Solutions

BLOCK 2:

ORDINARY DIFFERENTIAL EQUATIONS

## Pretest

1. a. $\frac{dy}{dx} = y^2$.

b. $y = 0$.

2. $\frac{y}{x} = \tan[\ln|x| + c]$.

3. $r = ce^{\theta}$.

4. $y = c_1 + (c_2 + c_3x)e^{x} + (c_4 + c_5x)e^{-x}$.

5. $y = (c_1 + c_2)x + 3e^{4x} + \frac{1}{18}\cos 3x$.

6. $y = -\frac{1}{2}(1 + xe^{x}) + (\sinh x)\ln(1 + e^{x})$.

7. $y = x + \sum_{n=1}^{\infty} \frac{(2)(5)\ldots(3n - 1)}{(3n + 1)!} x^{3n+1}$.

8. $$\begin{cases} x = \frac{1}{2}(2 \sin t + e^{t} + \cos t - t \cos t) \\ y = \frac{1}{2}(-e^{t} - \sin t + \cos t + t \sin t) \end{cases}$$

## Unit 1: The Concept of a General Solution

### 2.1.1(L)

At first glance, this problem may seem to be a bit beneath our dignity. It is computationally very simple, yet it serves to highlight the notion of what is meant by the general solution of a differential equation.

To begin with, we see at once that

$$y = x^2 + c \tag{1}$$

implies that

$$\frac{dy}{dx} = 2x. \tag{2}$$

This tells us that every curve whose equation is given by (1) has the property that its slope at any point (x,y) is twice the value of the x-coordinate of the point.

In the lecture we shall tackle this same problem - but only from the perspective of starting with (2) and deriving (1). In essence, in the calculus of real-valued functions of a single real variable, we started with a function, $f(x,y) = 0$; and we then saw what the relationship was between x,y and the various derivatives of y with respect to x. In differential equations we start with the relationship between x,y, and the various derivatives of y with respect to x; and we then try to find out how x and y are related.

In general, the major problem is that many different functions can satisfy the same differential equation and we must therefore come to grips with the problem of finding <u>all</u> solutions to the given differential equation. This, in turn, gives rise to a few sub-questions. For example, when do we know that we have found all solutions? Intuitively, one would expect that "the" solution to a first order differential equation (if there is indeed a solution) should be a family of curves deter-

2.1.1(L) continued

mined up to an arbitrary constant (such a family of curves is called a 1-parameter family of curves). More generally, since each time we integrate (i.e., inverse-differentiate) we tack on an additional constant, we should expect that the solution of an nth order differential equation should contain n arbitrary constants. In other words, if there is a solution to an nth order differential equation, we expect that the solution is an n-parameter family of curves.

The interesting point is that things are not always as simple as our above discussion might seem to indicate. We have, for obvious reasons, elected to begin with a problem to which there is a "simple" solution, but as we shall see in the next few exercises this is not always the case.

Returning to our present example, let us observe that (2) represents a specific first order differential equation and (1) represents a 1-parameter family of curves which is a solution to (2). [All we have done in this exercise is to start with a "solution" and construct the corresponding differential equation.]

In Part 1 of our course we learned that any two differentiable functions whose derivatives were identical could differ by at most an additive constant.

Thus, in this exercise, if g is any function such that $g'(x) = 2x$, it follows that $g(x) = x^2 + c$; and we see that <u>every</u> solution of (2) is given by (1). Conversely, every member of (1) is a solution of (2).

We also see that the particular member of the family in (1) is uniquely determined once we know a point $(x_0,y_0)$ that is to be on the curve. For example, if we want the solution of (2) which passes through (2,1)* we go to (1) and replace x by 2 and

*In many important situations one wants a "local" solution to the equation. That is, just as in ordinary calculus, we are interested in what's happening "near" a given point $(x_0,y_0)$. We may not care at all what's happening "far away". As we develop this topic we shall see how the concept of a general solution incorporates the fact that we are interested in "local" behaviour.

2.1.1(L) continued

y by 1 to see if c is determined. In this case, $y = x^2 + c$ implies that $1 = 2^2 + c$. Hence, $c = -3$, whereupon the desired curve is $y = x^2 - 3$.

To put things in different words, the only curve with the property given in (2) and which passes through (2,1) is $y = x^2 - 3$. More generally, if we let $(x_0,y_0)$ denote an arbitrary point in the plane then the only member of (1) which passes through $(x_0,y_0)$ must satisfy $y_0 = x_0^2 + c$, whereupon

$$c = y_0 - x_0^2. \tag{3}$$

Notice that while $x_0$ and $y_0$ are arbitrarily given numbers, once they are chosen, the value of $y_0 - x_0^2$ makes the choice of c in (3) a unique number. In particular, if this value of c is introduced into (1) we obtain

$$y = x^2 + y_0 - x_0^2 \tag{4}$$

and (4) represents the only curve which satisfies equation (2) and passes through the point $(x_0,y_0)$.

For the above reason, the family (1) is called the GENERAL SOLUTION of equation (2). To put the concept of a general solution in better perspective, what we really mean is the following. Suppose that there is a (connected) region R in the xy-plane for which a differential equation is defined [R need not be the whole plane, although, as in this example, it may be. For example if the equation is $\frac{dy}{dx} = \sqrt{y - x}$ , we must have that $y \geq x$, otherwise $\frac{dy}{dx}$ would be imaginary. Thus, R must be the half-plane (or any subregion of it) $y \geq x$. Recall that $y \geq x$ is the portion of the plane on and above the line $y = x$].
Then a family of curves $\gamma$ is called the general solution of the equation in R if and only if:

1. Each member of $\gamma$ satisfies the differential equation.
2. For a given point $(x_0,y_0) \in R$, one and only one member of $\gamma$ passes through that point.

2.1.1(L) continued

3. No other curve which passes through $(x_0,y_0)$ satisfied the differential equation.

Summarizing these three criteria in terms of equation (2):

1. The family $\gamma$ defined by $\gamma = \{y = x^2 + c$: c is an arbitrary constant $\}$ satisfies $\frac{dy}{dx} = 2x$ throughout the entire plane (R).
2. For any point $(x_0,y_0)\varepsilon R$ one and only one member of $\gamma$ passes through $(x_0,y_0)$. That member, as shown by (4), is $y = x^2 + y_0 - x_0^2$.
3. No curve other than $y = x^2 + y_0 - x_0^2$ can pass through $(x_0,y_0)$ and still satisfy the given differential equation $\frac{dy}{dx} = 2x$.

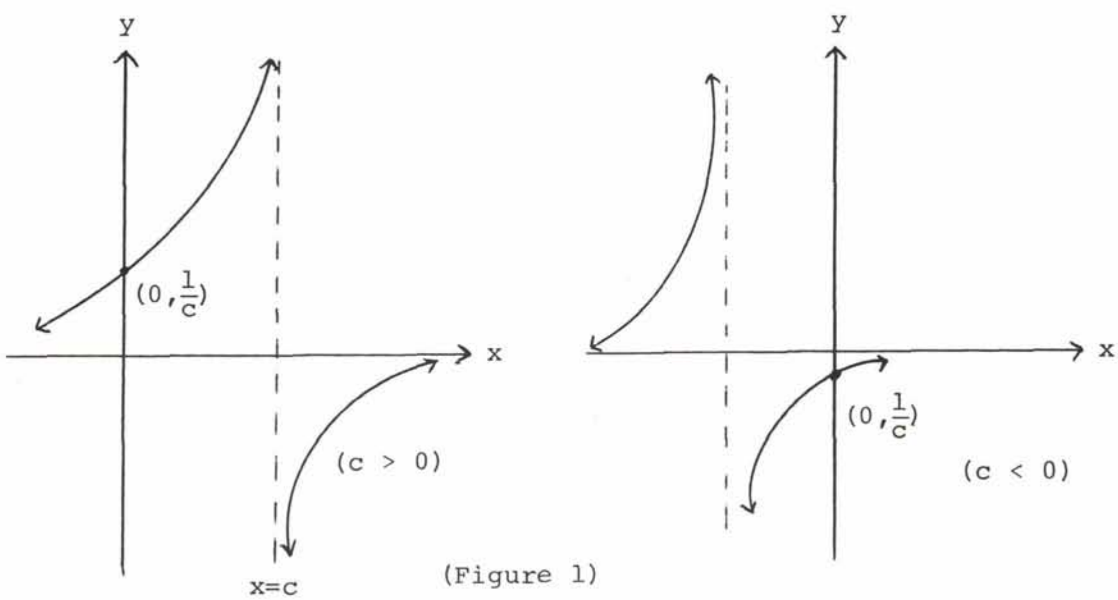
2.1.2(L)

The main purpose of this exercise is to show that the previous exercise was sufficiently contrived so as to observe certain fine points which occur when we define the concept of a general solution.

We have that our family $\gamma$ in this case is the set of curves (hyperbolas)

$$y = \frac{1}{c - x},\ x \neq c. \qquad (1)$$

Pictorially, a typical member of $\gamma$ is given by




(Figure 1)

2.1.2(L) continued

The "nastiness" of $\gamma$ is shown in Figure 1 by the fact that each member of $\gamma$ consists of two disjoint branches, separated by the line $x = c$. To emphasize our remark of the previous exercise concerning "localness", given any point $(x_o, y_o)$ on $y = \frac{1}{c - x}$, then $(x_o, y_o)$ is on the upper branch of the curve if $x_o < c$ (since then $y_o - 1/c - x_o$ is positive) and on the lower branch if $x_o > c$. If $x_o - c$, then $(x_o, y_o)$ is not on $y = 1/c - x$.

So much for the introductory geometry concerning our 1-parameter family defined by (1). Let us now turn our attention to finding the differential equation satisfied by (1). One technique is to differentiate (1) directly with respect to x to obtain

$$\frac{dy}{dx} = \frac{1}{(c - x)^2} = [\frac{1}{c - x}]^2 \quad x \neq c \tag{2}$$

and replacing $\frac{1}{c - x}$ by its value in (1), equation (2) becomes

$$\frac{dy}{dx} = y^2. \tag{3}$$

Thus, the family $\gamma$ defined by (1) is a 1-parameter family, each of whose members satisfies (3).

It is important to observe that (3) does not contain an arbitrary constant. That is, from a geometric point if view, (3) describes all curves with the property that the slope of the curve at <u>each</u> point on the curve is equal to the square of the y-coordinate of the point. What we have shown is that each member of $\gamma$ has this property. We have not shown whether there are any other curves with this property. At any rate the arbitrary constant arises when we examine solutions, not when we look at the equation.

Now we picked this particular exercise, among other reasons, because we know how to reverse the process and obtain (1) from (3). Namely, as we saw in Part 1 of our course, we may separate the variables in (3) to obtain

$$y^{-2}dy = dx, \tag{4}$$

and integrating (4) yields $-y^{-1} = x + c_1$, or $y^{-1} = -x + c$

2.1.2(L) continued

(where $c = -c_1$), or

$$y = \frac{1}{c - x} \quad , \ x \neq c \tag{5}$$

so that indeed (5) agrees with (1).

There is, however, one subtlely that occurred in obtaining (4) from (3) that is worth harping on. In going from (3) to (4) we divided by $y^2$ and since division by 0 is excluded, we may only conclude that (3) and (4) are equivalent when $y \neq 0$. In other words, $\frac{dy}{dx} = y^2$ is equivalent to $y^{-2}dy = dx$ when $y \neq 0$; but when $y = 0$, $\frac{dy}{dx} = y^2$ is still well-defined but $y^{-2}dy = dx$ is not.

Thus, (5) yields all the solutions of (3) as long as $y \neq 0$ [since then (3) and (4) are equivalent, and (5) yields all the solutions of (4)]. What we must then do is look at $y = 0$ separately. Indeed with $y \equiv 0$, $\frac{dy}{dx} \equiv 0$, so that $y = 0$ satisfies equation (3) but $y = 0$ is not a member of the family $\gamma$ defined by (1). That is, there is no constant c for which $\frac{1}{c - x} \equiv 0$ since the numerator of $\frac{1}{c - x}$ is always 1.

Thus, in terms of the definition of the general solution given in the previous exercise, (1) cannot be the general solution of (3) since (3) possesses at least one solution (namely, $y \equiv 0$) which does not belong to (1).

Notice from Figure 1 the connection between $y = 0$ and $y = \frac{1}{c - x}$. Namely no member of (1) contains even a single point whose y-coordinate is 0. That is no member of (1) intersects $y = 0$ (the x-axis).

This is where the choice of restricting the equation to a particular region R is important. For example, suppose we choose R so that for no point $(x_0, y_0) \varepsilon R$ does $y_0 = 0$. In other words R is a region which lies entirely above or entirely below the x-axis.

Then what is true is that for each point $(x_0, y_0) \varepsilon R$ one and only one member of (1) passes through $(x_0, y_0)$. In fact from (1) we

2.1.2(L) continued

see that $c - x = \frac{1}{y}$ $(y \neq 0)$

or

$$c = x + \frac{1}{y} = \frac{xy + 1}{y} \quad (y \neq 0). \tag{6}$$

Letting $x = x_o$ and $y = y_o$ in (6) we see that

$$c = \frac{x_o y_o + 1}{y_o} \tag{7}$$

which is a well-defined real number for each $(x_o, y_o) \varepsilon R$ since for $(x_o, y_o) \varepsilon R$, $y_o \neq 0$.

In summary, then, if R is any connected region which excludes any points on the x-axis, one and only one member of (1) passes through $(x_o, y_o) \varepsilon R$. From (7) this member is

$$y = \frac{1}{\frac{x_o y_o + 1}{y_o} - x}$$

or

$$y = \frac{y_o}{x_o y_o + 1 - y_o x}. \tag{8}$$

For example, letting $x_o = 2$ and $y_o = 1$ we see from (8) that

$$y = \frac{1}{3 - x}$$

is the only member of (1) that passes through (2,1).

What we have not yet proved (and the subtlety of this problem will be discussed in the next exercise) is whether there can be other solutions of (3) which pass through (2,1) but which do not belong to the family $y = \frac{1}{c - x}$.

If there are no other solutions of (3) which pass through $(x_o, y_o) \varepsilon R$ then (1) is the general solution of (3); otherwise it isn't.

2.1.2(L) continued

From a more affirmative point of view, what we have shown for sure in this problem is that given any point $(x_o, y_o)$ in the plane, there is at least one solution of (3) which passes through $(x_o, y_o)$. Namely, if $y_o \neq 0$, one solution is given by (8); and if $y_o = 0$, then the line $y = 0$ is a solution.

In fact, for those of us who may have been a bit more astute, we may have noticed that (8) covers the case $y_o = 0$, even though it was derived under the assumption that $y_o \neq 0$. Namely with $y_o = 0$, (8) becomes $y = \frac{0}{1} = 0$. In other words, for any point $(x_o, y_o)$ in the plane, equation (8) describes a curve which satisfies (3) and passes through $(x_o, y_o)$.

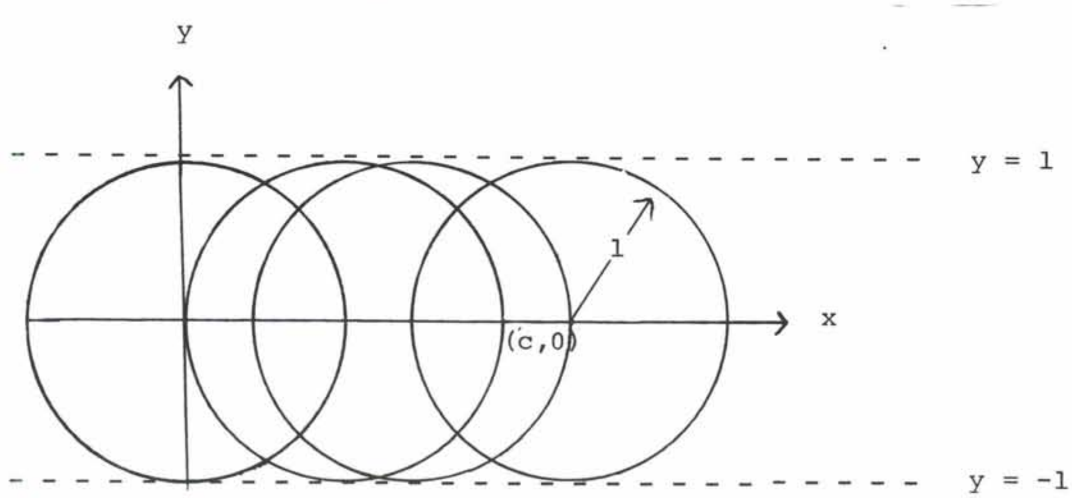
2.1.3(L)

Here we allow just about every possible subtlety to occur. We begin by observing that for any real number c,

$$(x - c)^2 + y^2 = 1 \qquad (1)$$

is the circle centered at (c,0) with radius 1.

In particular, the family defined by (1) is contained in the region $R = \{(x,y): |y| \leq 1\}$. Pictorially,




So for any point (x,y) on any of the circles defined by (1), $-1 \leq y \leq 1$.

2.1.3(L) continued

Let us return to the geometry later, but for now let us find the differential equation satisfied by each member of (1).

Differentiating (1) we obtain

$$2(x - c) + 2y\frac{dy}{dx} = 0$$

or

$$x - c + y\frac{dy}{dx} = 0$$

or

$$c = x + y\frac{dy}{dx}. \qquad (2)$$

Taking the value of c as determined from (2) and using it in (1) we obtain

$$\left(x - \left[x + y\frac{dy}{dx}\right]\right)^2 + y^2 = 1$$

or

$$y^2\left(\frac{dy}{dx}\right)^2 + y^2 = 1$$

or

$$y^2\left[\left(\frac{dy}{dx}\right)^2 + 1\right] = 1. \qquad (3)$$

Equation (3) is of interest on at least two counts.

1. Since $\left(\frac{dy}{dx}\right)^2 \geq 0$, $\left(\frac{dy}{dx}\right)^2 + 1 \geq 1$. Hence (3) cannot be satisfied if $y^2 > 1$ (since then $y^2\left[\left(\frac{dy}{dx}\right)^2 + 1\right] > 1$). Therefore, equation (3) does not make sense (i.e., it has no real solutions) unless it is restricted to a region R which is contained between the line $y = 1$ and $y = -1$.

More emphatically, there is <u>no</u> solution of (3) which passes through $(x_0, y_0)$ if $|y_0| > 1$.

2.1.3(L) continued

2. Equation (3) is of 2nd degree. That is, we have a quadratic equation in $\frac{dy}{dx}$. More explicitly we can solve (3) for $\frac{dy}{dx}$ to obtain

$$\frac{dy}{dx} = \pm \frac{\sqrt{1 - y^2}}{y}. \qquad (4)$$

In terms of the convention about single valuedness, the right side of (4) is not a (single-valued) function. Thus, (4) should be treated as the two equations

$$\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{y} \qquad (5)$$

and

$$\frac{dy}{dx} = -\frac{\sqrt{1 - y^2}}{y} \qquad (6)$$

where the right sides of both (5) and (6) are single-valued functions (recall that $\sqrt{1 - y^2}$ means the positive root unless otherwise specified).

3. Splitting (3), or equivalently, (4), into the two separate equations (5) and (6) is of very great conceptual importance if we are to understand the impact of our definition of general solution.

For example, suppose we want to find the solutions of (3) which pass through (0,0) and belong to (1). We solve for c by letting $x = y = 0$ in (1) to obtain $(0 - c)^2 + 0^2 = 1$ or $c^2 = 1$ or $c = \pm 1$. Thus, the two circles

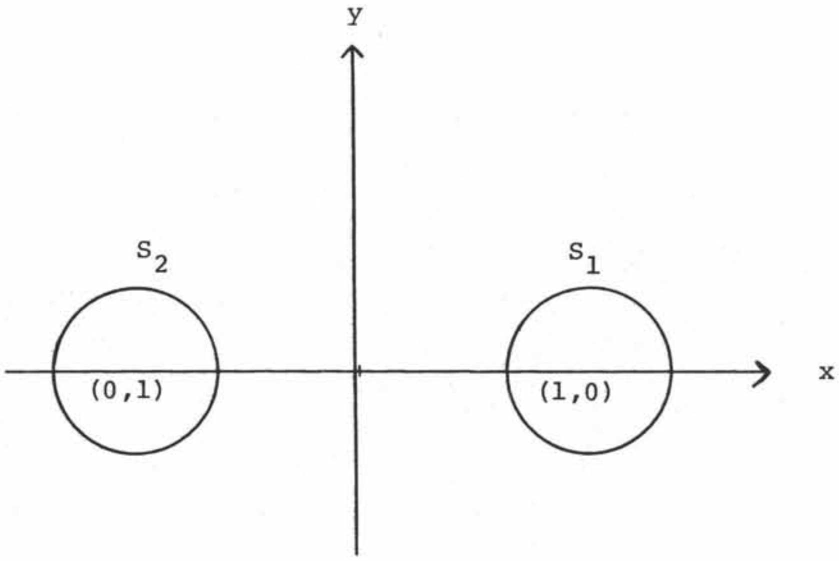
$$(x + 1)^2 + y^2 = 1 \qquad (7)$$

and

$$(x - 1)^2 + y^2 = 1 \qquad (8)$$

belong to (1), pass through (0,0), and satisfy (3). Pictorially,

2.1.3(L) continued




Notice, however, that only one of these curves satisfies (5) and the other (6). For example, if $y > 0$ (but no greater than 1) we see from (5) that $\frac{dy}{dx} > 0$ so that our curve should be rising as it passes through (0,0). $S_1$ has this property but $S_2$ doesn't.

In other words, while both $S_1$ and $S_2$ belong to (1) and satisfy (3), only $S_1$ satisfies (5) and only $S_2$ satisfies (6). Thus, if we require unique solutions we must treat higher degree equations as unions of <u>first degree</u> equations. Algebraically, this is often extremely difficult, perhaps even impossible explicitly.

4. The choice of (0,0) was not particularly a good one to illustrate the local property of a solution. Suppose we wanted to find all members of (1) which passed through $(\frac{1}{2}, \frac{\sqrt{15}}{4})$. We let $x = \frac{1}{2}$ and $y = \frac{\sqrt{15}}{4}$ in (1) to obtain

$$\left(\frac{1}{2} - c\right)^2 + \frac{15}{16} = 1$$

or

$$\left(\frac{1}{2} - c\right)^2 = \frac{1}{16}$$

2.1.3(L) continued

or

$$\frac{1}{2} - c = \pm \frac{1}{4}$$

or

$$c = \frac{1}{2} \pm \frac{1}{4}.$$

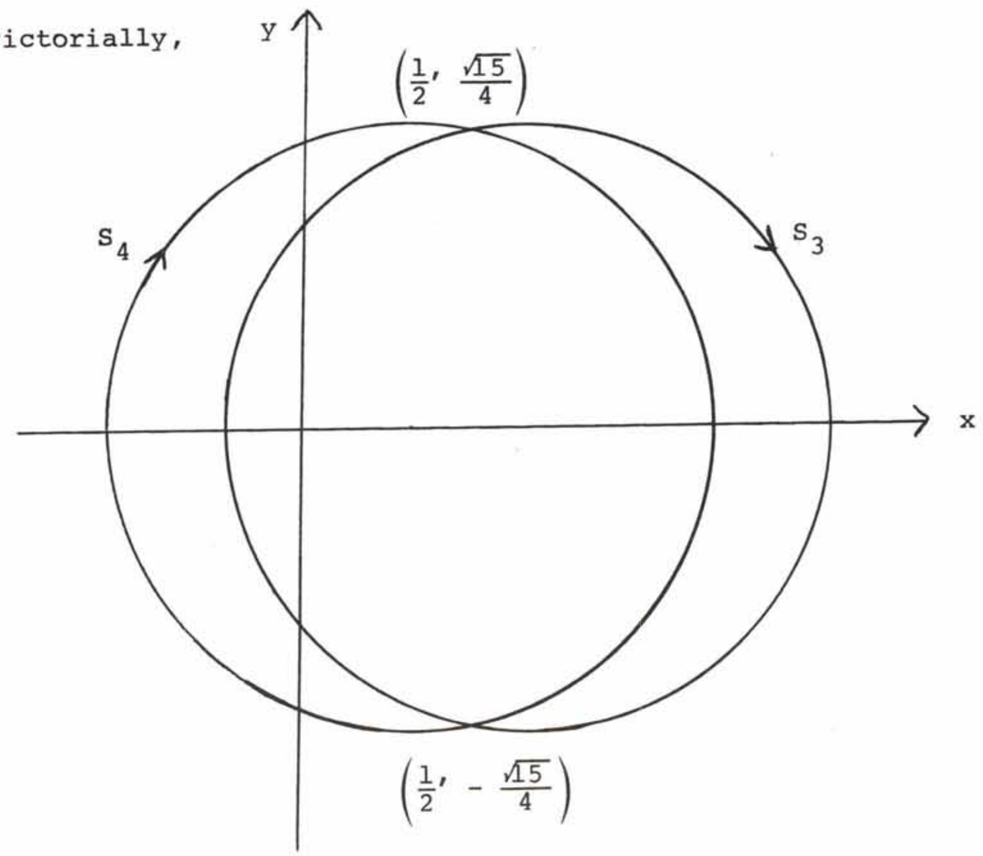
That is,

$$c = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

or

$$c = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Pictorially,



2.1.3(L) continued

Since $S_3$ has positive slope at the given point, it is a solution of (5); and since $S_4$ is falling as it passes through $(\frac{1}{2}, \frac{\sqrt{15}}{4})$ it satisfies (6). Thus, $S_3$ is the only solution if we view (S) as the given equation, and $S_4$ is the only solution if we view (6) as the given equation.

