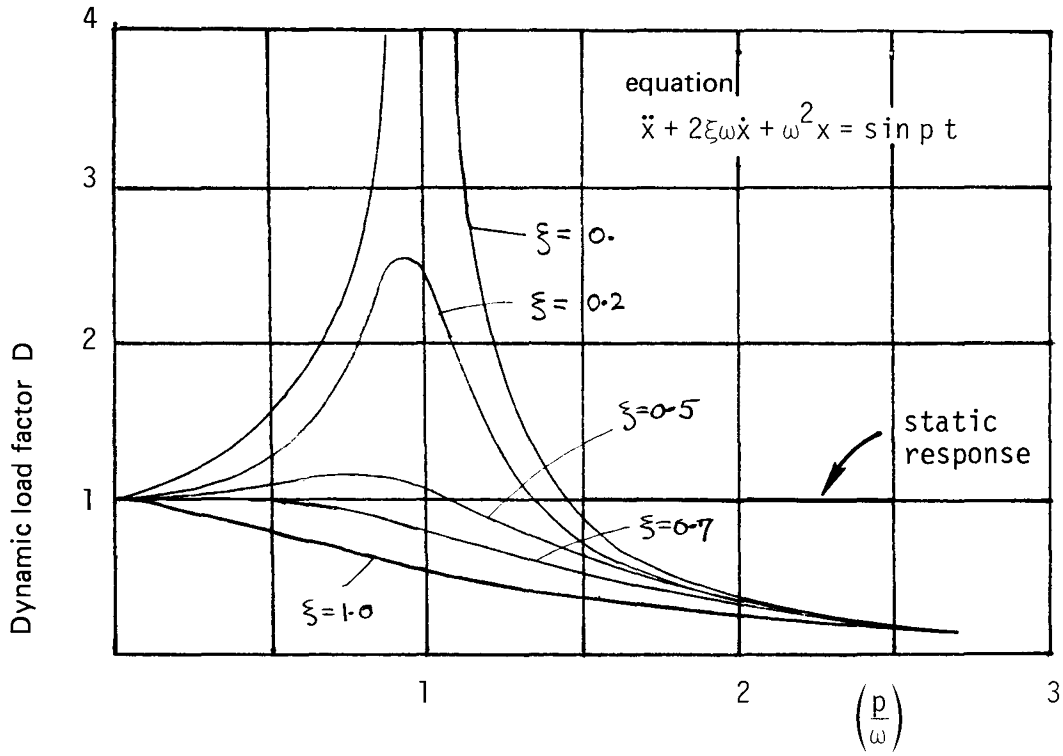

Fig. 9.4. The dynamic load factor

Hence we use

$$\underline{U}^p = \sum_{i=1}^{p} \underline{\phi}_i \, x_i(t)$$

where

$$\underline{U}^p \doteq \underline{U}$$

The error can be measured using

$$\epsilon^p(t) = \frac{\left\| \underline{R}(t) - \left(\underline{M}\,\ddot{\underline{U}}^p(t) + \underline{K}\,\underline{U}^p(t) \right) \right\|_2}{\left\| \underline{R}(t) \right\|_2} \tag{9.50}$$

Static correction

Assume that we used p modes to obtain $\underline{U}^p$, then let

$$\underline{R} = \sum_{i=1}^{n} r_i (\underline{M}\,\underline{\phi}_i)$$

Hence

$$r_i = \underline{\phi}_i^T\,\underline{R}$$

Then

$$\Delta\underline{R} = \underline{R} - \sum_{i=1}^{p} r_i\,(\underline{M}\,\underline{\phi}_i)$$

and

$$\underline{K}\,\Delta\underline{U} = \Delta\underline{R}$$

Analysis with Damping Included

Recall, we have

$$\ddot{\underline{X}}(t) + \underline{\Phi}^T\,\underline{C}\,\underline{\Phi}\,\dot{\underline{X}}(t) + \underline{\Omega}^2\,\underline{X}(t) = \underline{\Phi}^T\,\underline{R}(t) \qquad (9.43)$$

If the damping is proportional

$$\underline{\phi}_i^T\,\underline{C}\,\underline{\phi}_j = 2\omega_i\,\xi_i\,\delta_{ij} \qquad (9.51)$$

and we have

$$\ddot{x}_i(t) + 2\omega_i\,\xi_i\,\dot{x}_i(t) + \omega_i^2\,x_i(t) = r_i(t)$$

$$i = 1, \ldots, n \qquad (9.52)$$

A damping matrix that satisfies the relation in (9.51) is obtained using the Caughey series,

$$\underline{C} = \underline{M} \sum_{k=0}^{p-1} a_k [\underline{M}^{-1}\ \underline{K}]^k \qquad (9.56)$$

where the coefficients a_k , $k = 1, \ldots, p$, are calculated from the p simultaneous equations

$$\xi_i = \frac{1}{2}\left(\frac{a_0}{\omega_i} + a_1\omega_i + a_2\omega_i^3 + \ldots + a_{p-1}\omega_i^{2p-3}\right) \qquad (9.57)$$

A special case is Rayleigh damping,

$$\underline{C} = \alpha \underline{M} + \beta \underline{K} \qquad (9.55)$$

example:

Assume $\xi_1 = 0.02$; $\xi_2 = 0.10$

$\omega_1 = 2$ $\qquad \omega_2 = 3$

calculate α and β

We use

$$\underline{\phi}_i^T(\alpha\underline{M} + \beta\underline{K})\,\underline{\phi}_i = 2\omega_i\,\xi_i$$

or

$$\alpha + \beta\omega_i^2 = 2\omega_i\,\xi_i$$

Using this relation for ω_1, ξ_1 and ω_2, ξ_2, we obtain two equations for α and β:

$$\underline{\alpha} + 4\underline{\beta} = 0.08$$

$$\underline{\alpha} + 9\underline{\beta} = 0.60$$

The solution is $\alpha = -0.336$ and $\beta = 0.104$. Thus the damping matrix to be used is

$$\underline{C} = -0.336\,\underline{M} + 0.104\,\underline{K}$$

Note that since

$$\alpha + \beta\omega_i^2 = 2\omega_i\,\xi_i$$

for any i, we have, once α and β have been established,

$$\xi_i = \frac{\alpha + \beta\omega_i^2}{2\omega_i}$$

$$= \frac{\alpha}{2\omega_i} + \frac{\beta}{2}\,\omega_i$$

Response solution

As in the case of no damping
we solve p equations

$$\ddot{x}_i + 2\omega_i \xi_i \dot{x}_i + \omega_i^2 x_i = r_i$$

with

$$r_i = \underline{\phi}_i^T \underline{R}$$

$$x_i \big|_{t=0} = \underline{\phi}_i^T \underline{M}\, {}^0\underline{U}$$

$$\dot{x}_i \big|_{t=0} = \underline{\phi}_i^T \underline{M}\, {}^0\dot{\underline{U}}$$

and then

$$\underline{U}^p = \sum_{i=1}^{p} \underline{\phi}_i \, x_i(t)$$

Practical considerations

mode superposition analysis
is effective

- when the response lies in a few modes only, $p << n$
- when the response is to be obtained over many time intervals (or the modal response can be obtained in closed form).

  e.g. earthquake engineering
  vibration excitation
- it may be important to calculate $\varepsilon_p(t)$ or the static correction.

Resource: Finite Element Procedures for Solids and Structures
Klaus-Jürgen Bathe