$S_2$ and $S_4$ also meet at $(\frac{1}{2}, -\frac{\sqrt{15}}{4})$ but this may not be important to us if all we care about is what happens "near" $(\frac{1}{2}, \frac{\sqrt{15}}{4})$.

5. In summary, for any point $(x_o,y_o)$ such that $|y_o| \leq 1$ there is one and only one member of (1) which satisfies (5); and one and only one member of (1) which satisfies (6). More specifically, if we let $x = x_o$ and $y = y_o$ in (1) we obtain $(x_o - c)^2 + y_o^2 = 1$ or

$$x_o - c = \pm \sqrt{1 - y_o^2}\ .$$

Hence,

$$c = x_o \pm \sqrt{1 - y_o^2}. \qquad (9)$$

From (2)

$$c = x_o + y_o \frac{dy}{dx}\Big|_{(x_o,y_o)}$$

and comparing this with (9) yields

$$y_o \frac{dy}{dx} = \pm \sqrt{1 - y_o^2}\ .$$

Hence,

$$\frac{dy}{dx} = \pm \frac{\sqrt{1 - y_o^2}}{y_o}$$

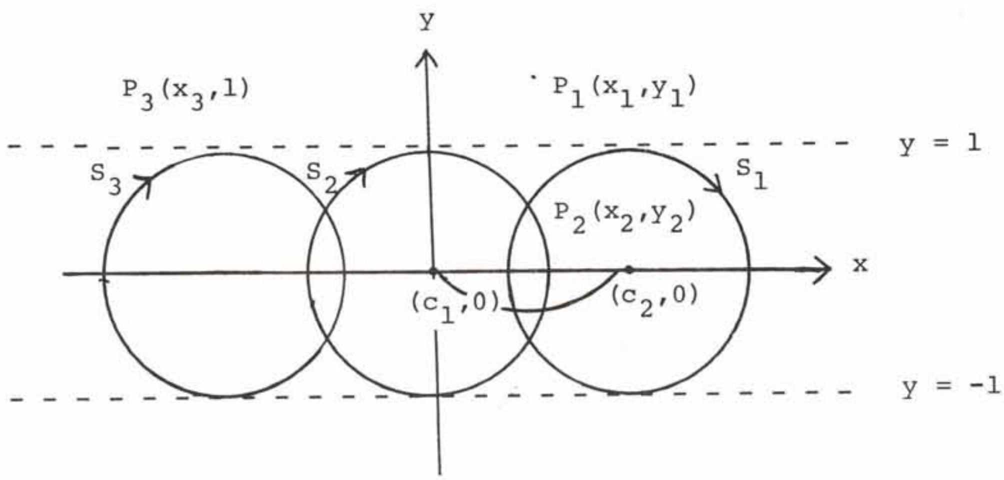
and the ambiguous sign is determined by whether we want the curve to be rising or falling at the given point. That is choosing eith (5) or (6) removes the ambiguous sign from (9).

2.1.3(L) continued

6. But now we come to still another subtle aspect not encountered previously. Suppose we pick an arbitrary point $(x_0,1)$ on the line $y = 1$. [A similar discussion applies to $(x_0,-1)$ on the line $y = -1$.] In this case, there is no ambiguity as to whether we view (3) as (5), since (9) shows that with $y_0 = \pm 1$, $c = x_0$. In other words, $(x - x_0)^2 + y^2 = 1$ is the only member of (1) which satisfies (3) and passes through $(x_0,1)$ [and also $(x_0,-1)$].

However the point $(x_0,1)$ lies on $y = 1$ and conversely every point $(c,1)$ on $y = 1$ belongs to a member of (1); namely $(x - c)^2 + y^2 = 1$. Thus, since every point on each member of (1) satisfies (3), and since each point on $y = 1$ belongs to a member of (1), it follows that the line $y = 1$ is itself a solution of (3) which passes through $(x_0,1)$. There is no way of choosing c in (1) to obtain $y =1$ [among other things, $y = 0$ is a line while (1) represents a circle for each choice of c]. Thus, for any point on the line $y = 1$, there are two solutions of (3) which pass through this point. One of these can be accounted for by either solving (5) or (6); but the other, $y = 1$, is a "mongrel" of sorts that sneaks in by virtue of the fact that each of its points happens to belong to some member of (1). The solution $y = 1$ is called a singular solution. The concept of a singular solution will be treated in more detail in the lecture as well as in the exercises which follow the lecture.

For now, however, let's summarize the results of this exercise pictorially.



2.1.3(L) continued

a. Since $P_1(x_1,y_1)$ has the property that $y_1 > 1$, no solution of (3) passes through $P_1$.

b. Two members of (1) are solutions of (3) which pass through $P_2(x_2,y_2)$ since $-1 < y_2 < 1$. To construct the two solutions, swing an arc of radius 1 centered at $P_2$ and this arc will meet the x-axis at two points, $(c_1,0)$ and $(c_2,0)$. The circles centered at these two points with radius 1 are the required members of (1).

$S_1$ is the solution corresponding to equation (5) (since the slope is positive at $P_2$), while $S_2$ is the solution which satisfies (6). In other words there are two solutions through $P_2$ from the family (1) because (3) is a second degree equation. Once we restrict our attention to either (5) or (6), there is only one member of (1), either $S_1$ or $S_2$ which passes through $P_2$.

c. There are two curves that we know of which satisfy (3) and pass through $P_3(x_3,1)$. One is the member of (1), $(x - x_3)^2 + y^2 = 1$; and the other is the line $y = 1$.*

As a final note, keep in mind that we are as yet unequiped to determine whether there are solutions of (3) which pass through $(x_o,y_o)$, $|y_o| \leq 1$, other than those described by the family (1) and the lines $y = \pm 1$.

2.1.4(L)

In the previous three exercises we introduced the notion of a general solution of a differential equation by beginning with a 1-parameter family of curves and then finding the differential equation which the family satisfied.

In this exercise we shall revisit the same topic, but now from the more conventional point of view of beginning with the first

*This should not be confused with our discussion of $y = 0$ and $y = 1/c - x$ of the previous exercise. There, a member of $y = 1/c - x$ passed through $(x_o,y_o)$ if $y_o \neq 0$ and $y = 0$ passed through $(x_o,y_o)$ if $y_o = 0$. But no point $(x_o,y_o)$ was satisfied by both $y = 0$ and $y = 1/c - x$.

2.1.4(L) continued

order differential equation and then finding a 1-parameter family of curves which is a solution of the equation. We shall then discuss whether the family of solutions is the general solution and in the cases in which it is not we shall talk about what the family lacks.

a. If we proceed mechanically in the same manner by which we solved this type of equation in Part 1 of our course, we obtain

$$\frac{dy}{dx} = x^2y \longrightarrow \tag{1}$$

$$\frac{dy}{y} = x^2dx. \tag{2}$$

Note #1:

We must now remember to check the case y = 0 later. Namely, the validity of going from (1) to (2) hinges on the fact that $y \neq 0$ since we are not allowed to divide by 0. Thus, (2) does not apply when y = 0. In other words, to be more precise, (2) should be replaced by (2') where:

$$\frac{dy}{y} = x^2dx, \quad \underline{y \neq 0} \tag{2'}$$

Integrating (2') we obtain

$$\ln |y| = \frac{1}{3}x^3 + c_1 \quad (y \neq c). \tag{3}$$

Note #2:

A common error is to write: $\ln y = \frac{1}{3}x^3 + c_1$ rather than (3). That is, one tends to say that for $u \neq 0$,

$$\int \frac{du}{u} = \ln u + c$$

rather than the more accurate statement that

$$\int \frac{du}{u} = \ln |u| + c.$$

2.1.4(L) continued

In many physical situations, u turns out to be positive (e.g., u is mass or absolute temperature, etc.), in which case there is no harm in replacing $|u|$ by u since for $u \geq 0$, $|u| = u$. However, to be on the safe side one should use (3) and not $\ln y = \frac{1}{3}x^3 + c_1$.

From (3) we conclude that

$$e^{\ln|y|} = e^{\frac{1}{3}x^3 + x_1} \tag{4}$$

and since $e^{\ln u} = u$ for all $u \neq 0$, conclude from (4) that $|y| = \exp[1/3\ x^3 + c_1]$ $(y \neq 0)$ or

$$|y| = e^{\frac{1}{3}x^3} e^{c_1} \quad (y \neq 0). \tag{5}$$

Since $c_1$ is an arbitrary constant (positive, negative, or zero), $e^{c_1} = c_2$ is an arbitrary positive constant.

<u>Note #3:</u>

For any real number, u, $e^u > 0$ so that (5) may be written as

$|y| = c_2 e^{\frac{1}{3}x^3}$, where $c_2$ is an arbitrary positive constant $(y \neq 0)$. (6)

Then, since $|u| = v \longrightarrow u = \pm v$ for any real numbers u and v, we conclude from (6) that

$$y = \pm c_2 e^{\frac{1}{3}x^3} \quad (y \neq 0). \tag{7}$$

Since $c_2$ was an arbitrary positive constant, $-c_2$ is an arbitrary negative constant. Hence, $\pm c_2$ denotes an arbitrary non-zero constant. That is, we may write (7) in the form

$y = c\, e^{\frac{1}{3}x^3}$ $(y \neq 0)$, where c is an arbitrary non-zero constant. (8)

2.1.4(L) continued

Note #4

Ironically the person who writes $\ln y = \frac{1}{3}x^3 + c_1$ rather than (3) usually obtains (8) much more quickly than we did, through the fortunate stroke of luck that two conceptual errors cancel each other. Namely, from $\ln y = \frac{1}{3}x^3 + c_1$ he concludes that

$$e^{\ln y} = e^{\frac{1}{3}x^3 + c_1} = e^{c_1}e^{\frac{1}{3}x^3},$$

hence that $y = e^{c_1}e^{\frac{1}{3}x^3}$. His final step is to say that since $c_1$ is an arbitrary constant so also is $e^{c_1}$ (forgetting that $e^{c_1}$ cannot be negative) and concludes that $y = c\, e^{1/3\, x^3}$ just as we did in (8). Notice, however, that if $y < 0$, $\ln y$ is not even real, hence $\ln y = \frac{1}{3}x^3 + c_1$ in terms of real numbers is a meaningless equation when $y < 0$.

Remembering that equation (8) applies only on the condition that $y \neq 0$ (which is why we augmented each equation in our derivation with the phrase "$y \neq 0$"), we must now look at the case $y = 0$ separately. We see that since y is identically zero so also is $\frac{dy}{dx}$. Thus, with $y = 0$, equation (1) reads: "0 = 0"; so we see that $y = 0$ is a solution of equation (1).

We next observe that $y = 0$ can be written in the form: $y = 0e^{\frac{1}{3}x^3}$ so that the case $y = 0$ may be included in equation (8) provided only that we remove the restriction that $c \neq 0$.

That is, the family defined by

$$y = ce^{\frac{1}{3}x^3}, \text{ where c is an arbitrary (real*) constant} \qquad (9)$$

is a 1-parameter family of solutions for equation (1).

---

*After all our talk about complex numbers in the previous Block, it may be difficult to remember that our present discussion usually is introduced in the calculus of real variables. Hence, unless otherwise specified, all numbers are assumed to be real in our treatment of Block 2.

2.1.4(L) continued

As a check on equation (9) we see that

$$y = c\,e^{\frac{1}{3}x^3} \longrightarrow$$

$$\frac{dy}{dx} = x^2 c\,e^{\frac{1}{3}x^3}$$

$$= x^2\left(c\,e^{\frac{1}{3}x^3}\right)$$

$$= x^2 y \text{ (including the case } y = 0\text{).}$$

b. In the language of sets we have shown that

$$\{y: y = c\,e^{\frac{1}{3}x^3},\ c \text{ arb.const.}\} \subseteq \{y: y' = x^2 y\}\ ;$$

or stated in terms of functions (where $y = f(x)$) we have shown that every function of the form:

$f(x) = ce^{\frac{1}{3}x^3}$ where c is an arbitrary constant is a solution of the differential equation

$$f'(x) = x^2 f(x), \tag{10}$$

where (10)is obtained by replacing y by f(x) in (1).

What has not yet been done by us is to investigate whether equation (1) [or (10)] can have solutions which are not of the form $y = f(x) = c\exp[1/3\ x^3]$.

It is at this point that we invoke the theorem (stated without proof in the lecture), which we shall use as an axiom in our course, that: If

$$\frac{dy}{dx} = g(x,y) \tag{11}$$

and if $g(x,y)$ and $g_y(x,y)$ are defined and continuous in a region R, then for <u>each $(x_0,y_0)\varepsilon R$</u> there is one and only one curve that satisfies (11) and passes through $(x_0,y_0)$.

2.1.4(L) continued

In our present exercise the role of $g(x,y)$, as defined in (11), is played by $x^2y$. That is, from (1)

$$\frac{dy}{dx} = x^2y = g(x,y).$$

Therefore,

$$g(x,y) = x^2y \text{ and } g_y(x,y) = x^2.$$

Since $x^2y$ and $x^2$ are continuous functions of x and y, the theorem tells us that through each point $(x_o,y_o)$ in the xy-plane there is one <u>and only one</u> curve whose equation satisfies equation (1).

c. With this in mind we choose an arbitrary point $(x_o,y_o)$ in the plane and see if there exists a value of c which makes a member of the family in (9) [i.e., the family $y = ce^{1/3x^3}$] pass through $(x_o,y_o)$.

<u>Note #5</u>:

By our "axiomatic theorem" if there is a member of (9) which passes through $(x_o,y_o)$ then <u>no other curve</u> which satisfies (1) can pass through $(x_o,y_o)$. On the other hand, if we find a point $(x_o,y_o)$ with the property that no member of (9) passes through it, then there must be a different curve which satisfies (1) and passes through $(x_o,y_o)$, since every point $(x_o,y_o)$, must have a solution of (1) which passes through it. Thus, our strategy in this part of the exercise will be try to show that at each $(x_o,y_o)$ in the plane there is a member of (9) which passes through $(x_o,y_o)$, whereupon the theorem "blocks out" the existence on any other solution which passes through the given point $(x_o,y_o)$.

At any rate, letting $x = x_o$ and $y = y_o$ in (9) we obtain that

$$y_o = ce^{\frac{1}{3}x_o^3}$$

so that

2.1.4(L) continued

$$c = y_o e^{-\frac{1}{3} x_o^3} . \qquad (13)$$

Obtaining (12) from $y_o = ce^{\frac{1}{3} x_o^3}$ required that we divide both sides of the equation by

$$e^{\frac{1}{3} x_o^3}$$

and since

$$e^{\frac{1}{3} x_o^3}$$

can never be 0 for any real number $x_o$, the validity of (12) holds for every choice of $x_o$.

Substituting (12) into (9) we obtain the result that

$$y = (y_o e^{-\frac{1}{3} x_o^3}) e^{\frac{1}{3} x^3}$$

or

$$y = y_o e^{\frac{1}{3}(x^3 - x_o^3)} = (y_o e^{-\frac{1}{3} x_o^3}) e^{\frac{1}{3} x^3} . \qquad (13)$$

By our key theorem, then (13) names the only curve which passes through $(x_o, y_o)$ and satisfies (1).

Now we can say that the 1-parameter family:

$$y = ce^{\frac{1}{3} x^3} \text{, c is an arbitrary,} \qquad (9)$$

is the general solution of the differential equation

$$\frac{dy}{dx} = x^2 y , \qquad (1)$$

since,

1. each member of (9) is a solution of (1);
2. for each point $(x_o, y_o)$ in the plane, one and only one member of (9) passes through this point, namely

2.1.4(L) continued

$$y = (y_0 e^{-\frac{1}{3}x_0^3})\, e^{\frac{1}{3}x^3}; \text{ and}$$

3. no other curve which passes through $(x_0, y_0)$ can satisfy (1), or from a different perspective, if another curve satisfies (1) it doesn't pass through $(x_0, y_0)$.

2.1.5(L)

Under the heading of ignorance is bliss, the present exercise would have been early disposed of back in Part 1 of our course. Namely, given

$$\frac{dy}{dx} = 3y^{\frac{2}{3}} \qquad (1)$$

we would separate variables to obtain

$$\frac{1}{3}\, y^{-\frac{2}{3}}\, dy = dx. \qquad (2)$$

Hence,

$$y^{\frac{1}{3}} = x + c,$$

or

$$y = (x + c)^3. \qquad (3)$$

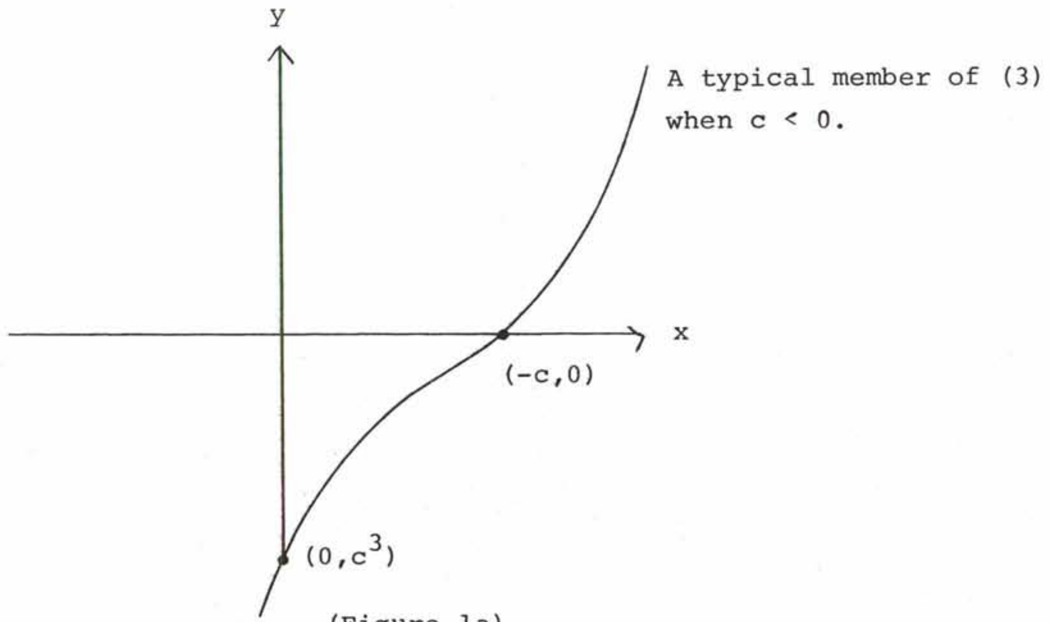
To be sure, going from (1) to (2) required that $y \neq 0$, so that the more careful among us might have recognized that

$$y = 0 \qquad (4)$$

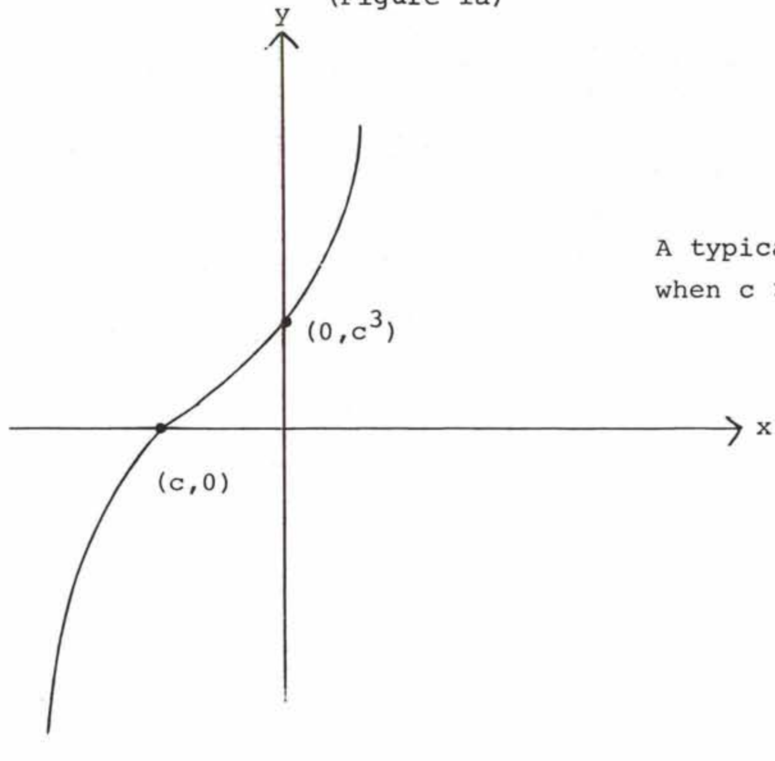
is a solution of (1), but is not a member of (3). [This should not be confused that for each member of (3), $y = 0$ when $x = -c$. In other words, each member of (1) meets the x-axis at $(-c,0)$, but $y = 0$ means $y = 0$ for all x; i.e., the x-axis.]

2.1.5(L) continued

Pictorially,




(Figure 1a)




(Figure 1b)

Thus, no member of (3) is the x-axis (y = 0).

2.1.5(L) continued

Moreover, if we wanted the member of (3) which passed through a given point $(x_o,y_o)$, we would let $x = x_o$ and $y = y_o$ in (3) to obtain

$$y_o = (x_o + c)^3$$

or

$$\sqrt[3]{y_o} = x_o + c$$

or

$$c = \sqrt[3]{y_o} - x_o. \tag{5}$$

Equation (5) shows that there is one member of (3) which passes through $(x_o,y_o)$, namely

$$y = (x + \sqrt[3]{y_o} - x_o)^3. \tag{6}$$

This much is hopefully old-hat. What we learned in the lecture is that since

$$3y^{\frac{2}{3}} \text{ and } \frac{\partial(3y^{\frac{2}{3}})}{\partial y}$$

exist and are continuous except when $y = 0$ (in which case

$$\frac{\partial(3y^{\frac{2}{3}})}{\partial y} = 2y^{-\frac{1}{3}} = \infty),$$

our fundamental theorem quarantees that (3) is the general solution of (1) provided that (1) is defined in a region R which includes no points on $y = 0$.

What this means, for example, is suppose we want a solution of (1) which passes through (2,1). Then from (6) we see that

$$y = (x + \sqrt[3]{1} - 2)^3$$

or

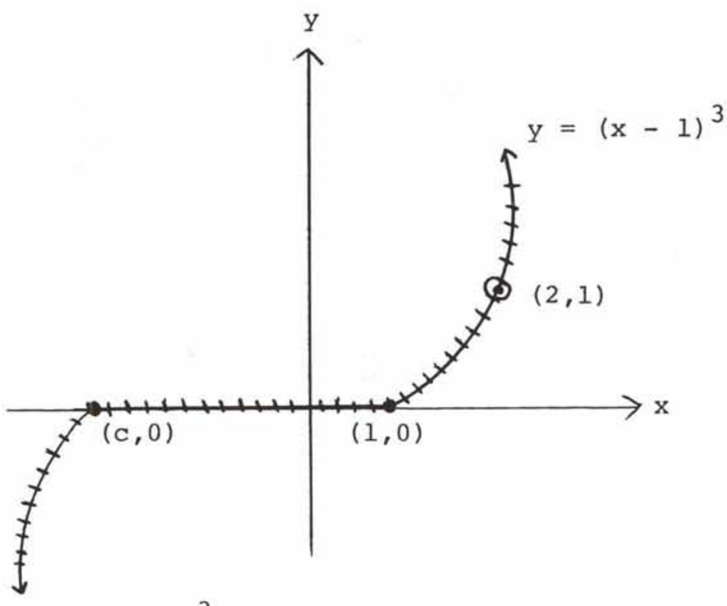
2.1.5(L) continued

$$y = (x - 1)^3 \tag{7}$$

is one such solution.

Moreover, from the theorem of the lecture we know that if we stay "sufficiently close" to (2,1) [i.e., in a neighborhood R which doesn't touch the x-axis] then (7) is the only solution of (1) which passes through (2,1), as long as we stay within R.

What happens if we leave R? This is where (4) becomes crucial. Namely, $y \equiv 0$ is a solution of (1). What we may then do is take the curve $y = (x - 1)^3$, chop it off when it meets the x-axis at (1,0), then "run" along the x-axis in the negative sense from (1,0) to any point (c,0) where $c < 1$. Then at (c,0), we pick up the curve $y = (x - c)^3$. Pictorially,



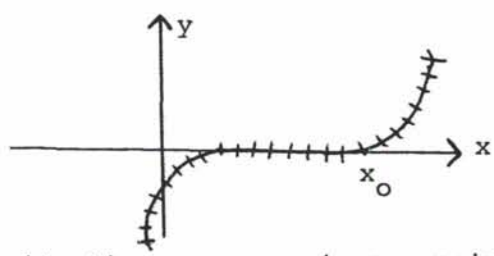

(Figure 2)

Every curve of the type depicted in Figure 2 is a solution of (1) and passes through (2,1). This doesn't contradict our fundamental theorem, however, since each curve of the type in Figure 2 contains at least a portion of $y = 0$.

2.1.5(L) continued

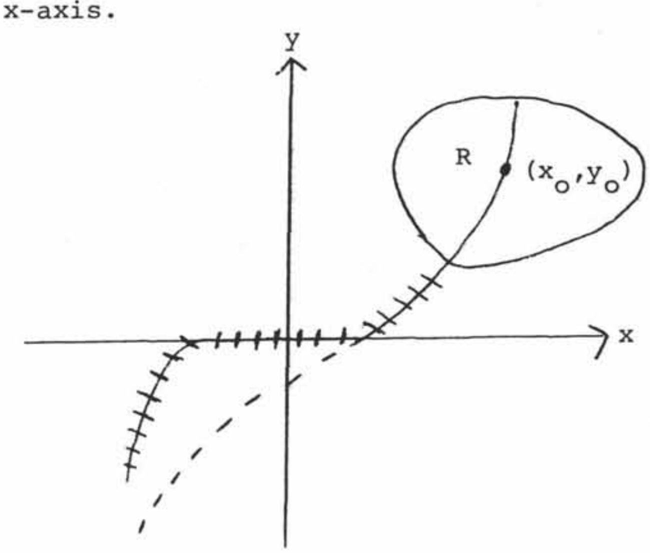
What is true is that the infinitely many solutions of (1) which pass through (2,1) all look the same (namely they are all $y = (x - 1)^3$) if the solution is restricted to a neighborhood R of (2,1) which does not include a segment of the x-axis.

As for any point $(x_0,0)$ on the x-axis there are many solutions of (1) that pass through $(x_0,0)$. One of these is $y = (x - x_0)^3$. The rest have the form




In most real life situations we are given a point $(x_0,y_0)$ and a particular differential equation and what we seek are all solutions of the equation "near" $(x_0,y_0)$ which pass through $(x_0,y_0)$.

Summarized pictorially, let $P_0(x_0,y_0)$ by any point not on the x-axis and let R be any neighborhood of $P_0$ which includes no part of the x-axis.




1. Equation (1) has only one solution in R which passes through $(x_0,y_0)$.
2. Several solutions may exist but they are indistinguishable when restricted to R.

2.1.6

a. Given

$$x \frac{dy}{dx} - 3y = 0 \qquad (1)$$

we may write

$$\frac{dy}{dx} = \frac{3y}{x}. \qquad (x \neq 0) \qquad (2)$$

Letting $f(x,y) = 3y/x$ we have that $f_y(x,y) = 3/x$; and since both $3y/x$ and $3/x$ exist and are continuous except when $x = 0$, we may conclude from the theorem that if R is any region of the xy-plane which does not intersect the y-axis (i.e., $x = 0$); then for each $(x_o,y_o)\varepsilon R$ there is one and only one curve c which passes through $(x_o,y_o)$ and satisfies (1).

b. If we separate variables in (2) we obtain

$$\frac{dx}{x} = \frac{dy}{3y}. \quad (y \neq 0)^* \qquad (3)$$

Hence,

$$\ln |x| + c_1 = \frac{1}{3} \ln |y|$$

or

$$\ln |y| = 3 \ln |x| + c_2$$

or

$$|y| = e^{3 \ln |x| + c_2} = e^{\ln |x|^3 + c_2} = e^{c_2} e^{\ln|x|^3}.$$

Hence,

$$|y| = c_3 |x|^3, \quad c_3 > 0 \ (\text{since } c_3 = e^{c_2}).$$

*The condition that $x \neq 0$ is already tacitly assumed by assuming that we are in a region R for which the general solution exists.

2.1.6 continued

Therefore,

$y = cx^3$, where $c = \pm c_3$, or, $c \neq 0$ is an arbitrary constant. (4)

By observing, $y = 0$ is handled by letting $c = 0$ in (4). We see that

$$y = cx^3, \text{ c an arbitrary constant} \tag{5}$$

is the general solution of equation (1) in R provided that R includes no points on the y-axis.

In particular, for the general point $(x_o,y_o)$ with $x_o \neq 0$, we see from (5) that $y_o = cx_o^3$, whence

$$c = \frac{y_o}{x_o^3} \tag{6}$$

[and notice in (6) how glaringly it stands out that $x_o \neq 0$].

That is, for $(x_o,y_o)\varepsilon R$, the only solution of equation (1) which passes through $(x_o,y_o)$ is

$$y = \frac{y_o}{x_o^3} x^3 \quad (x_o \neq 0). \tag{7}$$

c. Letting $x_o = y_o = 1$ in (7) we see that the curve c is defined by

$$y = x^3. \tag{8}$$

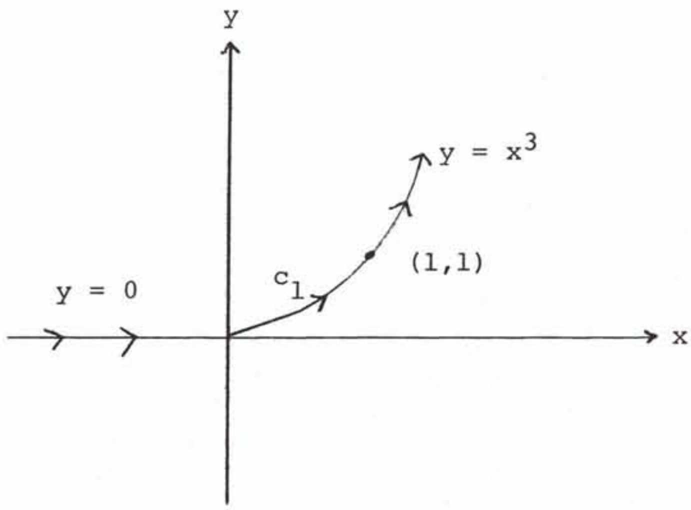
Note:
Returning to the case $x = 0$, let us observe that $y \equiv 0$ is trivially a solution of equation (1). Thus, if we let R intersect the y-axis, we find that (8) is not the only solution of (1) which passes through (1,1). In particular, if we define the curve $c_1$ by

$$y = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x \leq 1 \end{cases}$$

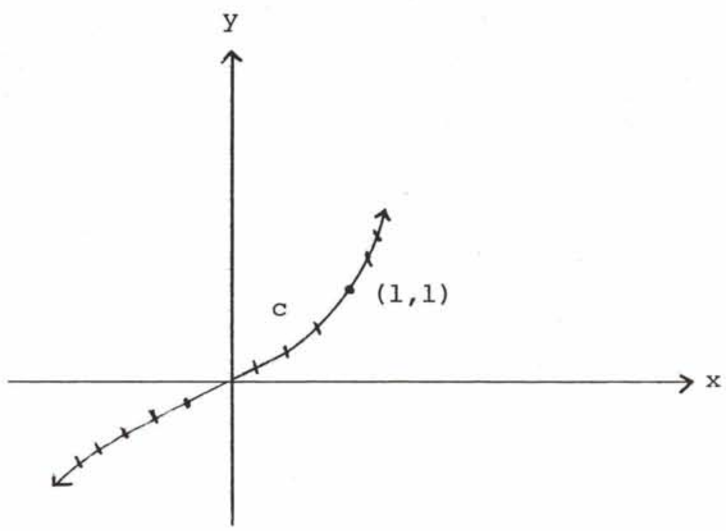
then $c_1$ also satisfies (1), passes through (1,1) but it is not the same curve as c.

2.1.6 continued

Pictorially,




$C_1$ passes through (1,1) and satisfies equation (1).




C passes through (1,1) and satisfies equation (1).

Notice, however, that in any region R which does not contain a segment of the x-axis we cannot distinguish between c and $c_1$.

2.1.7(L)

Our main purpose of introducing the concept of an envelope here is to show one source of a singular solution to a differential equation. Namely, if the one parameter family of curves, $y = f(x,c)$, possesses an envelope E, say $y = g(x)$; then if the 1-parameter family is a solution of a first-order differential equation, so also is the envelope, since each point on the envelope is also a point on at least one of the members of the 1-parameter family.

Before we begin to compute the envelope of a given family, let us first indicate the plausibility of the recipe for computing the envelope. Briefly outlined, if we assume that the family has an envelope, we look at any point $(x_0,y_0)$ that belongs to the envelope, E. Since the equaton for E is $y = g(x)$, we have that

$$y_0 = g(x_0) \tag{1}$$

since $(x_0,y_0)$ must satisfy the equation for E.

But we also know that there exists a value for c such that

$$y_0 = f(x_0,c) \tag{2}$$

since the definition of envelope requires that each point on E be a point on at least one member of the 1-parameter family.

Notice that we may look at (2) as a function of c. That is, for a fixed $(x_0,y_0)$ on E, we may replace x by $x_0$ and y by $y_0$ in each member of the 1-parameter family. For example, if the 1-parameter family is given by $y = cx - c^2$, then if we replace x and y by $x_0$ and $y_0$ we obtain

$$y_0 = cx_0 - c^2 \tag{3}$$

which is an implicit function of c. Moreover, if we differentiate (3) with respect to c, remembering that $x_0$ and $y_0$ are constants, we obtain

$$0 = x_0 - 2c.$$

2.1.7(L) continued

As illustrated in our parenthetical remark, we may differentiate (2) with respect to c [and this is why we assume that f(x,c) is differentiable. That is, the requirement of differentiability is not necessary to talk about the envelope of a family, but the technique to be used for finding the envelope will use the differentiability property] to obtain

$$0 = f'(x_o,c) \quad (4)$$

where in (4) the differentiation is taken with respect to c.

Of course, we do not know the point $(x_o,y_o)$ explicitly as yet since we are only assuming that there is an envelope (so even if the envelope does exist we do not know its equation - in fact that is what we are trying to find in this exercise) and that $(x_o,y_o)$ named an arbitrary point of the envelope. Thus, $x_o$ and $y_o$ are actually "unknowns" and are better written as x and y.

If we now revisit equations (2) and (4) in this light, we see that since (x,y) must satisfy both (2) and (4), if the envelope exists, it must be that the equation of the envelope is obtained by solving the following pair of equations simultaneously:

$$y = f(x,c) \text{ and } 0 = f_c(x,c). \quad (5)$$

Note #1:
Since x is not a constant we must replace $f'(x_o,c)$ by $f_c(x,c)$.

Note #2:
Notice that equation (5) makes no reference to g. This is as it should be, since it is f that is explicitly given, while g is used only to refer to the envelope, assuming in the first place that such an envelope exists.

Note #3:
Observe that (5) only tells us that if the envelope E exists it must satisfy the conditions stated in (5). It does not say that if we solve (5) by eliminating and finding y as a function of x

2.1.7(L) continued

that the resulting curve is the envelope of the family $y = f(x,c)$. All we are saying is that if the envelope exists, it is defined by (5).

At any rate, with the hope that (5) now seems more than just a memorizable formula, we turn our attention to the exercise.

a. Since $y = cx - c^2$, we have, using the notation of (5), that

$$f(x,c) = cx - c^2. \tag{6}$$

Hence,

$$f_c(x,c) = x - 2c. \tag{7}$$

Using (6) and (7) in (5), we see that if there is an envelope E, it is defined by

$$\left.\begin{aligned} y &= cx - c^2 \\ 0 &= x - 2c \end{aligned}\right\} \tag{8}$$

From the second equation in (8) we see that $c = x/2$, and replacing c by $x/2$ in the first equation [thus eliminating c from (8)] we obtain

$$y = \frac{x}{2}(x) - \left(\frac{x}{2}\right)^2$$

or

$$y = \frac{1}{4}x^2. \tag{9}$$

Notice that the Clairaut equation used in the lecture had $y = cx - c^2$ as a 1-parameter solution while (9) shows that the singular solution $y = 1/4x^2$ is indeed the envelope of the family.

b. Let us point out first that if the 1-parameter family is in the more implicit form $f(x,y,c) = 0$, then (5) is replaced by the system

2.1.7(L) continued

$$\left.\begin{aligned} f(x,y,c) &= 0 \\ f_c(x,y,c) &= 0 \end{aligned}\right\} \qquad (5')$$

With this in mind, the equation $(x - c)^2 + y^2 = 1$ may be written in the form

$$(x - c)^2 + y^2 - 1 = 0 \qquad (10)$$

whereupon in the language of (5'),

$$f(x,y,c) = (x - c)^2 + y^2 - 1. \qquad (11)$$

Hence,

$$f_c(x,y,c) = -2(x - c),$$

so that equating $f_c(x,y,c)$ to zero yields $-2(x - c) = 0$ or

$$x = c. \qquad (12)$$

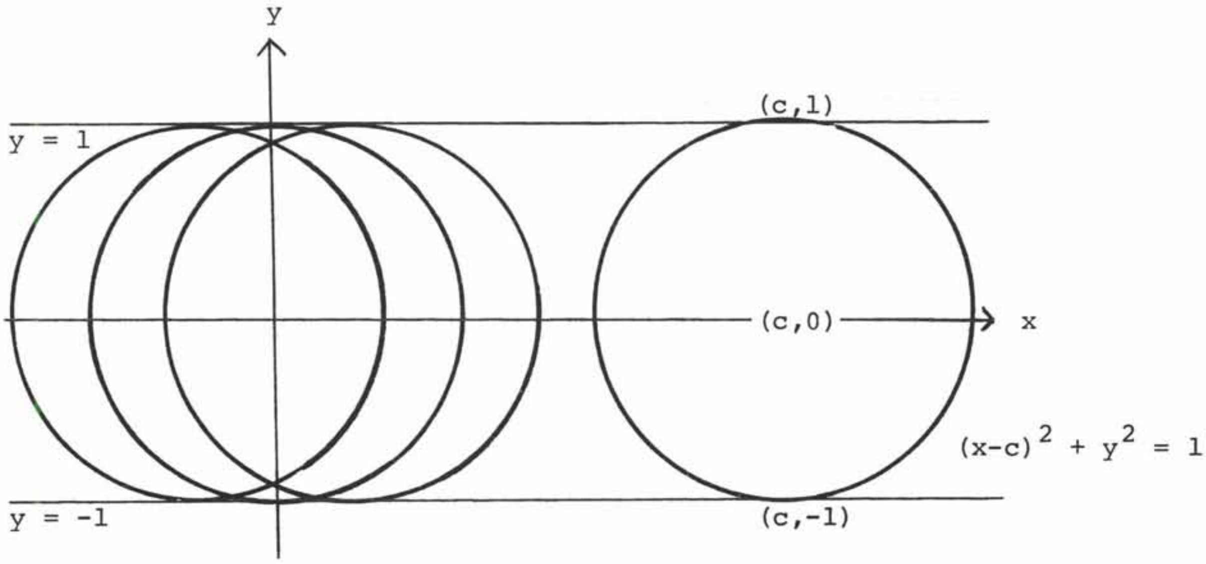
By virtue of (12), we may substitute x for c in (10) to obtain $0^2 + y^2 - 1 = 0$ or $y^2 = 1$ or

$$y = \pm 1. \qquad (13)$$

Deriving (13) from (10) and (12) is equivalent to solving (5'). We thus see that the lines $y = 1$ and $y = -1$ are envelopes to the family $(x - c)^2 + y^2 = 1$. This accounts for the fact that $y = \pm 1$ was a singular solution of the differential equation satisfied by $(x - c)^2 + y^2 = 1$ in our discussion of Exercise 2.1.3.

2.1.7(L) continued

Pictorially,




2.1.8(optional)

a. Differentiating

$$(x - c)^2 + y^2 = 4c + 4 \qquad (1)$$

with respect to x yields

$$2(x - c) + 2y \frac{dy}{dx} = 0.$$

Hence,

$$x - c + y \frac{dy}{dx} = 0,$$

or

$$c = x + y \frac{dy}{dx}. \qquad (2)$$

Replacing c in (1) by its value in (2) yields,

$$\left[x - \left(x + y \frac{dy}{dx}\right)\right]^2 + y^2 = 4\left(x + y \frac{dy}{dx}\right) + 4,$$

or

$$y^2\left(\frac{dy}{dx}\right)^2 + y^2 = 4x + 4y \frac{dy}{dx} + 4. \qquad (3)$$

2.1.8 continued

b. We rewrite (3) so as to emphasize that it is a quadratic equation in dy/dx. Namely,

$$y^2\left(\frac{dy}{dx}\right)^2 - 4y\left(\frac{dy}{dx}\right) + (y^2 - 4x - 4) = 0. \tag{4}$$

Applying the quadratic formula to (4), we obtain

$$\frac{dy}{dx} = \frac{4y \pm \sqrt{16y^2 - 4y^2(y^2 - 4x - 4)}}{2y^2},$$

or

$$\frac{dy}{dx} = \frac{4y \pm 2y\sqrt{4 - (y^2 - 4x - 4)}}{2y^2}$$

or

$$\frac{dy}{dx} = \frac{2 \pm \sqrt{8 - y^2 + 4x}}{y}, \quad (y \neq 0). \tag{5}$$

Equation (5), of course, is equivalent to the two first order, first degree equations:

$$\frac{dy}{dx} = \frac{2 + \sqrt{8 - y^2 + 4x}}{y}, \quad (y \neq 0) \tag{6}$$

and

$$\frac{dy}{dx} = \frac{2 - \sqrt{8 - y^2 + 4x}}{y}, \quad (y \neq 0). \tag{7}$$

To apply the fundamental theorem, we let

$$f(x,y) = \frac{2 + \sqrt{8 - y^2 + 4x}}{y}, \quad (y \neq 0) \tag{8}$$

so that (6) becomes

$$\frac{dy}{dx} = f(x,y).*$$

*Analogous results will hold, of course, if we let $f(x,y) = 2 - \sqrt{8 - y^2 + 4x}/y$ and use equation (7).

2.1.8 continued

Since $f(x,y)$ will not be real unless $8 - y^2 + 4x \geq 0$, we see that we must first restrict our region R to obey

$$8 - y^2 + 4x \geq 0, \text{ for all } (x,y) \varepsilon R. \tag{9}$$

Equation (9) may be written in the somewhat more suggestive form

$$y^2 \leq 4x + 8, \tag{10}$$

from which we conclude that R is within the parabola $y^2 = 4x + 8$, including the parabola itself.

While this restriction of R as above makes $f(x,y)$ real and continuous, the theorem requires that $f_y(x,y)$ as well, be real and continuous. Now from (8),

$$f_y(x,y) = \frac{y\left[\frac{1}{2}(8 - y^2 + 4x)^{-\frac{1}{2}}(-2y)\right] - (2 + \sqrt{8 - y^2 + 4x})}{y^2}$$

$$= \frac{\frac{-y^2}{\sqrt{8 - y^2 + 4x}} - (2 + \sqrt{8 - y^2 + 4x})}{y^2}$$

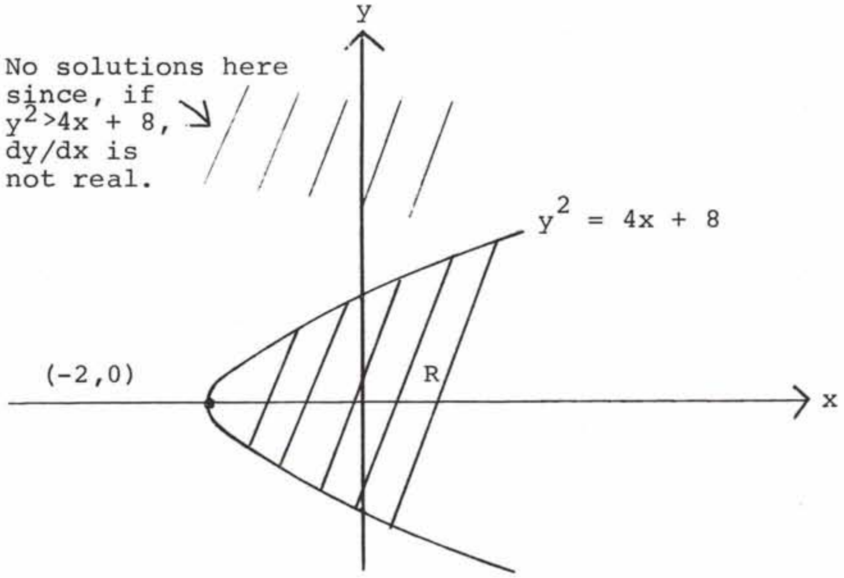
$$= \frac{-y^2 - 2\sqrt{8 - y^2 + 4x} - (8 - y^2 + 4x)}{y^2\sqrt{8 - y^2 + 4x}}, \tag{11}$$

and since $y^2 = 4x + 8$ makes the denominator of (11) vanish, we must make the additional restriction that $y^2 \neq 4x + 8$ so that $f_y(x,y)$ will be real and continuous.

Combining this with (10) we see that the largest region R in which the general solution exists is $y^2 < 4x + 8$, $y \neq 0$; which defines R to be the interior of the parabola $y^2 = 4x + 8$.

2.1.8 continued

Pictorially,




For each point $(x_o,y_o)$ in R there is one and only one solution of (6) which passes through $(x_o,y_o)$.*

c. Since $(3,4)\varepsilon R$, we expect that c will be determined once we let $x = 3$ and $y = 4$ in (1). This yields $(3 - c)^2 + 4^2 = 4c + 4$ or $9 - 6c + c^2 + 16 = 4c + 4$ or $c^2 - 10c + 21 = 0$, or

$$(c - 3)(c - 7) = 0. \tag{12}$$

From (12) we see that either $c = 3$ or $c = 7$.

With $c = 3$, equation (1) becomes

$$(x - 3)^2 + y^2 = 16, \tag{13}$$

while with $c = 7$, equation (1) becomes

$$(x - 7)^2 + y^2 = 32. \tag{14}$$

*Similarly one and only one solution of (7) passes through $(x_o,y_o)$. Since these two solutions need not be the same we see that for a unique solution of (3) we must restrict our attention to either (6) or (7) but not both.

2.1.8 continued

Both (13) and (14) represent circles which pass through (3,4). The slope of (13) at (3,4) is

$$2(x - 3) + 2y \frac{dy}{dx}\bigg|_{(3,4)} = 0$$

or

$$\frac{dy}{dx}\bigg|_{(3,4)} = -\frac{(x - 3)}{y}\bigg|_{(3,4)} = 0.$$

This agrees with the value of

$$\frac{dy}{dx}\bigg|_{(3,4)}$$

as given by (7). On the other hand the slope of (14) at (3,4) is given by

$$2(x - 7) + 2y \frac{dy}{dx}\bigg|_{(3,4)} = 0$$

so that

$$\frac{dy}{dx}\bigg|_{(3,4)} = -\frac{(x - 7)}{y}\bigg|_{(3,4)} = 1$$

which agrees with the value of

$$\frac{dy}{dx}\bigg|_{(3,4)}$$

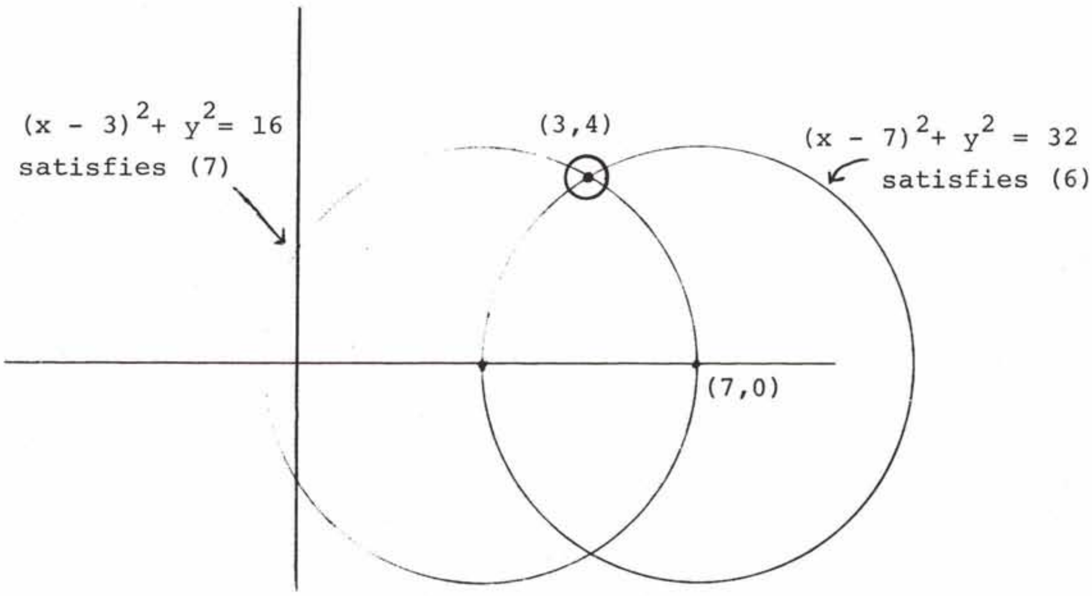
as given by (6).

In summary, then, the circle (14) is the only solution of (6) which passes through (3,4); while the circle (13) is the only solution of (7) which passes through (3,4). Since the only solutions of (3) are solutions of either (6) or (7), we have that (13) and (14) are the only solutions of (3) which pass through (3,4). The ambiguity vanishes once we agree to replace equation

2.1.8 continued

(5) by the two separate equations (6) and (7).

Pictorially,




d. We rewrite (1) as

$$(x - c)^2 + y^2 - 4c - 4 = 0 \tag{15}$$

and let

$$f(x,y,c) = (x - c)^2 + y^2 - 4c - 4. \tag{16}$$

Then

$$f_c(x,y,c) = -2\,(x - c) - 4. \tag{17}$$

Using (16) and (17) in the system

$$\left.\begin{array}{l} f(x,y,c) = 0 \\ f_c(x,y,c) = 0 \end{array}\right\}$$

yields

$$(x - c)^2 + y^2 - 4c - 4 = 0 \tag{18}$$

2.1.8 continued

and

$$-2(x - c) - 4 = 0. \qquad (19)$$

From (19), $x - c = -2$ or $c = x + 2$, whereupon (18) becomes $4 + y^2 - 4(x + 2) - 4 = 0$ or

$$y^2 = 4(x + 2) = 4x + 8. \qquad (19)$$

Thus, $y^2 = 4(x + 2)$ must also satisfy (3) since it is the envelope of a family of solutions of (3). As a check, (19) yields

$$2y\,\frac{dy}{dx} = 4$$

or

$$y\,\frac{dy}{dx} = 2$$

whereupon (3) becomes

$$4 + y^2 = 4x + 8 + 4$$

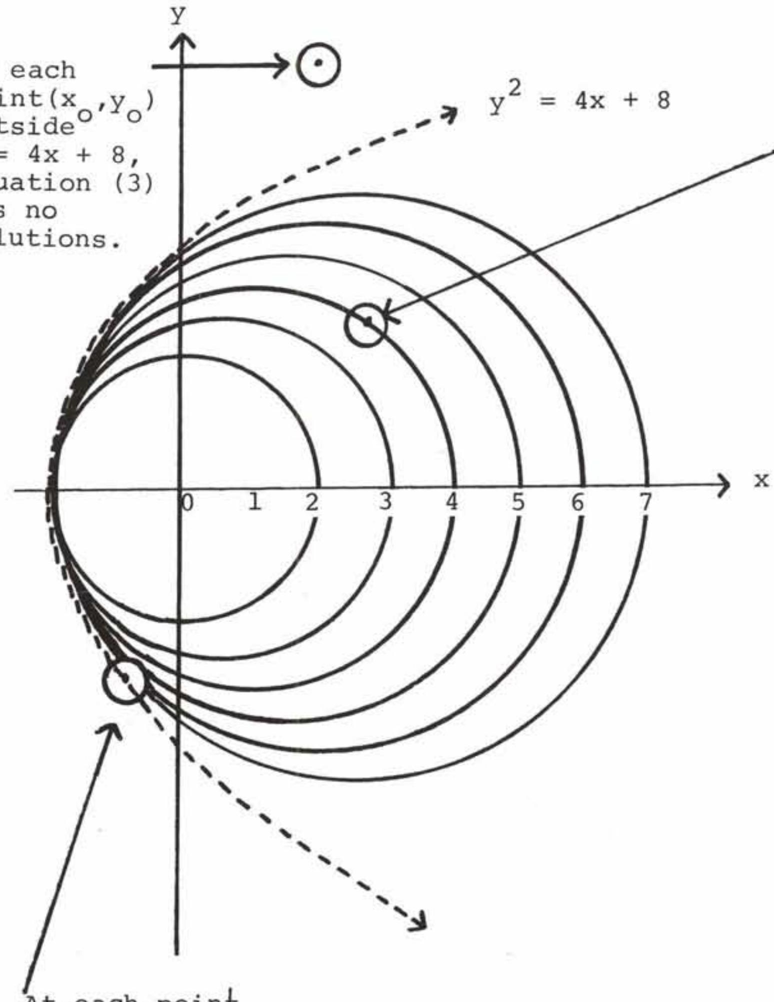
or

$$y^2 = 4x + 8$$

which checks with (19).

Again, pictorially, we have drawn the members of (1) with $c = 0,0,1,2,3,4,5,6,$ and 7 to obtain

2.1.8 continued



2.1.9 (optional)

a. Given the Clairaut equation

$$y = x\frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^4 \qquad (1)$$

we know from the lecture that we may replace dy/dx by c in (1) to obtain

$$y = cx - \frac{1}{4}c^4 \qquad (2)$$

which is a 1-parameter family of solutions of (1).

b. Letting $f(x,c) = cx - \frac{1}{4}c^4$, we see that

$$f_c(x,c) = x - c^3. \qquad (3)$$

Hence, using (2) and (3) in the system

$$y = f(x,c)$$
$$f_c(x,c) = 0$$

we obtain

$$\left.\begin{aligned} &y = cx - \tfrac{1}{4}c^4 \\ &\text{and} \\ &x - c^3 = 0. \end{aligned}\right\} \qquad (4)$$

The bottom equation in (4) indicates that $c = x^{\frac{1}{3}}$, and replacing c by $x^{1/3}$ in the top equation of (4) [i.e., in (2)], we obtain

$$\begin{aligned} y &= (x^{\frac{1}{3}})x - \frac{1}{4}(x^{\frac{1}{3}})^4 \\ &= x^{\frac{4}{3}} - \frac{1}{4}x^{\frac{4}{3}} \end{aligned}$$

so that

$$y = \frac{3}{4}x^{\frac{4}{3}} \qquad (5)$$

2.1.9 continued

is the envelope of (2).*

c. Hence,

$$y = \frac{3}{4} x^{\frac{4}{3}}$$

should also be a solution of (1).

To check this we see from (5) that

$$\frac{dy}{dx} = x^{\frac{1}{3}}$$

whereupon

$$x \frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^4 = x^{\frac{4}{3}} - \frac{1}{4} x^{\frac{4}{3}}$$

$$= \frac{3}{4} x^{\frac{4}{3}}$$

$$= y \text{ [by (5)].}$$

d. 1. $y = 3/4x^{4/3} = 3/4(\sqrt[3]{x})^4, \geq 0$ for all x. Hence, $y = 3/4x^{4/3}$ never goes below the x-axis.

2. $dy/dx = x^{1/3} = \sqrt[3]{x}$ .

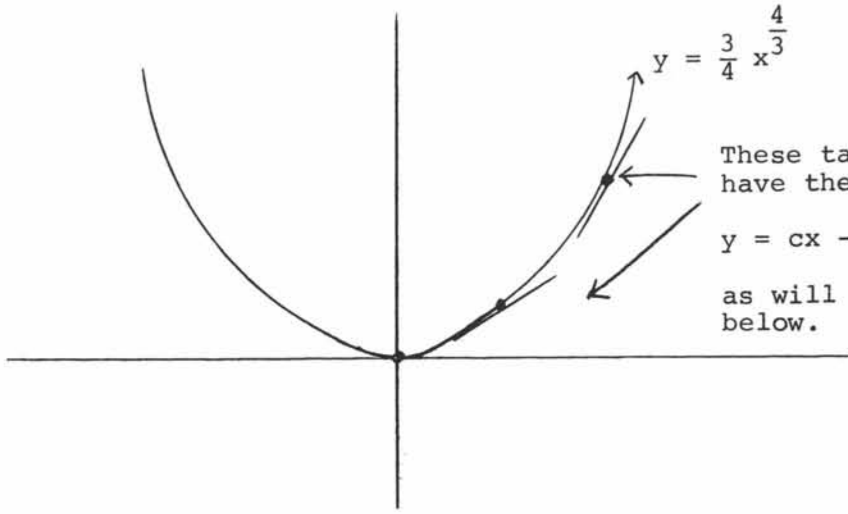
Hence, dy/dx and x have the same sign for each x. Consequently, $y = 3/4x^{4/3}$ rises when $x > 0$ and falls when $x < 0$.

*Actually, all we have shown is that if (2) has an envelope, it is given by (5). Technically speaking, we must still check that (5) is the envelope and this is laborious. However in part (c) we prove indirectly that (5) is the envelope of (2) since it satisfies the same differential equation as each member of (2); and in part (d) we show directly that (5) is the envelope of (2).

2.1.9 continued

3. $\frac{d^2y}{dx^2} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3(\sqrt[3]{x})^2}$, $x \neq 0$, so $\frac{d^2y}{dx^2}$ is never negative.

Hence, $y = \frac{3}{4}x^{4/3}$ always "holds water". Therefore,




Since $dy/dx = x^{1/3}$, the slope of $y = 3/4x^{4/3}$ at $(x_o,y_o)$ is given by $x_o^{1/3}$. Hence, the equation of the line tangent to $y = 3/4x^{4/3}$ at $(x_o,y_o)$ is

$$\frac{y - y_o}{x - x_o} = x_o^{\frac{1}{3}}$$

or

$$y - y_o = x_o^{\frac{1}{3}}x - x_o^{\frac{4}{3}}$$

or

$$y = x_o^{\frac{1}{3}}x - x_o^{\frac{4}{3}} + y_o \qquad (6)$$

and since for $(x_o,y_o)$ on $y = 3/4x^{4/3}$, $y_o = 3/4x_o^{4/3}$, (6) becomes

$$y = x_o^{\frac{1}{3}}x - \frac{1}{4}x_o^{\frac{4}{3}}$$

2.1.9 continued

$$= x_o^{\frac{1}{3}}x - \frac{1}{4}(x_o^{\frac{1}{3}})^4. \qquad (7)$$

Then, since $x_o$ is fixed, so also is $x_o^{1/3}$. Therefore, letting $c = x_o^{1/3}$ in (7) we obtain $y = cx - 1/4c^4$ which is certainly a member of (2).

Conversely, the fact that $c = x_o^{1/3}$[from (4)] implies that $x_o = c^3$, whereupon it is easier seen that $y = cx - 1/4c^4$ is tangent to $y = 3/4x^{4/3}$ at $(c^3, 3/4c^4)$; thus establishing that (5) is the envelope of (2).

In particular,

1. Equation (1) has at least* one solution that passes through $(x_o,y_o)$ if $(x_o,y_o)$ lies below $y = 3/4x^{4/3}$. Namely there is a member of (2) that does the job.

2. Equation (2) has at least two solutions that pass through $(x_o,y_o)$ if $(x_o,y_o)$ is on $y = 3/4x^{4/3}$. Namely $y = 3/4x^{4/3}$, itself, as well as a member of (2).

e. Here we present a general attack for solving the general Clairaut equation. The attack works for all such equations but our specific illustration is restricted to the equation in the present exercise. The key idea is to differentiate the equation with respect to x, and for the sake of notational convenience to let $u = dy/dx$. Thus, from

$$y = x\frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^4$$

*The fundamental theorem of the lecture requires that our equation has the form $dy/dx = f(x,y)$. Our present equation is of the 4th degree and such equations are difficult to write in this form (and even worse, if the degree exceeds 4, there is no guarantee that the equation can be written in the required form). This uniqueness is harder to come by, but this is discussed in the next part of this exercise.

2.1.9 continued

we obtain

$$y = xu - \frac{1}{4} u^4 \tag{8}$$

and differentiating (8) with respect to x yields:

$$\frac{dy}{dx} = \frac{d}{dx} (xu) + \frac{d}{dx}\left( - \frac{1}{4} u^4\right)$$

or

$$\frac{dy}{dx} = x \frac{du}{dx} + u - u^3 \frac{du}{dx} . \tag{9}$$

Then, since $u = dy/dx$, (9) becomes

$$u = x \frac{du}{dx} + u - u^3 \frac{du}{dx}$$

or

$$0 = (x - u^3) \frac{du}{dx}. \tag{10}$$

From (10) it follows that either

$$\frac{du}{dx} = 0 \tag{11}$$

or

$$x - u^3 = 0. \tag{12}$$

If (11) holds, $u = c$ and since $u = dy/dx$, we conclude that

$$\frac{dy}{dx} = c$$

yields a solution of (1); and a trivial check verifies that this is indeed correct.

If (12) holds, we have that $dy/dx = u = x^{1/3}$ and replacing $dy/dx$ by $x^{1/3}$ in (1) yields $y = x(x^{1/3}) - 1/4(x^{1/3})^4$ or $y = 3/4\ x^{4/3}$ which agrees with our earlier-found envelope of the family given

2.1.9 continued

by (2).

Since (11) and (12) are the only possible solutions of

$$y = x\,\frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^4$$

we see that every solution of (1) comes from (2) and (5). In other words;

1. No solution of (1) passes through $(x_0,y_0)$ if $(x_0,y_0)$ is above $y = 3/4x^{4/3}$.

2. There are exactly* two solutions of (1) which pass through $(x_0,y_0)$ if $(x_0,y_0)$ is on $y = 3/4x^{4/3}$.

3. There is one solution of (1) if $(x_0,y_0)$ is below $y = 3/4x^{4/3}$.

Since we are not always able to solve for dy/dx explicitly, the safest way to state the result is:

1. If $(x_0,y_0)$ is above $y = 3/4x^{4/3}$,equation (1) has no solutions which pass through $(x_0,y_0)$.

2. If $(x_0,y_0)$ is below $y = 3/4x^{4/3}$, every solution of (1) which passes through $(x_0,y_0)$ belongs to the 1-parameter family defined by (2).

3. If $(x_0,y_0)$ is on $y = 3/4x^{4/3}$ then there is in addition to any members of (2) which pass through $(x_0,y_0)$, the solution $y = 3/4x^{4/3}$.

---

*Here we are assuming that we have solved (1) for dy/dx and are restricting our answer to one such factor (just as we did in the previous exercise).

## Unit 2: Special Types of First Order Equations

### 2.2.1(L)

$$(2xy + x^3)dx + (x^2 + y^2 + 1)dy = 0 \qquad (1)$$

has the form $Mdx + Ndy = 0$ with

$$M = 2xy + x^3 \text{ and } N = x^2 + y^2 + 1. \qquad (2)$$

From (2) it follows that $M_y = 2x = N_x$ so that (1) is exact.

Hence, there exists $f(x,y)$ such that

$$df = (2xy + x^3)dx + (x^2 + y^2 + 1)dy \qquad (3)$$

and since

$$df = f_x dx + f_y dy \qquad (4)$$

we may equate (3) and (4) to conclude that

$$f_x = 2xy + x^3 \qquad (5)$$

and

$$f_y = x^2 + y^2 + 1. \qquad (6)$$

From (5) we see that

$$f(x,y) = x^2y + \frac{1}{4}x^4 + g(y), \qquad (7)$$

whence

$$f_y(x,y) = x^2 + g'(y). \qquad (8)$$

Equating $f_y$ in (6) to its value in (7) yields $x^2 + g'(y) = x^2 + y^2 + 1$ or $g'(y) = y^2 + 1$. Hence,

$$g(y) = \frac{1}{3}y^3 + y + c_1. \qquad (9)$$

2.2.1(L) continued

Replacing g(y) in (7) by its value in (9), we conclude that

$$f(x,y) = x^2y + \frac{1}{4}x^4 + \frac{1}{3}y^3 + y + c_1. \qquad (10)$$

Moreover, since (1) and (3) together imply that df = 0; or $f(x,y) = c_2$, we see from (10) that

$$x^2y + \frac{1}{4}x^4 + \frac{1}{3}y^3 + y + c_1 = c_2 \qquad (11)$$

is a solution of (1). Moreover, since both $c_1$ and $c_2$ are arbitrary constants we may "amalgamate" them and thus write (11) as

$$x^2y + \frac{1}{4}x^4 + \frac{1}{3}y^3 + y = c_3 \text{ (where } c_3 = c_2 - c_1\text{)}. \qquad (12)$$

To clear (12) of fractions, we may multiply both sides by twelve to obtain

$$12x^2y + 3x^4 + 4y^3 + 12y = c \text{ (where } c = 12c_3\text{)}. \qquad (13)$$

To find the member(s) of (13) which pass through the point $(x_0,y_0)$ we replace x by $x_0$ and y by $y_0$ in (13) to obtain

$$12x_0{}^2y_0 + 3x_0{}^4 + 4y_0{}^3 + 12y_0 = c \qquad (14)$$

whence we conclude that

$$12x^2y + 3x^4 + 4y^3 + 12y = 12x_0{}^3y_0 + 3x_0{}^4 + 4y_0{}^3 + 12y_0 \qquad (15)$$

is the only member of (13) which passes through $(x_0,y_0)$. Moreover, since $(x_0,y_0)$ could denote any point in the plane and since (15) is uniquely determined and well-defined for each choice of $x_0$ and $y_0$, we may conclude from (15) that there is one and only one member of (13) which passes through a given point $(x_0,y_0)$; and this member is defined by (15).

Up to this point, our discussion has been a review of exact differentials which we did earlier in our course. To tie in the concept of general solution as discussed in the previous

2.2.1(L) continued

section, we observe that (1) may be written equivalently as

$$\frac{dy}{dx} = -\frac{(2xy + x^3)}{x^2 + y^2 + 1} \qquad (16)$$

where the equivalence of (1) and (16) hinges on the fact that $x^2 + y^2 + 1$ is never 0*.

Applying the theorem of Lecture 7.010 to (16) with

$$f(x,y) = -\frac{(2xy + x^3)}{x^2 + y^2 + 1}$$

we see that since $f_y(x,y)$ exists and is continuous (since f is the quotient of two continuously differentiable functions whose denominator never vanishes) in the entire plane, there is one and only one solution of (16) [or, equivalently, of (1)] which passes through $(x_0,y_0)$.

Since (15) is such a solution, it is the only one and we therefore conclude that (13) is the general solution of (1).

As a final note, let us observe that in any region R where M/N and $\partial(-M/N)/\partial y$ exist and are continuous, the equation

$$\frac{dy}{dx} = -\frac{M}{N} \qquad (17)$$

has a general solution. However, (17) and the equation Mdx + Ndy = 0 are not equivalent if the region R includes points for which N(x,y) = 0. In such a case, once we solve (17) we must tackle separately those special cases for which N(x,y) = 0.

*In the more general case, when we re-write Mdx + Ndy = 0 in the form dy/dx = -M/N, we must be prepared to expect singular points wherever N(x,y) = 0.

2.2.2(L)

A very special case of a first order exact differential equation is the equation

$$f(x)dx + g(y)dy = 0 \tag{1}$$

in which the variables are already separated. Trivially (1) is exact since with $M = f(x)$ and $N = g(y)$ we have

$$M_y = \frac{\partial f(x)}{\partial y} = 0 = \frac{\partial g(y)}{\partial x} = N_x.$$

Indeed when (1) holds, a solution is $F(x) + G(y) = c$ where $F'(x) = f(x)$ and $G'(y) = g(y)$.

A more sophisticated version of (1) is seen in

$$(1 + y^2)dx + (1 + x^2)\ dy = 0. \tag{2}$$

Namely, as (2) now stands, it is not exact, since

$$\frac{\partial(1 + y^2)}{\partial y} = 2y$$

and

$$\frac{\partial(1 + x^2)}{\partial x} = 2x$$

so that $M_y \not\equiv N_x$.

In particular, then, the variables in (2) are not separated.

However, the variable in (2) are separable. That is, there is an equivalent way of writing (2) in which the variables are separated. Namely, we multiply both sides of (2) by

$$\frac{1}{(1 + x^2)(1 + y^2)}$$

to obtain

2.2.2(L) continued

$$\frac{dx}{1 + x^2} + \frac{dy}{1 + y^2} = 0. \qquad (3)*$$

Note:
In the language used in the text as well as in the lecture, the fact that multiplying both sides of (2) by $1/(1 + x^2)(1 + y^2)$ yielded an equivalent exact differential equation (namely (3)), means that $1/(1 + x^2)(1 + y^2)$ is called an integrating factor of (2).

We may now integrate (3) by sight to obtain

$$\text{arc tan } x + \text{arc tan } y = c_1. \qquad (4)$$

Note:
A knowledge of various trigonometric identities allows us to re-write (4) more "algebraically" if we so desire. Namely, letting tan u = x and tan v = y (i.e., u = arc tan x and v = arc tan y) we have:

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \qquad (5)$$

but from (4) $u + v = c_1$. Hence, (5) implies

$$\tan c_1 = \frac{x + y}{1 - xy}$$

or letting $\tan c_1 = c$, we obtain

$$x + y = c(1 - xy) \qquad (6)$$

which is equivalent to (4).

*In general, the process of rewriting $f(y)dx + g(x)dy = 0$ in the form $dx/g(x) + dy/f(y) = 0$ does not yield an identity since those points at which either $g(x) = 0$ or $f(y) = 0$ must be investigated separately. We have avoided this problem in our use of $1 + x^2$ and $1 + y^2$ since neither of these two expressions can ever be zero.

2.2.2(L) continued

The fact that (4) [or (6)] is the general solution of (2) follows from the fundamental theorem when we write (2) in the form

$$\frac{dy}{dx} = -\frac{(1 + y^2)}{(1 + x^2)}.$$

2.2.3(L)

If the first order equation can be written in the special form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right), \tag{1}$$

[i.e., dy/dx is a function of both x and y but the variables occur only in the form y/x; so that we may view f as a function of the single variable v where v = y/x]

it seems suggestive* to let v = y/x in (1). If we do this then y = xv, whereupon dy/dx = x dv/dx + v, and (1) becomes

$$x\frac{dv}{dx} + v = f(v). \tag{2}$$

Equation (2) happens to have the variables separable - a solution which we can already handle! In particular, from (2) we obtain

$$x\,dv + v\,dx = f(v)dx$$

or

$$x\,dv = [f(v) - v]dx. \tag{3}$$

Then, provided $x \neq 0$ and $f(v) - v \neq 0$, (3) is equivalent to

*Much of the technique for solving differential equations involves trying certain substitutions - some fairly obvious, others rather sophisticated - in the hope of replacing the given equation by an equivalent equation which happens to be easier for us to handle.

2.2.3(L) continued

$$\frac{dv}{f(v) - v} = \frac{dx}{x} \; * \qquad (4)$$

and in (4) the variables are separated.

We then integrate (4) and find v in terms of x. Once this is done we need only replace v by y/x to obtain the final result.

It should be noted that there is no need for memorizing (4) since in a problem of type (1) the substitution v = y/x (or y = xv) leads us to (4) directly.

Thus, in the present exercise, with

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \frac{y^2}{x^2} \qquad (5)$$

we let v = y/x. Hence , y =vx and therefore

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

so that (5) becomes

$$v + x\frac{dv}{dx} = 1 + v + v^2$$

or

$$x\frac{dv}{dx} = 1 + v^2.$$

Therefore,

$$\frac{dv}{1 + v^2} = \frac{dx}{x} \; (x \neq 0),$$

so that

$$\text{arc tan } v = \ln|x| + c \qquad (6)$$

*We must remember, however, that if x = 0 or f(v) - v = 0, (3) is still defined but (4) isn't. Hence, these special cases, as usual, must be treated separately.

2.2.3(L) continued

and since $v = y/x$, (6) may be written as

$$\arc\tan \frac{y}{x} = \ln|x| + c. \quad (7)$$

In particular, if we let $x = y = 1$ in (7) we obtain

$$\arc\tan 1 = \ln 1 + c$$

or

$$\frac{\pi}{4} = c.$$

Hence, with this value of c, (7) becomes

$$\arc\tan \frac{y}{x} = \ln|x| + \frac{\pi}{4} \quad (8)$$

and if we now take into account that R is the half-plane $x > 0$, (8) becomes

$$\arc\tan \frac{y}{x} = \ln x + \frac{\pi}{4}.$$

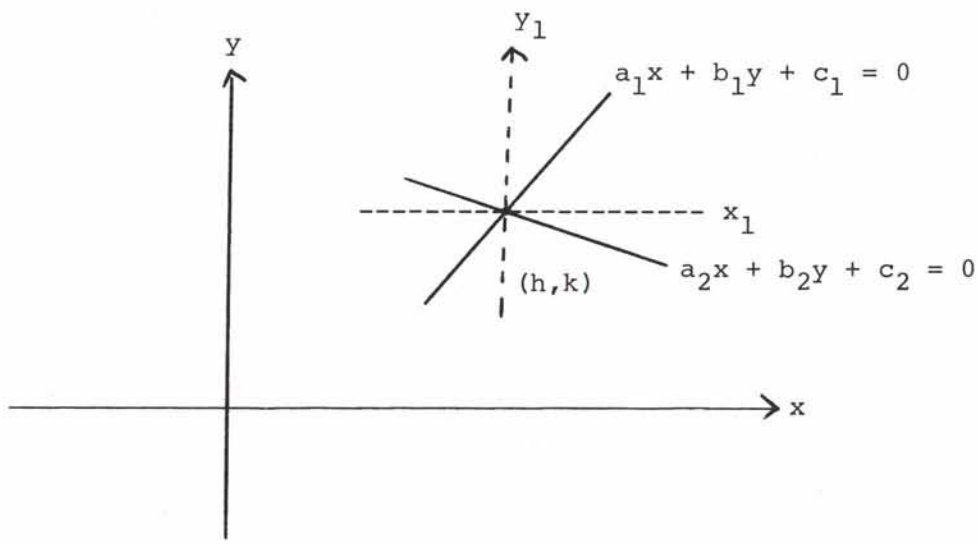
2.2.4 (optional)

a. Here we have an equation which almost has the form $dy/dx = f(y/x)$ but it is spoiled only by the fact that $c_1$ and $c_2$ do not have to be zero. The technique here is to try for a change of variables in the form

$$\left.\begin{array}{l} x = x_1 + h \\ y = y_1 + k \end{array}\right\} \quad (1)$$

Geometrically, this is equivalent to translating our coordinate system for (0,0) to (h,k). Pictorially,

2.2.4 continued




The method fails if $a_1b_2 - a_2b_1 = 0$ (i.e., if the lines are parallel) but in that case, letting $u = a_1x + b_1y$ works well for us (see part (c)).

At any rate, replacing x and y in

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \tag{2}$$

by then values in (1), we obtain

$$\frac{d(y_1 + k)}{d(x_1 + h)} = \frac{a_1(x_1 + h) + b_1(y_1 + k) + c_1}{a_2(x_1 + h) + b_2(y_1 + k) + c_2}$$

or

$$\frac{dy_1}{dx_1} = \frac{a_1x_1 + b_1y_1 + (a_1h + b_1k + c_1)}{a_2x_1 + b_2y_1 + (a_2h + b_2k + c_2)}. \tag{3}$$

Looking at (3) we see that if we choose h and k such that

$$\left.\begin{aligned} a_1h + b_1k + c_1 = 0 \\ a_2h + b_2k + c_2 = 0 \end{aligned}\right\} \tag{4}$$

then, equation (3) has the effect of eliminating $c_1$ and $c_2$ from equation (2).

2.2.4 continued

In other words, if $a_1b_2 - a_2b_1 \neq 0$, we may solve (4) uniquely for h and k to obtain

$$h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \tag{5}$$

$$k = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\ . \tag{6}$$

With h and k as given by (5) and (6), equation (3) becomes

$$\frac{dy_1}{dx_1} = \frac{a_1x_1 + b_1y_1}{a_2x_1 + b_2y_1}\ . \tag{7}$$

Dividing numerator and denominator of (7) by $x_1$, we obtain

$$\frac{dy_1}{dx_1} = \frac{a_1 + b_1\left(\frac{y_1}{x_1}\right)}{a_2 + b_2\left(\frac{y_1}{x_1}\right)} \quad (x_1 \neq 0)$$

$$= f\left(\frac{y_1}{x_1}\right). \tag{8}$$

Equation (8) is then solved by letting $v = y_1/x_1$; after which we replace $x_1$ and $y_1$ by $x - h$ and $y - k$ where h and k are as given in (5) and (6).

b. Given

$$\frac{dy}{dx} = \frac{10 - 2x + 2y}{3x - y - 9}\ , \tag{9}$$

we first observe the right side of (9) is continuously differentiable except when $3x - y - 9 = 0$ (or in more familiar form, $y = 3x - 9$). In other words (9) has no solution on the line $y = 3x - 9$ but in any region R,which excludes this line,it has a unique general solution.

In particular using the notation in (a) we have

2.2.4 continued

$a_1 = -2,\ b_1 = 2,\ c_1 = 10$
$a_2 = 3,\ b_2 = -1,\ c_2 = -9$

so that from (5) and (6),

$$h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

$$= \frac{2(-9) - (-1)(10)}{(-2)(-1) - 3(2)}$$

$$= 2 \qquad (5')$$

and

$$k = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

$$= \frac{3(10) - (-2)(-9)}{(-2)(-1) - 3(2)}$$

$$= -3. \qquad (6')$$

Thus, (7) becomes

$$\frac{dy_1}{dx_1} = \frac{-2x_1 + 2y_1}{3x_1 - y_1} = \frac{-2 + 2\left(\frac{y_1}{x_1}\right)}{3 - \left(\frac{y_1}{x_1}\right)} \qquad (10)$$

and letting $v = y_1/x_1$, or $y_1 = vx_1$, (10) becomes

$$v + x_1\frac{dv}{dx_1} = \frac{-2 + 2v}{3 - v}$$

or

$$x_1\frac{dv}{dx_1} = \frac{-2 + 2v}{3 - v} - v$$

$$= \frac{-2 + 2v - 3v + v^2}{3 - v}$$

$$= \frac{v^2 - v - 2}{3 - v} = \frac{(v - 2)(v + 1)}{3 - v}.$$

2.2.4 continued

Hence,

$$\frac{(3 - v)\,dv}{(x - 2)(v + 1)} = \frac{dx_1}{x_1}. \tag{11}$$

Applying partial fractions to 3 - v/(v - 2)(v + 1), we obtain

$$\frac{A}{v - 2} + \frac{B}{v + 1} = \frac{3 - v}{(v - 2)(v + 1)} \tag{12}$$

so that with $v \neq 2$

$$A + \frac{B(v - 2)}{v + 1} = \frac{3 - v}{v + 1}$$

and letting $v \to 2$, we obtain

$$A = \frac{1}{3}. \tag{13}$$

Similarly if $v \neq -1$, (12) may be written as

$$\left(\frac{v + 1}{v - 2}\right)A + B = \frac{3 - v}{v - 2}$$

and letting $v \to -1$,

$$B = \frac{4}{-3} \tag{14}$$

Combining (13) and (14) with (12) we have

$$\frac{1}{3}\left(\frac{1}{v - 2}\right) - \frac{4}{3(v + 1)} = \frac{3 - v}{(v - 2)(v + 1)},$$

so that (11) becomes

$$\frac{1}{3}\int\frac{dv}{v - 2} - \frac{4}{3}\int\frac{dv}{v + 1} = \int\frac{dx_1}{x_1} + c_1$$

or

$$\frac{1}{3}\ln|v - 2| - \frac{4}{3}\ln|v + 1| = \ln|x_1| + \ln c_1 \quad (c_1 > 0)$$

or

2.2.4 continued

$$\ln |v - 2|^{\frac{1}{3}} - \ln |v + 1|^{\frac{4}{3}} = \ln c_1 |x_1|$$

or

$$\ln \frac{|v - 2|^{\frac{1}{3}}}{|v + 1|^{\frac{4}{3}}} = \ln c_1 |x_1|$$

or

$$\frac{|v - 2|^{\frac{1}{3}}}{|v + 1|^{\frac{4}{3}}} = c_1 \ |x_1| \ , \ c_1 > 0$$

or

$$\frac{|v - 2|}{|v + 1|^4} = c_1^{\,3}|x_1|^3, \ c_1^{\,3} > 0$$

and if we remove the absolute value signs, we have

$$\frac{v - 2}{(v + 1)^4} = c\, x_1^{\,3}, \text{ where } c = \pm\, c_1^{\,3}. \qquad (15)$$

Remembering that $v = y_1/x_1$, (15) becomes

$$\frac{(\frac{y_1}{x_1} - 2)}{(\frac{y_1}{x_1} + 1)^4} = c\, x_1^{\,3}$$

or

$$\frac{[\frac{y_1 - 2x_1}{x_1}]}{\frac{(y_1 + x_1)^4}{x_1^{\,4}}} = c\, x_1^{\,3}$$

or

$$[\frac{y_1 - 2x_1}{x_1}][\frac{x_1^{\,4}}{(y_1 + x_1)^4}] = c\, x_1^{\,3}$$

or

2.2.4 continued

$$y_1 - 2x_1 = c(x_1 + y_1)^4, \; x_1 + y_1 \neq 0. \tag{16}$$

Finally recalling that $x = x_1 + h$ and $y = y_1 + k$ and using the fact that from (5') and (6'), $h = 2$ and $k = -3$, equation (16) becomes

$$[y + 3 - 2(x - 2)] = c[(x - 2) + (y + 3)]^4$$

or

$$\underline{y - 2x + 7 = c(x + y + 1)^4}$$

While the computation is itself somewhat involved, notice that our main message is that we are trying to show how we modify existing recipes to find new recipes for solving problems which are different from the usual ones. In other words, much of the idea behind solving differential equations hinges on the ability (be it logical or hit-and-miss) of reducing unsolved equations to equivalent, more familiar equations which we have already solved.

c. Recognizing that $2(2x + 3y) = 4x + 6y$, we let $u = 2x + 3y$. In this way

$$(2x + 3y + 4)dx - (4x + 6y + 1)dy = 0 \tag{17}$$

becomes

$$(u + 4)dx - (2u + 1)dy = 0 \tag{18}$$

and since $u = 2x + 3y$, $du = 2dx + 3dy$

or

$$dx = \frac{du - 3dy}{2}$$

so that (18) becomes

$$(u + 4)\frac{(du - 3dy)}{2} - (2u + 1)dy = 0$$

2.2.4 continued

or

$$(u + 4)(du - 3dy) - 2(2u + 1)dy = 0$$

or

$$udu + 4du - 3udy - 12dy - 4udy - 2dy = 0$$

or

$$(u + 4)du - (7u + 14)dy = 0$$

or

$$dy = \frac{1}{7}\left(\frac{u + 4}{u + 2}\right)dy \quad (u \neq -2)$$

or

$$dy = \frac{1}{7}\left(1 + \frac{2}{u + 2}\right)du.$$

Hence,

$$y = \frac{1}{7}(u + 2\ln|u + 2|) + c_1$$

and since $u = 2x + 3y$, we have

$$y = \frac{1}{7}(2x + 3y + 2\ln|2x + 3y + 2|) + c_1$$

or

$$7y = 2x + 3y + 2\ln|2x + 3y + 2| + 7c_1$$

or

$$4y = 2x + 2\ln|2x + 3y + 2| + 7c_1$$

or

$$\ln|2x + 3y + 2| = 2y - x + c \quad \left(c = -\frac{7}{2}c_1\right). \qquad (19)$$

2.2.4 continued

d. Here we simply wish to reinforce how particular solutions are obtained from the general solution. With $x = -2$ and $y = 1$, equation (19) becomes

$$\ln|-4 + 3 + 2| = 2 - (-2) + c$$

or

$$\ln 1 = c + 4.$$

Hence, $c = -4$ so that (19) becomes

$$\ln|2x + 3y + 2| = 2y - x - 4.$$

More generally, if we let $x = x_o$ and $y = y_o$ in (19) we obtain

$$\ln|2x_o + 3y_o + 2| = 2y_o - x_o + c$$

or

$$c = \ln|2x_o + 3y_o + 2| + x_o - 2y_o$$

which is well-defined except when $2x_o + 3y_o + 2 = 0$. This is the reason that $u + 2 = 0$ (i.e., $2x + 3y + 2 = 0$) is excluded. In summary, then, in any region R which does not contain a portion of the line $2x + 3y + 2 = 0$, (19) is the general solution of (17).

2.2.5(L)

The problem with finding integrating factors is that we must solve a partial differential equation rather than an ordinary differential equation. That is, if $Mdx + Ndy$ is not exact, the test for finding u such that $uMdx + uNdy$ is exact requires that

$$\frac{\partial(uM)}{\partial y} = \frac{\partial(uN)}{\partial x}. \qquad (1)$$

2.2.5(L) continued

Since, in general, u, M, and N are functions of x and y, equation (1) means that

$$uM_y + u_yM = uN_x + u_xN \ . \qquad (2)$$

a. In this part of the exercise we try to determine u knowing that $u = u(x)$; i.e. $\partial u/\partial y = 0$. With this knowledge, (2) becomes:

$$uM_y + 0 = uN_x + \frac{du}{dx}N^*$$

or

$$N\frac{du}{dx} = u(M_y - N_x)$$

or

$$\frac{du}{dx} = \frac{u(M_y - N_x)}{N}$$

or

$$\frac{du}{u} = [\frac{M_y - N_x}{N}]dx. \qquad (4)$$

Equation (4) is not even meaningful unless $M_y - N_x/N$ is a function of x alone. In this even, if we let

$$P(x) = \frac{M_y - N_x}{N} \ ,$$

we obtain from (4) that

$$\frac{du}{u} = P(x)dx$$

whence

$$\ln|u| = \int P(x)dx + c_1$$

*If $u = u(x)$, then $u_x = du/dx$.

2.2.5(L) continued

or

$$u = c\, e^{\int P(x)\,dx}. \tag{5}$$

Since all we require is a single integrating factor, we may let c = 1 in (5) to obtain

$$u = e^{\int P(x)\,dx}. \tag{6}$$

b. Given

$$(y - xe^x)dx - xdx = 0, \tag{7}$$

we let

$$\begin{cases} M = y - xe^x \\ N = -x. \end{cases}$$

Then

$$\begin{cases} M_y = 1 \\ N_x = -1, \end{cases}$$

so (7) is not exact.

However.

$$\frac{M_y - N_x}{N} = \frac{1 - (-1)}{-x} = -\frac{2}{x}$$

so that from (6),

$$u = e^{\int -\frac{2dx}{x}} = e^{-2|\ln|x} = e^{\ln|x|^{-2}} = \frac{1}{x^2}$$

is an integrating factor of (7).

Indeed, if we multiply both sides of (7) by $1/x^2$ we obtain

2.2.5(L) continued

$$\left(\frac{y}{x^2} - \frac{e^x}{x}\right)dx - \frac{1}{x}\,dy = 0 \qquad (8)$$

and

$$\frac{\partial\left(\frac{y}{x^2} - \frac{e^x}{x}\right)}{\partial y} = \frac{1}{x^2} = \frac{\partial\left(-\frac{1}{x}\right)}{\partial x}$$

so that (8) is exact.

In fact, we may write (8) as

$$\left(\frac{y\,dx}{x^2} - \frac{1}{x}\,dy\right) - \frac{e^x dx}{x} = 0$$

or

$$d(-yx^{-1}) = \frac{e^x dx}{x}$$

or

$$-\frac{y}{x} = \int \frac{e^x dx}{x} + c_1$$

or

$$y = -x \int \frac{e^x dx^*}{x} + c_x. \qquad (9)$$

*(9) is called the solution of (7) even if we do not evaluate $\int\frac{e^x dx}{x}$ more explicitly. The point is that (except at x= 0) $e^x/x$ is a continuous, hence integrable, function which defines a function f(x) implicitly by

$$f'(x) = \frac{e^x}{x}.$$

2.2.5(L) continued

c. A first order differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{10}$$

is said to be linear in y. (This will be explained in more detail later in the Block.)

We may rewrite (10) in the form

$$[P(x)y - Q(x)]\,dx + dy = 0. \tag{11}$$

Letting

$$\begin{cases} M = P(x)y - Q(x) \\ N = 1 \end{cases}$$

we see that

$$M_y = P(x)$$
$$N_x = 0$$

so that (11) is not exact unless $P(x) \equiv 0$ .

Yet,

$$\frac{M_y - N_x}{N} = \frac{P(x) - 0}{1} = P(x)$$

so that

$$e^{\int P(x)dx}$$

is an integrating factor for (11), hence also for (10).

In fact, if we multiply both sides of (10) by $e^{\int P(x)dx}$

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx} \tag{12}$$

or

$$\frac{d}{dx}[ye^{\int P(x)dx}] = Q(x)e^{\int P(x)dx} \tag{12'}$$

2.2.5(L) continued

so that

$$ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}\,dx + c. \qquad (13)$$

One should never memorize (13) [which could easily be a traumatic experience]. Rather, one should perform the various operations as they occur. For example:

d. With

$$\frac{dy}{dx} - \frac{y}{x} = x^5 \quad (x > 0) \qquad (14)$$

We have that $P(x) = -\frac{1}{x}$ whence $\int P(x)dx = -\ln|x| = \ln|x|^{-1}$ so that

$$e^{\int P(x)dx} = e^{\ln|x|^{-1}} = \frac{1}{|x|}$$

or since $x > 0$, $e^{\int P(x)dx} = \frac{1}{x}$.

Multiplying both sides of (14) by $\frac{1}{x}$ yields

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = x^4 \qquad (15)$$

or

$$\frac{d(yx^{-1})}{dx}{}^{*} = x^4 \qquad (16)$$

Hence,

$$yx^{-1} = \frac{1}{5}x^5 + c.$$

*As a quick way of getting from (15) to (16), compare the process of getting from (12) to (12'). That is, once we multiply both sides by integrating factor

$e^{\int P(x)dx}$,

the left side of the equations is precisely $\frac{d}{dx}[y\,e^{\int P(x)dx}]$.

2.2.5(L) continued

Therefore,

$$y = \frac{1}{5}x^6 + cx. \qquad (17)$$

Moreover, for each $(x_o,y_o)$ in the plane, (17) yields

$$y_o = \frac{1}{5}x_o^{\,6} + cx_o.$$

Hence,

$$c = \frac{y_o - \frac{1}{5}x_o^{\,6}}{x_o} \quad \text{(provided } x_o \neq 0\text{)}.$$

Thus, (14) has a solution of the form (17) through each point $(x_o,y_o)$ provided $x_o \neq 0$.

If we write (14) as

$$\frac{dy}{dx} = x^5 + \frac{y}{x}$$

and let $f(x,y) = x^5 + \frac{y}{x}$, we see that both $f_x(x,y)$ and $f(x,y)$ are continuously differentiable except when $x = 0$, so that (17) is the general solution of (14).

2.2.6(L)

a. Given

$$\frac{dy}{dx} + p(x)y = q(x).$$

Then

$$\frac{dy}{dx} = f(x,y)$$

where

$$f(x,y) = q(x) + p(x)y. \qquad (1)$$

2.2.6(L) continued

Thus, $f(x,y)$ is continuous provided $p(x)$ and $q(x)$ are continuous. Moreover, from (1) $f_y(x,y) = p(x)$, which is also continuous if $p(x)$ is continuous.

Hence,

$$\frac{dy}{dx} + p(x)y = q(x)$$

has a general solution in any region R in which $p(x)$ and $q(x)$ are continuous.

b. Given

$$y = f(x) + c\,g(x), \tag{2}$$

where f and g are differentiable, we see from (2) that

$$\frac{dy}{dx} = f'(x) + cg'(x)$$

so that

$$c = \frac{\frac{dy}{dx} - f'(x)}{g'(x)}, \text{ where } g'(x) \neq 0. \tag{3}$$

Putting (3) into (2) yields

$$y = f(x) + \left[\frac{\frac{dy}{dx} - f'(x)}{g'(x)}\right] g(x)$$

or

$$\frac{g'(x)y}{g(x)} = \frac{f(x)g'(x)}{g(x)} + \frac{dy}{dx} - f'(x),\ g(x) \neq 0$$

or

$$\frac{dy}{dx} - \frac{g'(x)}{g(x)}\,y = \frac{f(x)g'(x)}{g(x)} + f'(x). \tag{4}$$

2.2.6(L) continued

Letting $p(x) = -\frac{g'(x)}{g(x)}$ and $q(x) = \frac{f(x)g'(x)}{g(x)} + f'(x)$, we see that (4) takes the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

which is linear in y.

Note:
What (b) shows is the structure of the first order linear differential equation. In the previous exercise we essentially showed that every first order linear differential equation had its solution in the form $f(x) + cg(x)$.

In part (b) we have shown that the converse is also true. That is, if $y = f(x) + cg(x)$ is a 1-parameter family of curves. Then, if suitable restrictions are made on f and g, this 1-parameter family is the solution of a first order linear differential equation.

c. Notice that

$$\frac{dx}{dy} + \frac{x}{y} = y^6, \ (y > 0) \qquad (5)$$

is linear in x (rather than y).

We solve such an equation just as before, only now $e^{\int p(y)dy}$ is the integrating factor. In the present exercise, $p(y) = \frac{1}{y}$; so that $\int p(y)dy = \ln y$. Consequently, $e^{\int p(y)dy} = e^{\ln y} = y \ (y > 0)$.

Multiplying both sides of (5) by y we obtain

$$y\frac{dx}{dy} + x = y^7$$

or

$$\frac{d(yx)}{dy} = y^7$$

or

2.2.6(L) continued

$$yx = \frac{1}{8}y^8 + c.$$

Hence,

$$x = \frac{1}{8}y^7 + cy.$$

d. Given

$$\frac{dy}{dx} + \frac{y}{x} = x^3y^4 \qquad (6)$$

(which is not linear since the right side depends on y as well as on x), we divide both sides of (6) by $y^4$ to obtain

$$y^{-4}\frac{dy}{dx} + \frac{y^{-3}}{x} = x^3 \quad (y \neq 0). \qquad (7)$$

If we now let $u = y^{-3}$, then $\frac{du}{dx} = -3y^{-4}\frac{dy}{dx}$, or, $y^{-4}\frac{dy}{dx} = -\frac{1}{3}\frac{du}{dx}$. Hence (7) becomes

$$-\frac{1}{3}\frac{du}{dx} + \frac{u}{x} = x^3$$

or

$$\frac{du}{dx} - \frac{3}{x}u = -3x^3 \text{ *}$$

and (8) is linear in u. In fact, an integrating factor of (8) is

$$e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3} \quad (x > 0).$$

Multiplying both sides of (8) by $x^{-3}$, we obtain

$$x^{-3}\frac{du}{dx} - 3x^{-4}u = -3$$

*In our discussions on linear equations, it is crucial that we use the form $dy/dx + p(x)y = q(x)$. For example, given $r(x)\ dy/dx + p(x)y = q(x)$ we could write this as $dy/dx + p(x)/r(x)\ y = q(x)/r(x)$, but then we must worry about where $r(x) = 0$ since this case can lead to singular solutions.

2.2.6(L) continued

or

$$\frac{d(ux^{-3})}{dx} = -3.$$

Therefore, $ux^{-3} = -3x + c$, or

$$u = -3x^4 + cx^3. \tag{9}$$

Then, since $u = y^{-3}$, (9) becomes $y^{-3} = -3x^4 + cx^3$

or

$$y = \sqrt[3]{\frac{1}{-3x^4 + cx^3}}$$

<u>Note</u>:
The general form of the Bernoulli equation is

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \text{ where } n \neq 0 \text{ or } 1 \tag{10}$$

[i.e., when $n = 0$ or $n = 1$, the equation is linear).

The technique for solving (10) is to multiply both sides by $y^{-n}$ to obtain

$$y^{-n}\frac{dy}{dx} + p(x)y^{-n+1} = q(x). \tag{11}$$

We then let $u = y^{-n+1}$ in (11) [so that $\frac{du}{dx} = (-n+1)y^{-n}\frac{dy}{dx}$, or, $y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$ (and since $n \neq 1$, $1 - n \neq 0$)] to obtain

$$\frac{1}{1-n}\frac{du}{dx} + p(x)u = q(x)$$

or

$$\frac{du}{dx} + p(x)[1-n]u = (1-n)q(x) \tag{12}$$

which is linear in u. We solve (12) using $e^{\int(1-n)p(x)dx}$ as an integrating factor; after which we replace u by $y^{-n+1}$ to obtain the first answer.

2.2.6(L) continued

From a learning point of view, notice how a bit of clever maneuvering allows us to transform a Bernoulli equation into the more familiar linear equation.

2.2.7 (optional)

The main aim of this exercise is to indicate how we often (when we are lucky enough) may guess a proper substitution even if the equation doesn't have a familiar form.

In the present example, we have

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 4y = 4x - 1 \qquad (1)$$

which is a quadratic equation in $\frac{dy}{dx}$. In fact, equation (1) happens to be solvable for $\frac{dy}{dx}$ even more conveniently in this case. Namely, we may rewrite (1) as

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 1 = 4x - 4y$$

so that

$$\left(\frac{dy}{dx} - 1\right)^2 = 4(x - y)$$

or

$$\frac{dy}{dx} - 1 = \pm 2\sqrt{x - y} ,$$

and this leads to

$$\frac{dy}{dx} = 1 \pm 2\sqrt{x - y} . \qquad (2)$$

In order to keep things single-valued, we, as usual, elect to view (2) as the two separate equations

$$\frac{dy}{dx} = 1 + 2\sqrt{x - y} \qquad (3)$$

and

2.2.7 continued

$$\frac{dy}{dx} = 1 - 2\sqrt{x - y}\ . \qquad (4)$$

The major point of this exercise is simply that in either (3) or (4), the right side includes x and y only in the form x-y. Thus while (1), in either the form (3) or form (4), does not fall into any previously-studied category of first order equations, we might perhaps expect the substitution, u = x-y, to be helpful. Making this substitution, we see that $\frac{du}{dx} = 1 - \frac{dy}{dx}$ ; or $\frac{dy}{dx} = 1 - \frac{du}{dx}$ . With this in mind, equation (3) becomes

$$1 - \frac{du}{dx} = 1 + 2\sqrt{u}$$

or

$$\frac{du}{dx} = -2\sqrt{u}\ ; \qquad (3')$$

while (4) becomes

$$\frac{du}{dx} = 2\sqrt{u}\ . \qquad (4')$$

Thus, from either (3') or (4') we see that the substitution u = x - y has reduced (1) to two first order, first degree differential equations in which the variables are separable.

From the point of view of what we are trying to teach in this exercise (i.e., the "clever" use of change of variables), equations (3') and (4') are sufficient to make our point.

Nevertheless, to reinforce some of earlier ideas and practice various computations, let us carry this exercise further, if only as a review.

From (3') we obtain

$$\frac{du}{2\sqrt{u}} = -dx$$

so that

2.2.7 continued

$$\sqrt{u} = -x + c \,. \tag{5}$$

Recalling that $u = x - y$, equation (5) becomes

$$\sqrt{x - y} = -x + c \tag{6}$$

so that (6) is a 1-parameter family of solutions of (3).

Notice from (6) that c is not real unless $x - y \geq 0$. That is, no member of (3) rises above the line $y = x$.

This is in accord with the fundamental theorem of Lecture 7.010 since with $f(x,y) = 1 + 2\sqrt{x - y}$, we see from (3) that $x - y \geq 0$ for $f(x,y)$ to exist. Moreover, since $f_y(x,y) = -1/\sqrt{x - y}$ we see that $x - y$ may lead to a (possibly) singular solution of (3). Indeed, with $y = x$ equation (3) is satisfied and $y = x$ does not belong to the 1-parameter family defined by (6).

In summary:

1. If $(x_o,y_o)$ is above the line $y = x$, equation (3) [or for that matter, also (4)] has no solution since then $\frac{dy}{dx}$ is non-real.

2. If $(x_o,y_o)$ is below the line $y = x$ [i.e., $x_o - y_o > 0$], then by the fundamental theorem, one and only one curve satisfied (3) and passes through $(x_o,y_o)$. This curve is a member of (6) and corresponds to the value of c when $x = x_o$ and $y = y_o$. That is, with $x = x_o$ and $y = y_o$, equation (6) yields

$$\sqrt{x_o - y_o} = -x_o + c$$

or

$$c = x_o + \sqrt{x_o - y_o} \,. \tag{7}$$

Replacing c in (6) by its value in (7) we conclude that if $y_o > x_o$, then

$$\sqrt{x - y} = -x + x_o + \sqrt{x_o - y_o} \tag{8}$$

2.2.7 continued

is the only curve which passes through $(x_0,y_0)$ and satisfies (3).

3. If $x_0 = y_0$, then (7) indicates that $c = x_0$. Hence,

$$\sqrt{x - y} = -x + x_0$$

passes through $(x_0,y_0)$ and satisfies (3); but in addition so also $y = x$. In fact, $y = x$ is the envelope of (6) as we shall show shortly.

Our results concerning the solutions of (3) may be easier to grasp in terms of a graph.

To begin with, (6) might be easier to recognize without the square root. Thus, let us square both sides of (6) to obtain

$$x - y = (-x + c)^2 = x^2 - 2cx + c^2$$

or

$$y = -x^2 + (1 + 2c)x - c^2. \tag{9}$$

We must be careful to note that (9) includes more than (6). Namely, if $a = b$ and we square both sides to obtain $a^2 = b^2$, notice that the latter equality includes $a = -b$ as well.

In other words, (9) contains the result of squaring both sides of $\sqrt{x - y} = -(-x + c)$ or $\sqrt{x - y} = x - c$, as well. To keep track of what's happening, notice from (3) that since $\sqrt{x - y} \geq 0$ , $\frac{dy}{dx}$ must be at least 1.

With this in mind, we may now graph a typical member of (6). Namely, each member of (9) is a parabola. In particular, we obtain from (9) that

$$y' = -2x + (1 + 2c) \tag{10}$$

and

2.2.7 continued

$$y'' = -2. \tag{11}$$

From (11) we see that our parabola always "spills water". From (10) we see that $y' \geq 1 \leftrightarrow$

$$-2x + (1 + 2c) \geq 1 \leftrightarrow$$
$$-2x \geq 2c \leftrightarrow$$
$$x \leq c. \tag{12}$$

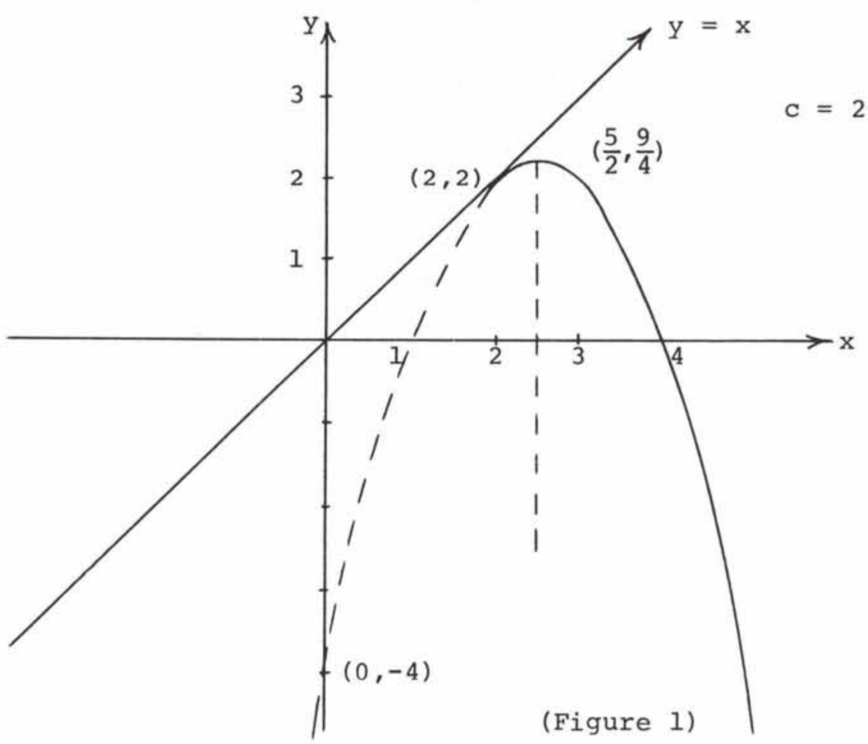
Recall that each member of (6) is the corresponding member of (9) limited to $\frac{dy}{dx} \geq 1$. In other words, from (12) we see that for a given c, only that portion of (9) for which $x \leq c$ belongs to (6).

In any event, continuing to plot the member of (9) first we have that $y' = 0 \leftrightarrow x = \frac{1 + 2c}{2}$ ; and when $x = 1 + 2c$, (9) yields

$$y = -\frac{(1 + 2c)^2}{4} + (1 + 2c)\frac{(1 + 2c)}{2} - c^2 = \frac{(1 + 2c)^2}{4} - c^2$$

$$= \frac{1 + 4c}{4} .$$

In any event,

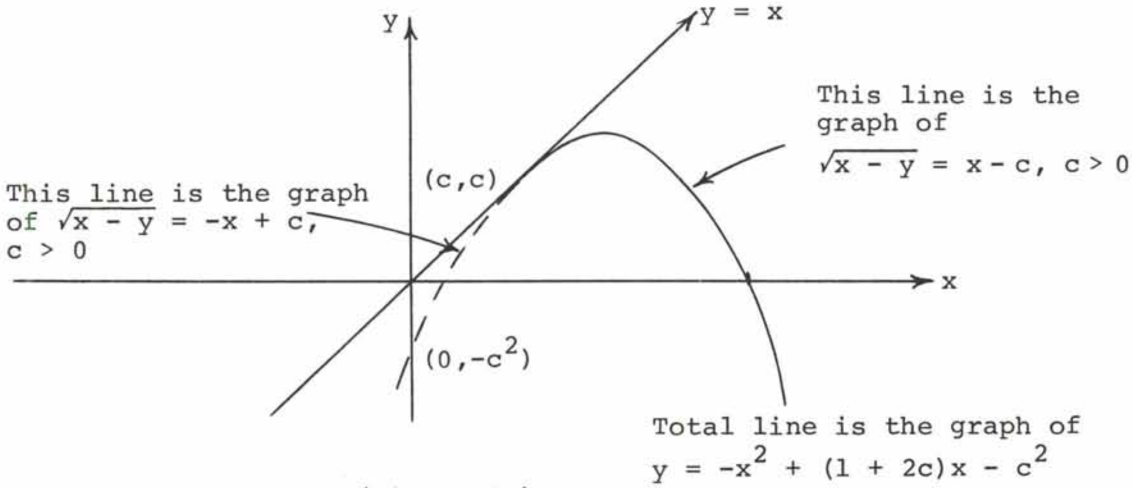



(Figure 1)

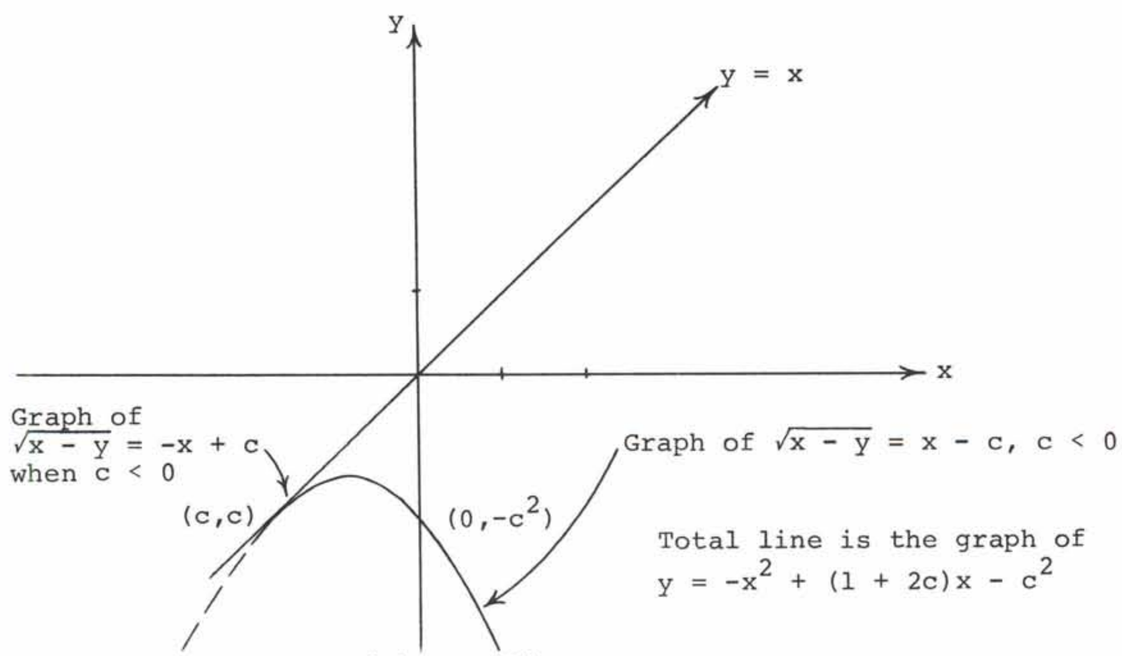
2.2.7 continued

$y = -x^2 + (1 + 2c)x - c^2$ for $c = 2$. Only the hatched portion makes up the graph of (6); i.e., $\sqrt{x - y} = -x + 2$.

Using Figure 1 as a hint, coupled with the fact that $y = x$ is a solution of (3), let us replace y by x in (9) [on in(6)] to obtain that $y = x$ intersects $\sqrt{x - y} = -x + c$ only at $(c,c)$. Moreover, when $x = c$ we see from (10) that $y' = 1$ so that $y = x$ is tangent to $\sqrt{x - y} = -x + c$ at $(c,c)$. We also see that the high point of $y = -x^2 + (1 + 2c)x - c^2$ occurs at $\left(\frac{1 + 2c}{2}, \frac{1 + 4c}{4}\right) = \left(c + \frac{1}{2}, c + \frac{1}{4}\right)$. Thus:




(Figure 2a)




(Figure 2b)

2.2.7 continued

Figure 2 shows that each point of $y = x$ belongs to a member of (6); hence, $y = x$ is a solution of (3).

A similar analysis holds for (4). In fact for (4), since $\sqrt{x - y} \geq 0$, $dy/dx \leq 1$. More specifically, solving (4') yields

$$\sqrt{u} = x + c_1$$

so that

$$\sqrt{x - y} = x + c_1. \qquad (13)$$

Squaring both sides yields

$$x - y = x^2 + 2xc_1 + c_1^2$$

or

$$y = -x^2 + (1 - 2c_1)x - c_1^2. \qquad (14)$$

Notice that (14) is the same as (9) with $c = -c_1$.

In other words, the dotted portions of the curves in Figure 2 correspond to the graphs of (18) while the solid portions correspond to the graphs of (6).

As a final concrete example, let us find all solutions of (1) which pass through (1,0).

Letting $x = 1$ and $y = 0$ in (6) yields

$$\sqrt{1 - 0} = -1 + c$$

or

$$1 = -1 + c.$$

Hence $c = 2$.

2.2.7 continued

Therefore,

$$\sqrt{x - y} = -x + 2 \qquad (15)$$

is the only member of (6) which satisfies (3). Hence, equation (15) is the only solution of (3) which passes through (1,0). In terms of (a), (15) is the appropriate portion of the parabola $y = -x^2 + 5x - 4$.

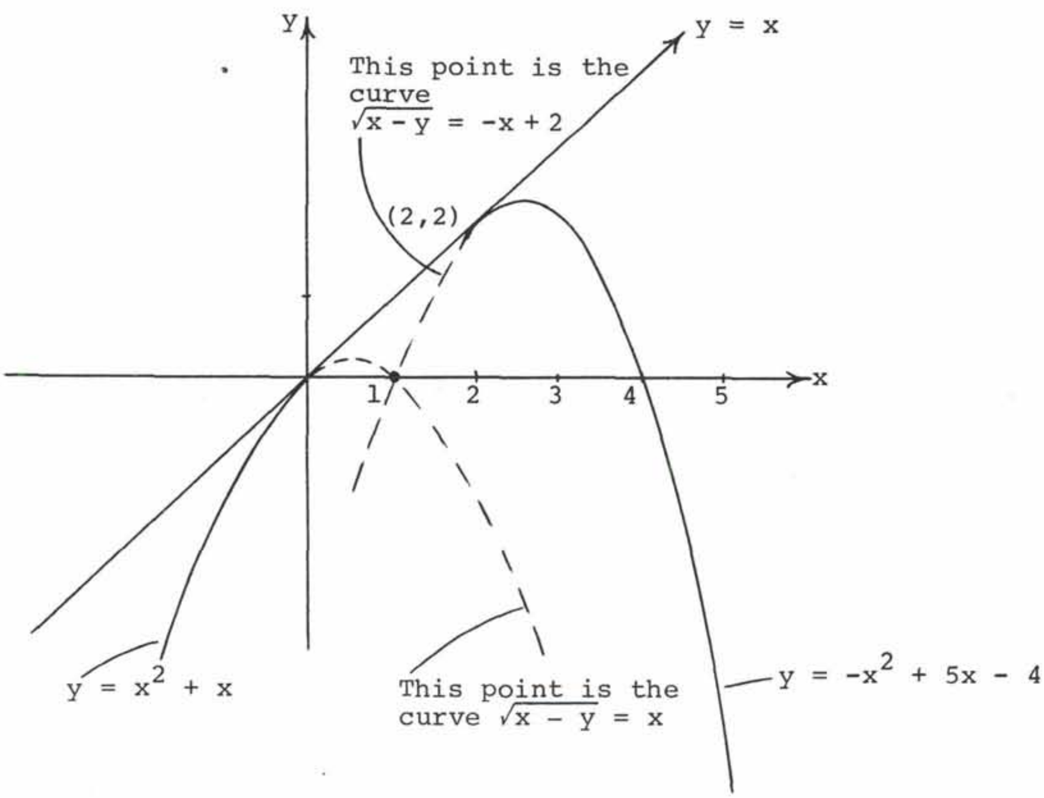
Similarly letting $x = 1$ and $y = 0$ in (13) yields

$$\sqrt{1 - 0} = 1 + c_1$$

so that $c_1 = 0$.

Therefore, $\sqrt{x - y} = x$ is the only member of (13) that satisfies (4); hence, the only solution of (4) which passes through (1,0). Notice from (14) that $c_1 = 0$ yields $y = -x^2 + x$.

Graphing $y = -x^2 + x$ and $y = -x^2 + 5x - 4$ in a single diagram, we have:



2.2.7 continued

Thus, while two parabolas are solutions of (1) which pass through (1,0), at (1,0) only one of the solutions satisfies (3) while the other satisfies (4).

2.2.8(L)

The main aim of this exercise is to show how certain second order differential equations may be reduced to two successive first order equations. The general second order equation is a relationship between x, y, y', and y"; and the technique we have in mind is one that works when either x or y (and it works even better if both x and y are missing) is missing from the equation.

In either case, if we let $p = y'$, then $p'$ ( $= \frac{dp}{dx}$ ) $= y''$. Hence, if y is missing, this substitution gives us a relationship between x, p, and $\frac{dp}{dx}$, which is a first order equation involving p as the dependent variable and x as the dependent variable. We then solve this equation to find p in terms of x; and once this is done we replace p by y' and solve the resulting first order equation for y in terms of x.

If x is missing, we must be a bit tricky, since then the resulting equation involves y, p, and $\frac{dp}{dx}$, where both p and y are functions of x. To resolve this dilemma of too many variables, we invoke the chain rule to rewrite $\frac{dp}{dx}$ as $(\frac{dp}{dy})(\frac{dy}{dx})$; and observing that $p = \frac{dy}{dx}$, we see that we may write $\frac{dp}{dx}$ as $p\frac{dp}{dy}$*, and the resulting equation now involves only y, p, and $\frac{dp}{dy}$. This is the technique that we shall examine in this exercise.

*For those who remember our elementary physics course, recall that in the usual kinematics problems, we either wrote the acceleration as dv/dt or as v dv/dx [i.e., dv/dt = dv/dx dx/dt] depending on whether the acceleration was given in terms of x or in terms of t. The method being described in this exercise is simply a generalization of this technique.

2.2.8(L) continued

a. We are given that

$$y''e^{y'} = e^x. \quad (1)$$

Letting $p = y'(= dy/dx)$, we have that $y'' = p'$ so that (1) becomes

$$p'e^p = e^x$$

or

$$\frac{dp}{dx} e^p = e^x. \quad (2)$$

Separating variables in (2) yields $e^p dp = e^x dx$ so that

$$e^p = e^x + c \quad (3)$$

or

$$e^{\frac{dy}{dx}} = e^x + c. \quad (3')$$

Thus,

$$\frac{dy}{dx} = \ln(e^x + c). \quad (4)$$

Integrating (4) yields y as the desired function of x*.

Since we do not want to be bogged down at this stage of the game with techniques of integration, we use the fact that $dy/dx = 1$ when $x = 1$. From (3'), this implies $e^1 = e^1 + c$ so that $c = 0$ and with $c = 0$, equation (4) is simplified to

$$\frac{dy}{dx} = x, \quad (5)$$

*When the integration is performed we obtain a second arbitrary constant. Thus, our final solution has the form $y = f(x,c,c_1)$, and this shouldn't be too surprising since we are solving a 2 nd order equation (by integrating twice).

2.2.8(L) continued

from which

$$y = \frac{1}{2}x^2 + c_1 \qquad (6)$$

and since $(1, \frac{3}{2})$ is to be on the curve, we see from (6) that

$$\frac{3}{2} = \frac{1}{2} + c_1,$$

so that $c_1 = 1$; and (6) becomes

$$y = \frac{1}{2}x^2 + 1. \qquad (7)$$

Note #1

All we know about (7) is that it's a curve which passes through (1, 3/2) with slope 1, and satisfies equation (1). Since (1) is a second order equation, we do not know anything about the general solution of (1). However, we are sure that (7) is one such curve and that's all that's asked for in this problem.

Note #2

Equation (1) is a special case of the more general equation

$$y''f(y') = f(x) \qquad (8)$$

which becomes

$$\frac{dp}{dx}f(p) = f(x) \qquad (9)$$

under the substitution $p = dy/dx$.

While we may always separate variables in (9) to obtain $f(p)dp = f(x)dx$, this equation may be difficult to solve, depending on the choice of f, Symbolically, the solution always has the form

$$F(p) = F(x) + c, \text{ where } F' = f \qquad (10)$$

and knowing that there is a value $x_o$ for x at which $x = p$;

2.2.8(L) continued

i.e., $F(x_o) = F(p_o)$, we see from (10) that $F(p_o) = F(x_o) + c$ so that $c = 0$. Hence (10) becomes

$$F(p) = F(x). \tag{11}$$

If F happens to be 1-1 as in this case where $F(x) = e^x$, we deduce from (11) that $p = x$ or $dy/dx = x$. Even if F is not 1-1, one solution of (11) is $p = x$. In other words, $y = 1/2x^2 + c$ is always a one family of curves which satisfies $y''f(y') = f(x)$ and passes through at least one point where the slope equals the x-coordinate.

b. Given

$$yy'' = (y')^2 \tag{12}$$

we let $p = y'$ whereupon (12) becomes

$$y \frac{dp}{dx} = p^2. \tag{13}$$

If we write $dp/dx$ as $dp/dy\ dy/dx = dp/dy\ p$, (13) becomes

$$yp \frac{dp}{dy} = p^2. \tag{14}$$

If we assume that $p \neq 0$*, (14) becomes

$$y \frac{dp}{dy} = p,$$

and separation of variables then yields

$$\frac{dp}{p} = \frac{dy}{y}$$

or

$$p = c_1 y. \tag{15}$$

*If $p = 0$, then $y$ = constant and this clearly satisfies (12).

2.2.8(L) continued

Therefore, $dy/dx = c_1y$ or $dy/y = c_1dx$.

Hence,

$$\ln |y| = c_1x + c_2$$

or

$$y = c_3e^{c_1x}. \quad (16)$$

As a check that (16) is a 2-parameter (i.e., $c_1$ and $c_3$ are both arbitrary constants) family of solutions of (12), we have

$$\begin{aligned} y &= c_3e^{c_1x} \\ y' &= c_1c_3e^{c_1x} \\ y'' &= c_1^2c_3e^{c_1x}. \end{aligned} \quad (17)$$

Hence,

$$\begin{aligned} yy'' &= (c_3e^{c_1x})(c_1^2c_3e^{c_1x}) \\ &= c_1^2c_3^2e^{2c_1x} \\ &= [c_1c_3e^{c_1x}]^2 \\ &= (y')^2. \end{aligned}$$

Finally, to find the member of (16) which passes through (0,2) with slope 4, we have from (16) that $2 = c_3e^0$ or

$$c_3 = 2; \quad (18)$$

while from (17), $4 = c_1c_3e^0$.

2.2.8(L) continued

Hence

$c_1c_3 = 4,$

and since $c_3 = 2,$

$$c_1 = 2. \qquad (19)$$

Using the values of $c_1$ and $c_3$ given by (18) and (19) in (16), we obtain

$y = 2e^{2x}.$

Unit 3: (optional) Some Geometric Applications of First Order Equations

2.3.1(L)

Up to now, we have been starting with a differential equation and then finding its solution. The point is that in many cases, part of the problem is to develop the appropriate differential equation. That is, in real-life situations we are often given a certain situation which, hopefully, can be translated into a differential equation; after which, we hope that the equation is solvable.

The aim of this exercise is to give a brief glimpse into this aspect of differential equations.

We are told to find the family of orthogonal trajectories to the 1-parameter family of curves

$$y = x + ce^{-x}. \tag{1}$$

That is, we want that family of curves (if it exists) with the property that every intersection between a member of this family and a member of (1) is at right angles. Since two curves meet at right angles if and only if their slopes at the point are negative reciprocals of one another (except in the case where one of the lines is vertical since in that case we do not refer to the slope of the line, except to say that the soope is infinite), the procedure is to find dy/dx from (1) in a form that is free of the parameter c. Once this is done we take the negative reciprocal of dy/dx and use this as the slope of a member of the desired family. This procedure leads us to the differential equation which must be solved.

It is only after we have the equation that the discussion of this Unit becomes important. In still other words, as far back as part 1 of this course we knew how to write the differential equation for the family of orthogonal trajectories for a given 1-parameter family of curves. If, however, the equation was anything different from variables separable, we did not have the "technological know-how" to solve the equation. All that is

2.3.1(L) continued

happened with our discussion of this unit is that we are now able to solve a wider range of first order differential equations than we could have solved before.

a. Returning to (1) we have

$$c = e^x(y - x), \tag{2}$$

and differentiating (2) implicitly with respect to x yields

$$0 = e^x(y - x) + e^x\left(\frac{dy}{dx} - 1\right)$$

or since $e^x \neq 0$,

$$0 = (y - x) + \left(\frac{dy}{dx} - 1\right).$$

That is,

$$\frac{dy}{dx} = 1 + x - y \tag{4}$$

is the differential equation satisfied by the 1-parameter family (1).

Since the negative reciprocal of $1 + x - y$ is $\frac{-1}{1 + x - y}$, we see from (4) that the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = \frac{-1}{1 + x - y}. \tag{5}$$

From the fundamental theorem of Lecture 2.010, we see from (5) that there is one and only one solution of (5) that passes through an arbitrary point $(x_0,y_0)$ unless possibly when $(x_0,y_0)$ is on $1 + x - y = 0$ (that is we must be on our guard when we deal with the line $y = x + 1$, but we shall say more about this later).

Assuming then for the moment that the region R on which (5) is defined excludes any point of the form $(x_0, x_0 + 1)$ we know

2.3.1(L) continued

that (5) possesses a unique general solution and we must now try to find this solution.

Our point now is that in previous exercises we would have begun with equation (5) given. In this exercise we first had to derive equation (5). The so-called "cook-book technique" begins once (5) is derived in the sense that we try to match (5) with a specific type of equation for which we already have a "recipe".

One technique for solving (5) is to write it in the form $Mdx + Ndy = 0$ and see if it's exact*. Equation (5) then becomes

$$dx + (1 + x - y)\,dy = 0 \qquad (6)$$

in which case $M = 1$ and $N = 1 + x + y$. Therefore $M_y = 0$ and $N_x = 1$ so that (6) is not exact. Hopefully, however, we notice that

$$\frac{N_x - M_y}{M} = 1$$

so that $e^{\int 1dy} = e^y$ is an integrating factor of (6)**. In fact, if we multiply both sides of (6) by $e^y$, we obtain $e^y dx + e^y(1 + x - y)\,dy = 0$ or $e^y dx + e^y dy + xe^y dy - ye^y dy = 0$ or $(e^y dx + xe^y dy) + e^y dy - ye^y dy = 0$*** or $d(xe^y) + d(e^y) + d(e^y - ye^y) = 0$. Hence, $d(xe^y + e^y + [e^y - ye^y]) = 0$, or $xe^y + e^y + e^y - ye^y = c$, or

$$xe^y + (2 - y)\,e^y = c \qquad (7)$$

is the desired solution of (5).

[Aside:
There is no priority on the way of solving an equation. For example, one might have elected to write (5) as

*Unless another technique suggests itself, this approach is always a good idea, for if the equation is exact, we're home free. If it isn't, we might be able to "predict" an integrating factor. Even if this fails the time for the trial is not very great so we have not lost much as we now set out trying to find a different technique.

**Recall the result of Exercise 2.2.5.

***When in doubt, the longer technique of equating $f_x$ with M and $f_y$ with N may always be used to obtain the same result.

2.3.1(L) continued

$$\frac{dx}{dy} = -(1 + x - y)$$

or

$$\frac{dx}{dy} + x = y - 1$$

which is linear in x and which has $e^y$ as an integrating factor. Thus,

$$e^y \frac{dx}{dy} + xe^y = e^y(y - 1)$$

or

$$\frac{d(xe^y)}{dy} = ye^y - e^y$$

or

$$xe^y = ye^y - 2e^y + c$$

which agrees with (7).]

Equation (7) can, if we wish to, be written in the more explicit form

$$x = (y - 2) + ce^{-y}. \qquad (8)$$

b. If $y = x + ce^{-x}$ passes through (0,4) we have $4 = 0 + ce^0$ so that c = 4, and our member of (1) is

$$y = x + 4e^{-x}. \qquad (9)$$

Similarly, if (0,4) satisfies (8) we have $0 = (4 - 2) + ce^{-4}$ or $-2e^4 = c$. Hence, $x = (y - 2) - 2e^4e^{-y}$, or

$$x = y - 2 - 2e^{4 - y} \qquad (10)$$

is the member of (8) which passes through (0,4). As a check, we see from (9) that

2.3.1(L) continued

$$\left.\frac{dy}{dx}\right|_{(0,4)} = 1 - 4e^{-x}\Big|_{(0,4)} = 1 - 4e^{0} = -3 \tag{11}$$

and from (10) that

$$\left.\frac{dx}{dy}\right|_{(0,4)} = 1 + 2e^{4-y}\Big|_{(0,4)} = 1 + 2 = 3$$

so that

$$\left.\frac{dy}{dx}\right|_{(0,4)} = \frac{1}{3}\,. \tag{12}$$

A comparison of (11) and (12) shows that (9) and (10) intersect at right angles at (0,4).

c. We proceed as in (6), letting $x = 0$ and $y = 1$ in (1) to obtain $1 = 0 + ce^{0}$ or $c = 1$. Hence,

$$y = x + e^{-x} \tag{13}$$

is the member of (1) that passes through (0,1).

Similarly, if we let $x = 0$ and $y = 1$ in (8), we obtain $0 = (1 - 2) + ce^{-1}$ so that $c = e$, whereupon

$$x = (y - 2) + e^{1-y} \tag{14}$$

is the member of (8) which passes through (0,1).

Again, as a check, we see from (13) that

$$\left.\frac{dy}{dx}\right|_{(0,1)} = 1 - e^{-x}\Big|_{(0,1)} = 1 - 1 = 0\,, \tag{15}$$

while from (14) we have

$$\left.\frac{dx}{dy}\right|_{(0,1)} = 1 - e^{1-y}\Big|_{(0,1)} = 1 - e^{0} = 1 - 1 = 0$$

or

2.3.1(L) continued

$$\left.\frac{dy}{dx}\right|_{(0,1)} = \text{" } \infty \text{ "}. \tag{16}$$

Thus, (13) and (14) intersect at right angles at (0,1).

The interesting fact is that (0,1) belongs to the line $y = x + 1$, and we have seen that under this condition the solution to (5) need not be unique. For example, let us notice that (15) tells us that the line $x = 1$ also meets $y = x + e^{-x}$ at right angles. More generally, $y = x + ce^{-x}$. Therefore,

$$y' = 1 - ce^{-x} \tag{17}$$

and

$$y'' = ce^{-x}. \tag{18}$$

Since $c > 0$, (18) tells us that $y = x + ce^{-x}$ always "holds water". (17) tells us that $dy/dx = 0 \leftrightarrow 1 - ce^{-x} = 0 \leftrightarrow e^{-x} = 1/c \leftrightarrow e^{x} = c \leftrightarrow x = \ln c$*.

For $x = \ln c$ we find that

$$\begin{aligned} y &= \ln c + ce^{-\ln c} \\ &= \ln c + ce^{\ln \frac{1}{c}} \\ &= \ln c + c(\tfrac{1}{c}), \ (c > 0) \\ &= \ln c + 1 \\ &= x + 1. \end{aligned}$$

In other words, for $c > 0$, $y = x + ce^{-x}$ has a minimum value (hence, a horizontal tangent) at ($\ln c$, $\ln c + 1$) which is on the ln c $y = x + 1$.

*If $c \leq 0$, ln c is non-real. Hence dy/dx would never by zero, but this is discussed in more detail in the optimal remarks at the end of this lecture.

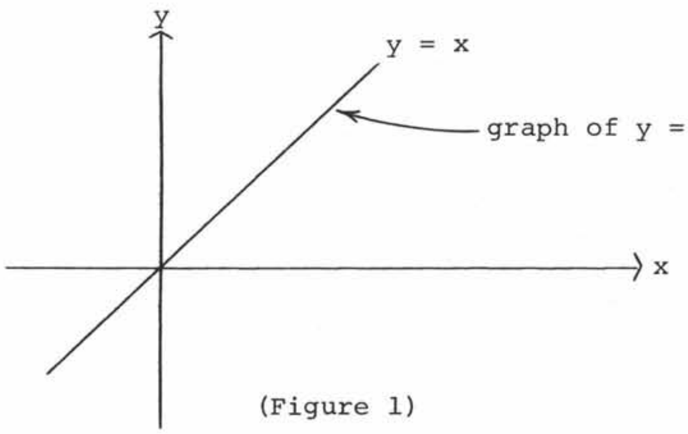
2.3.1(L) continued

Some Geometric Detail

The Graph of $y = x + ce^{-x}$

Case 1: $c = 0$

We then have the line $y = x$.




(Figure 1)

Case 2: $c > 0$

Then $y = x + ce^{-x} \rightarrow$

$\frac{dy}{dx} = 1 - ce^{-x} \rightarrow$

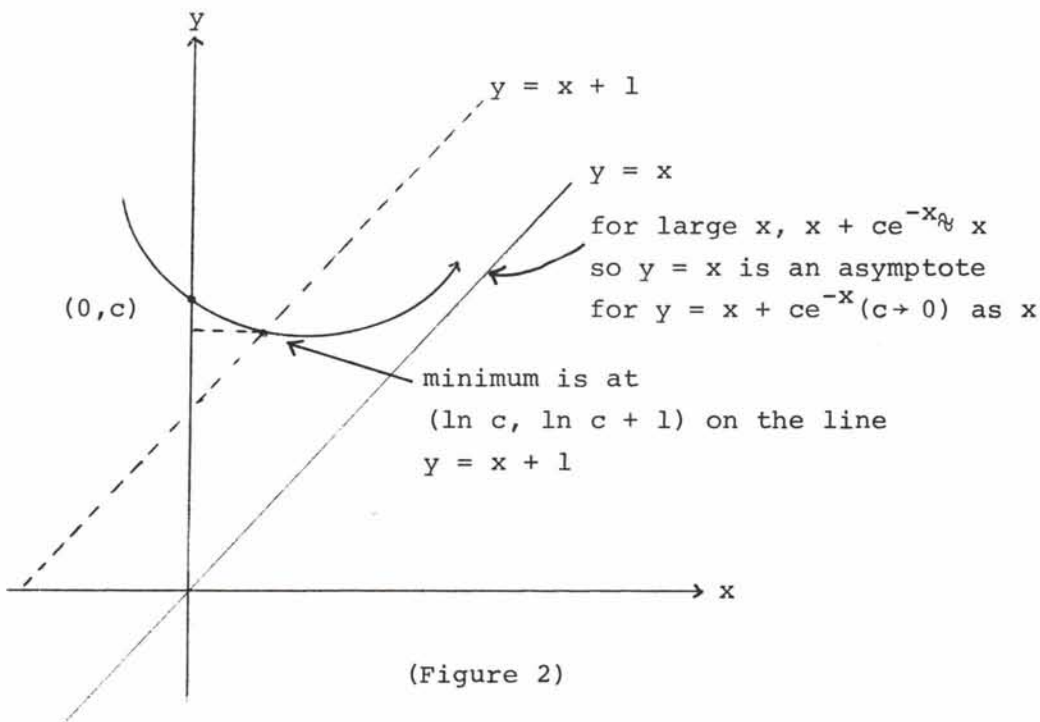
$\frac{d^2y}{dx^2} = ce^{-x}$ (> 0 for all x since $c > 0$ and $e^{-x} > 0$).

Thus, the curve always holds water, has its y-intercept at $y = 0 + ce^{-0} = c$, i.e., at $(0,c)$; and its minimum occurs when $1 - ce^{-x} = 0$ or $e^{-x} = \frac{1}{c}$ or $e^x = c$ or $x = \ln c$*. With $x = \ln c$, $y = x + ce^{-x}$ implies $y = \ln c + ce^{-\ln c}$ or $y = \ln c + ce^{\ln 1/c} = \ln c + c(1/c) = \ln c + 1$. Hence, the minimum of $y = x + ce^{-x}$ $(c > 0)$ occurs at the point $(\ln c, \ln c + 1)$ which is on the line $y = x + 1$, the special case we mentioned earlier in the problem.

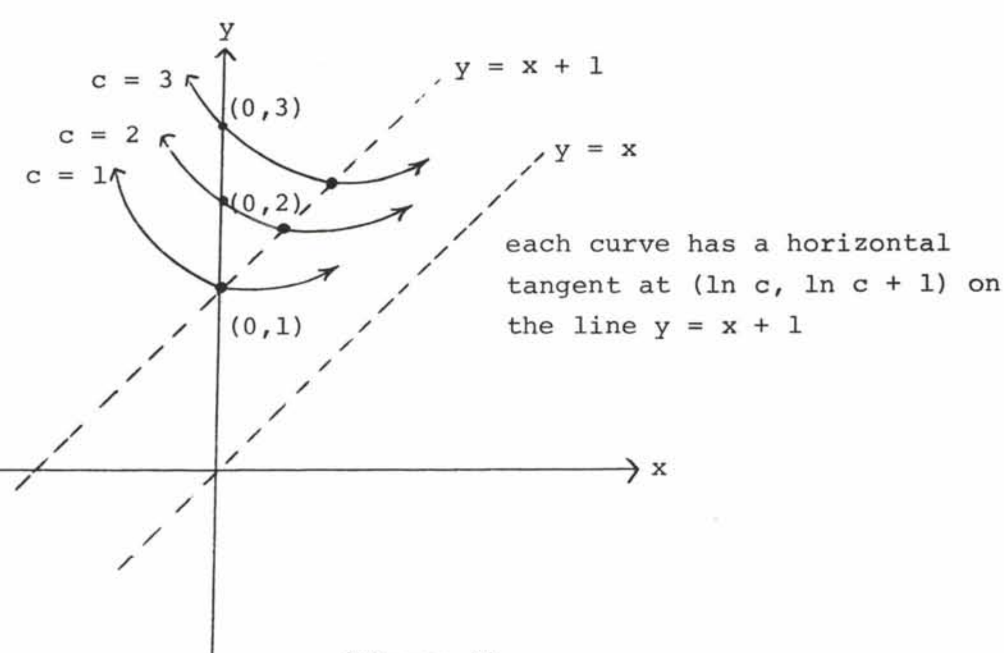
*Notice $\ln c$ is non-real if $c \leq 0$ so it is crucial that Case 2 requires that $c > 0$.

2.3.1(L) continued

Graphically, Case 2 is illustrated by




(Figure 2)




(Figure 3)

2.3.1(L) continued

Figure 2 is reproduced in Figure 3 for the special cases c = 1, 2 and 3.

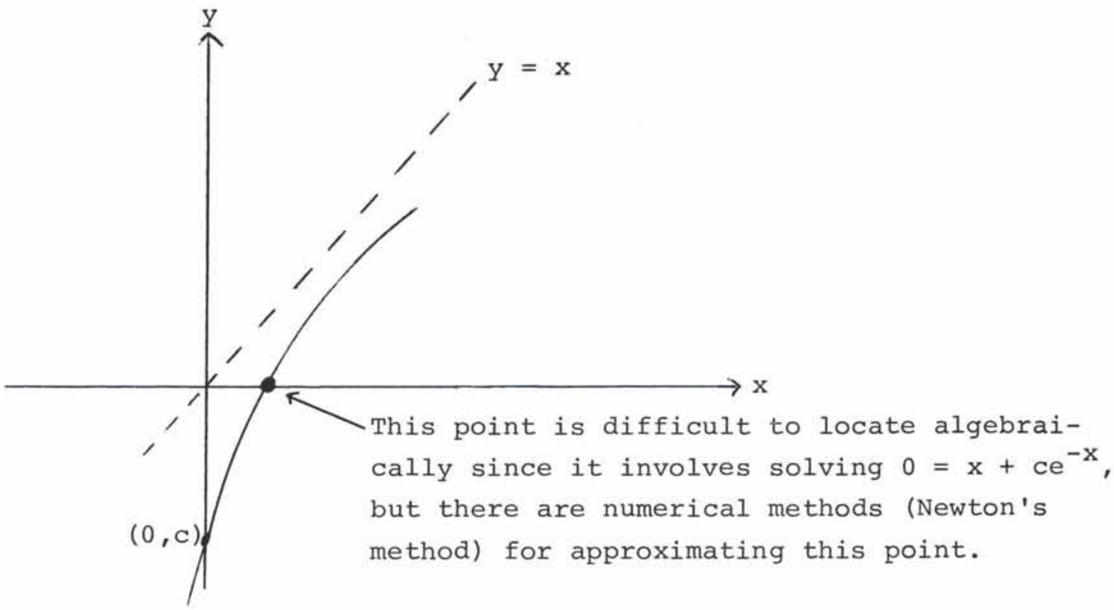
Case 3: $c < 0$
In this case we still have

$$y = x + ce^{-x}$$

$$y' = 1 - ce^{-x}$$

$$y'' = ce^{-x}$$

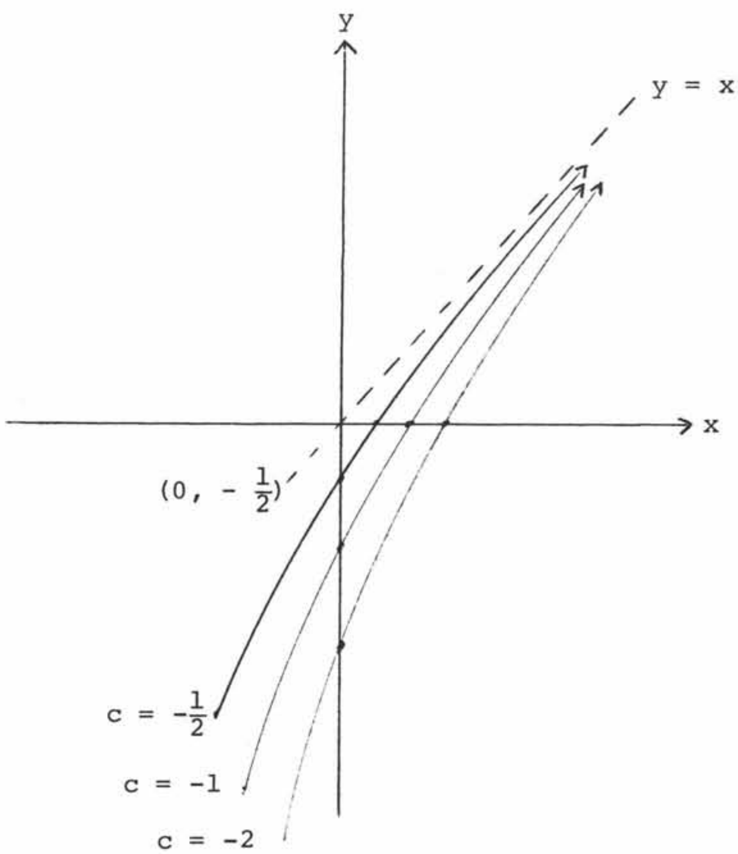
but since $c < 0$, $y'' < 0$ for all x, so now our curve always "spills water". Moreover, since $-c > 0$ and $e^{-x} > 0$, $1 - ce^{-x} > 0$, hence $y' > 0$ for all x so our curve is always rising. Finally, for large x, $x + ce^{-x} \approx x$ but since $c < 0$, $x + ce^{-x} < x$. Thus, we have that for a typical $c < 0$, the graph of $y = x + ce^{-x}$ is given by




(Figure 4)

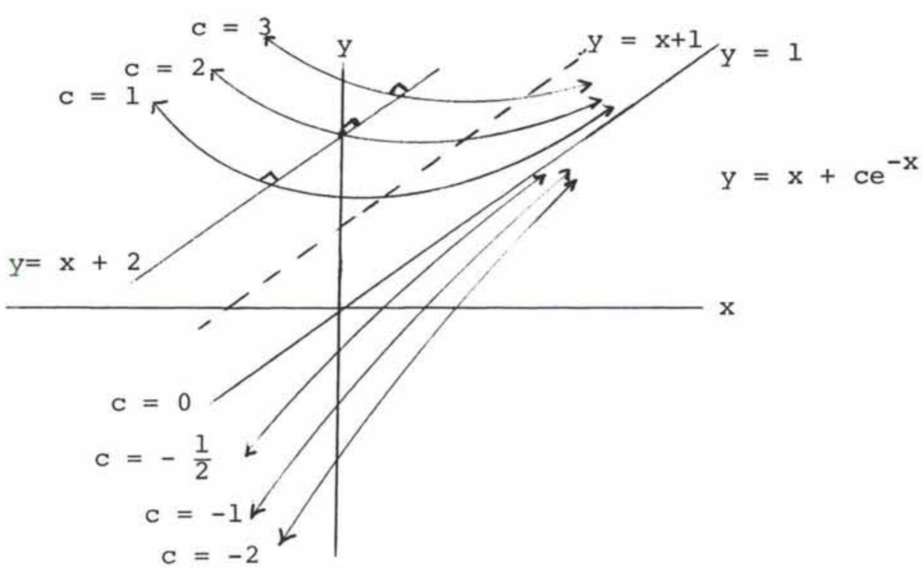
We reproduce Figure 4 for the special cases $c = -\frac{1}{2}$, $c = -1$, and $c = -2$.

2.3.1(L) continued




(Figure 5)

We may next combine Figure 1, 3 and 5 to obtain:




(Figure 6)

2.3.1(L) continued

Rather than draw the line $y = x + 2$ (in Figure 6) in a separate diagram, we have added it to Figure 6 and we hope that it seems clear that $y = x + 2$ meets every member of $y = x + ce^x$ $(c > 0)$ at right angles.

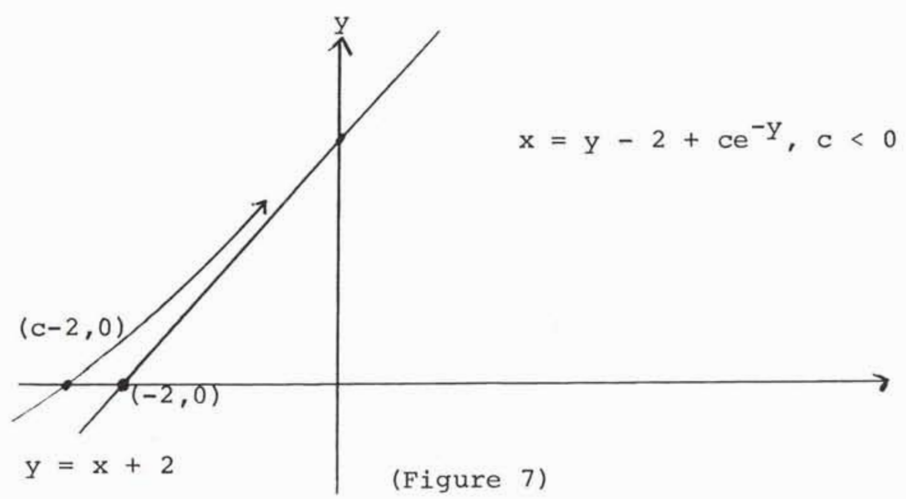
In a similar way we discuss the <u>Graph of $x = y - 2 + ce^{-y}$</u>

<u>Case 1</u>: $c = 0$
We then obtain $x = y - 2$ or $y = x + 2$.

<u>Case 2</u>: $c < 0$
Then

$$x = y - 2 + ce^{-y} \rightarrow$$
$$\frac{dx}{dy} = 1 - ce^{-y} \rightarrow$$
$$\frac{dy}{dx} = (1 - ce^{-y})^{-1} \rightarrow$$
$$\frac{d^2y}{dx^2} = -(1 - ce^{-y})^{-2}(ce^{-y}\frac{dy}{dx})$$
$$= -(1 - ce^{-y})^{-2}ce^{-y}(1 - ce^{-y})^{-1}$$
$$= \frac{-ce^{-y}}{(1 - ce^{-y})^3}.$$

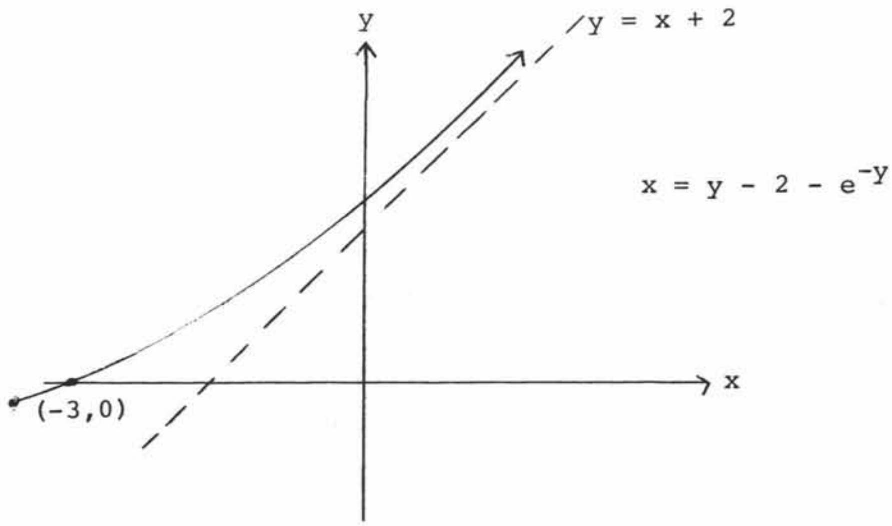
Since $c < 0$, $-c > 0$. Hence, $-ce^y$ and $1 - ce^{-y}$ are both positive. Thus, our curve is always rising and holding water.




(Figure 7)

2.3.1(L) continued

We illustrate Figure 7 for the case c = -1. We obtain




(Figure 8)

For the case $c > 0$, $1 - ce^{-y}$ can be positive, zero, or negative. In fact,

$$1 - ce^{-y} = 0 \leftrightarrow y = \ln c$$

[in which case $x = \ln c - 2 + ce^{-\ln c} = \ln c - 2 + ce^{\ln \frac{1}{c}} = \ln c - 1$ so that $dy/dx = \infty$ at $(\ln c - 1, \ln c)$ which is also on $y = x + 1$]

$$1 - ce^{-y} < 0 \leftrightarrow e^{-y} > \frac{1}{c}$$

$$\leftrightarrow e^{y} < c$$

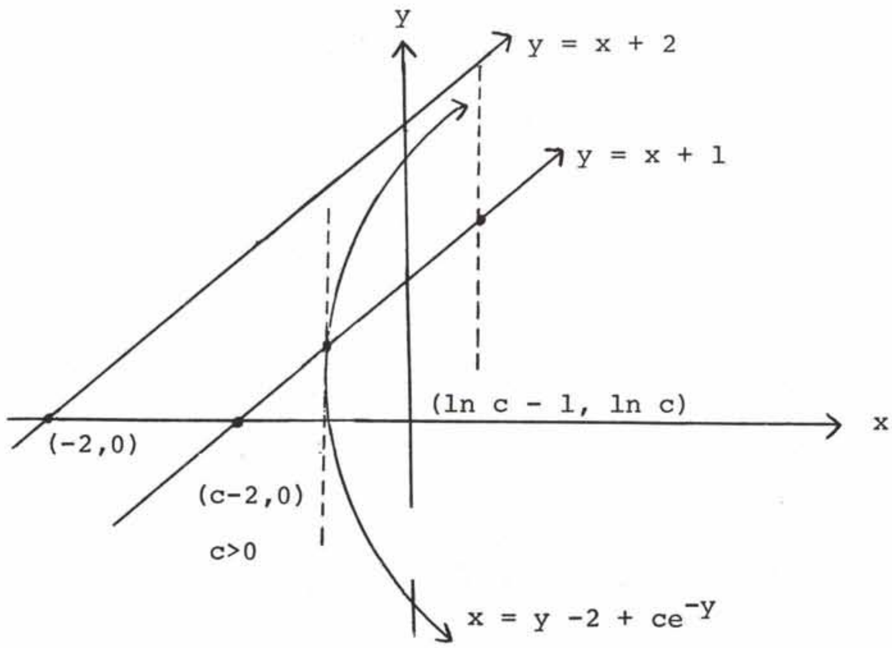
$$\leftrightarrow y < \ln c$$

and

$$1 - ce^{-y} > 0 \leftrightarrow y > \ln c.$$

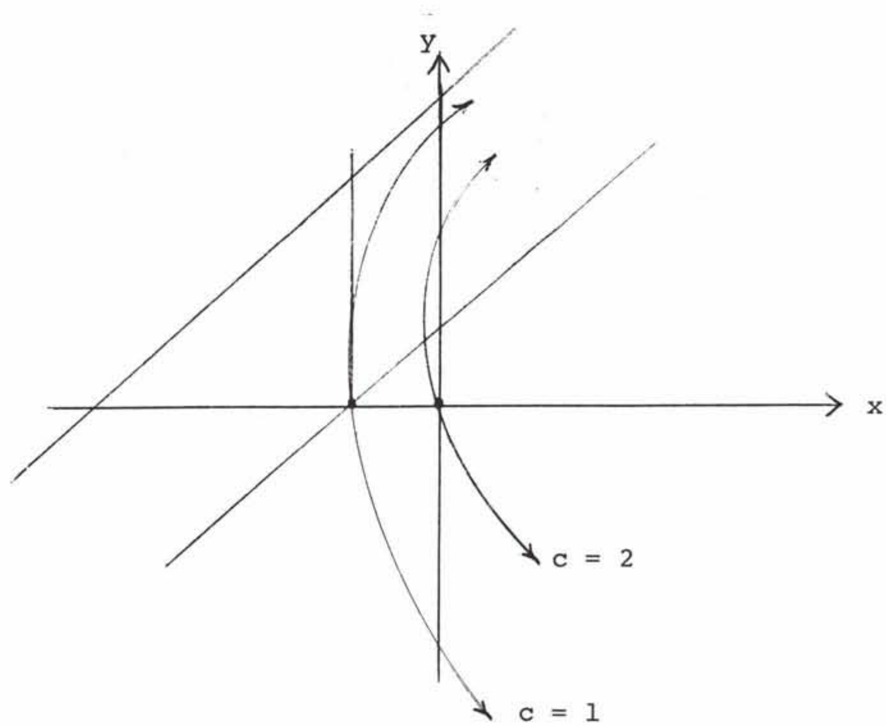
Thus,

2.3.1(L) continued



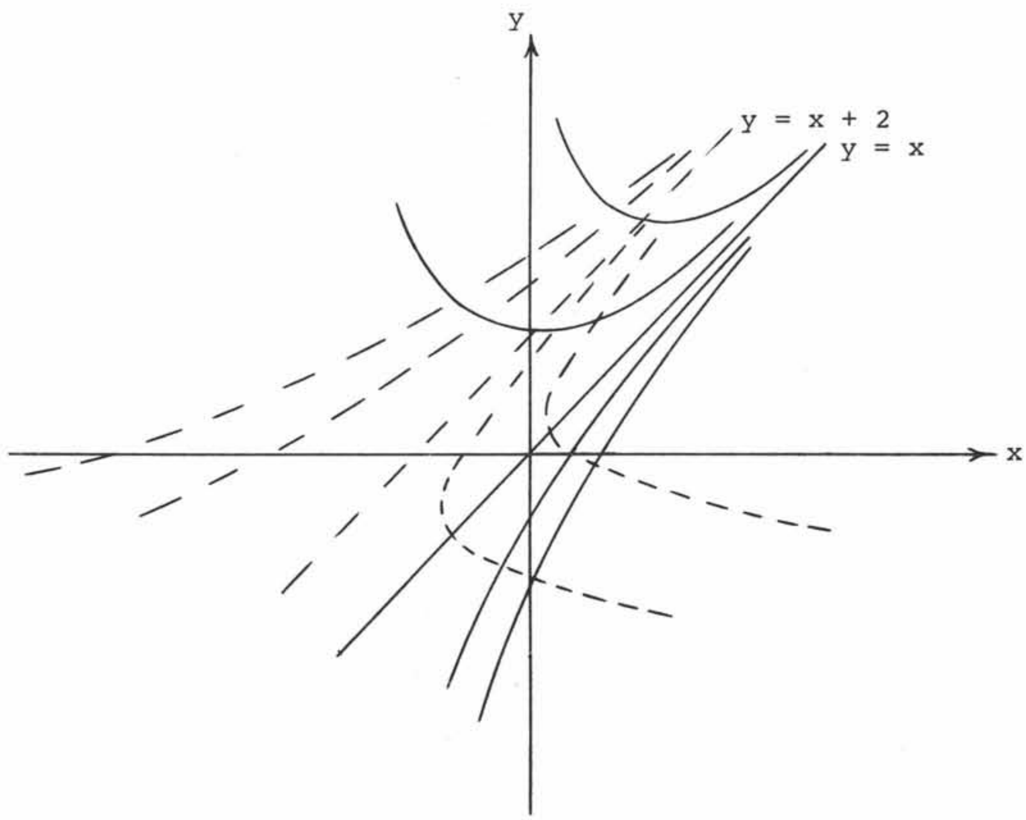

(Figure 9)

Letting c = 1 and c = 2, Figure 9 becomes




(Figure 10)

2.3.1(L) continued

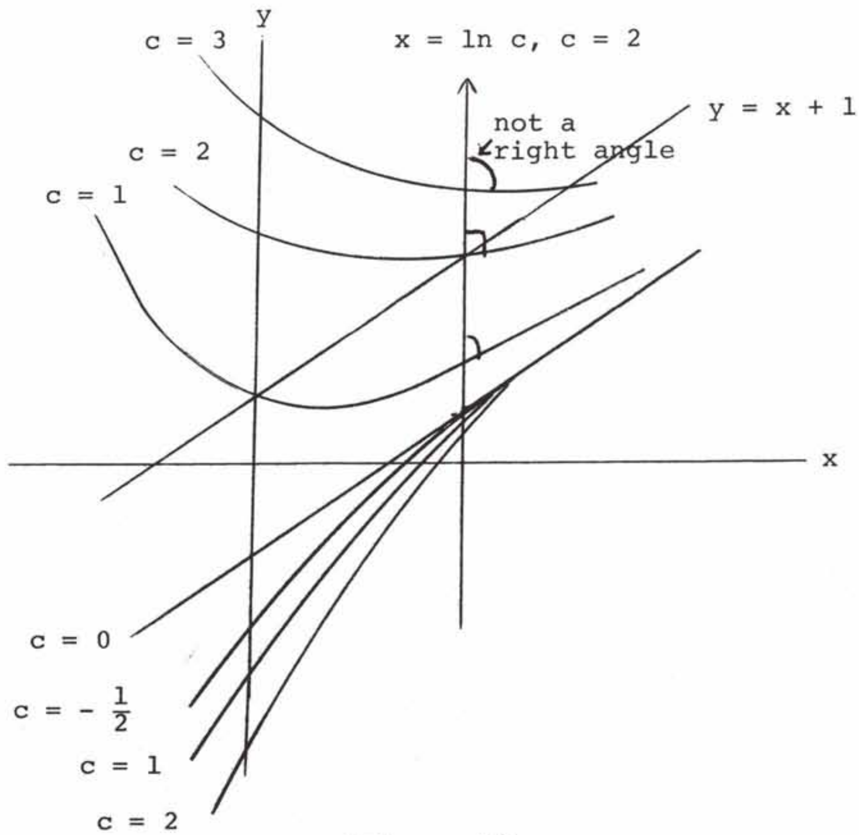



(Figure 11)

Dotted curves are members of $x = y - 2 + ce^{-y}$. Solid curves are members of $y = x + ce^{-x}$.

Finally, we see that for a given $c \neq 0$, $x = \ln c$ meets $y = x + ce^{-x}$ at right angles, but it intersects other members of $y = x + ce^{-x}$ not at right angles. That is,

2.3.1(L) continued




(Figure 12)

Thus, while the role of $y = x + 1$ is an interesting one, disregarding this case would not affect our answer to the problem since the family $x = \ln c$ is not orthogonal to the family $y = x + ce^{-x}$.

2.3.2

From

$$y^2 = cx \tag{1}$$

we obtain

$$2y \frac{dy}{dx} = c. \tag{2}$$

2.3.2 continued

Replacing c in (1) by its value in (2) yields

$$y^2 = \left(2y\frac{dy}{dx}\right)x \tag{3}$$

so that if $y \neq 0$* , equation (3) becomes $y = 2x\frac{dy}{dx}$ or

$$\frac{dy}{dx} = \frac{y}{2x} \quad \text{(provided } x \neq 0\text{**)}. \tag{4}$$

From (4) the differential equation for the orthogonal trajectories is given by

$$\frac{dy}{dx} = -\frac{2x}{y}. \tag{5}$$

In (5) the variables are separable and we obtain

$$ydy = -2xdx$$

or

$$\frac{1}{2}y^2 = -x^2 + c_1,\ c_1 \geq 0 \tag{6}$$

or

$$y^2 = -2x^2 + c_2,$$

or

$$2x^2 + y^2 = c_2,\ c_2 \geq 0\ \text{***} \tag{7}$$

---

*If $y = 0$, we simply have a particular solution of (1) given by $c = 0$.

**We must beware of the case $x = 0$, but from (3) if $x = 0$ then $y = 0$ and we have already disposed of this case.

***Notice that since $1/2y^2 \geq 0$ and $-x^2 < 0$, equation (6) requires that $c_1 \geq 0$. Hence, in (7), c is an arbitrary positive constant. With $c_2 = 0$, (7) is the single point (0,0).

2.3.2 continued

is the family of orthogonal trajectories, and this is a family of ellipses. In fact, in more "traditional" form equation (7) becomes

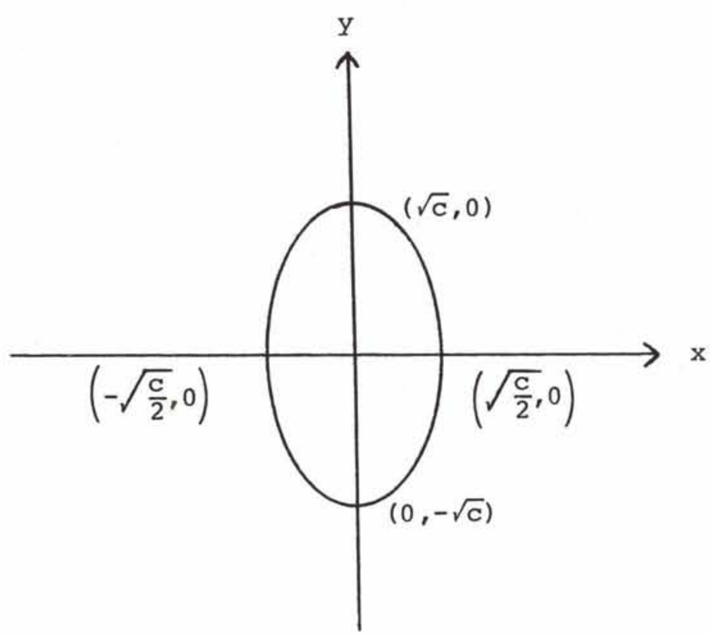
$$\frac{2x^2}{c} + \frac{y^2}{c} = 1$$

(where we have elected to write c rather than $c_2$ to facilitate notation. Notice that this c is not the same as the c given in the exercise.)

or

$$\frac{x^2}{\left(\sqrt{\frac{c}{2}}\right)^2} + \frac{y^2}{(\sqrt{c})^2} = 1. \qquad (8)$$

(8) represents the ellipse




A few notes:

1. If $(x_o,y_o)$ is any point in the plane, we see from (1) that

$$y_o^2 = cx_o$$

so that if $x_o \neq 0$,

2.3.2 continued

$$c = \frac{y_o^2}{x_o}.$$

Thus, if $(x_o, y_o)$ is any point not on the y-axis (i.e., $x_o \neq 0$) one and only one member of equation (1) passes through $(x_o, y_o)$ and this member is given by

$$y^2 = \left(\frac{y_o^2}{x_o}\right)x \tag{9}$$

and this member, as do all members of (1), passes through (0,0).

Notice, however, that if $y_o \neq 0$, no member of (1) passes through $(0, y_o)$ since from (1) $x_o = 0$ implies $y_o^2 = 0$ or $y_o = 0$.

Hence, if $x_o \neq 0$, one and only one member of (1) passes through $(x_o, y_o)$; and if $x_o = 0$, every member of (1) passes through $(x_o, y_o)$ if $y_o = 0$, but no member of (1) passes through $(x_o, y_o)$ if $y_o \neq 0$.

2. As for family (7), given any point $(x_o, y_o)$ we see that $2x_o^2 + y_o^2 = c$. Hence, c is a well-defined number for all points $(x_o, y_o)$. Thus, for each point $(x_o, y_o)$ in the plane (excluding the origin since $c \gneq 0$) one and only member of (5) passes through $(x_o, y_o)$ and this is given by

$$2x^2 + y^2 = 2x_o^2 + y_o^2. \tag{10}$$

3. As a check on (9) and (10), let us pick a point, (1,2) and see what happens. Letting $x_o = 1$, $y_o = 2$, equation (9) becomes

$$y^2 = 4x \tag{11}$$

and equation (10) becomes

$$2x^2 + y^2 = 6. \tag{12}$$

From (11),

2.3.2 continued

$$2y \left.\frac{dy}{dx}\right|_{(1,2)} = 4$$

or

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \left.\frac{4}{2y}\right|_{(1,2)} = 1. \qquad (13)$$

From (12),

$$4x + 2y\frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx} = -\frac{2x}{y}$$

or

$$\left.\frac{dy}{dx}\right|_{(1,2)} = -\frac{2}{2} = -1. \qquad (14)$$

Comparing (13) and (14) we see that (11) and (12) intersect at right angles at (1,2).

4. Solving (11) and (12) simultaneously we get

$$2x^2 + 4x = 6 \text{ (since } y^2 = 4x)$$

or

$$x^2 + 2x - 3 = 0.$$

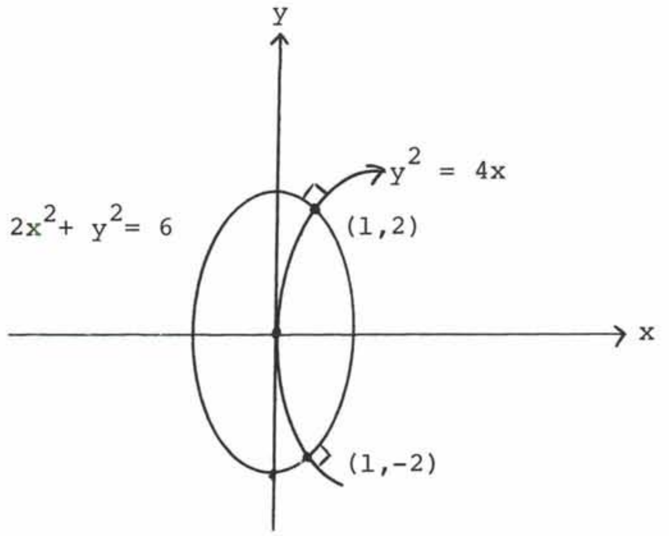
Hence, $x = 1$ or $x = -3$.

We may discard $x = -3$ since it does not satisfy (11). But when $x = 1$ we see that in both (11) and (12) $y = 2$ or $y = -2$. Thus, (11) and (12) also intersect at (1,-2). To be orthogonal trajectories, <u>every</u> intersection must be at right angles.

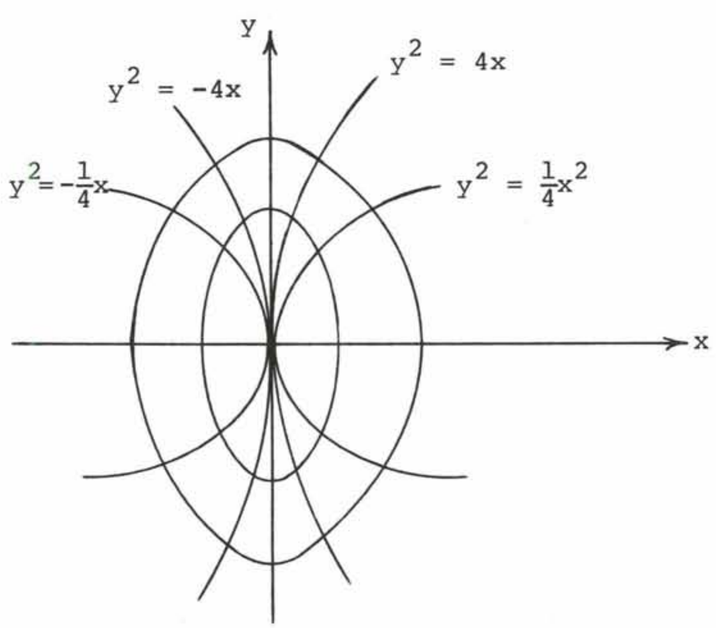
2.3.2 continued

The symmetry of both $y^2 = 4x$ and $2x^2 + y^2 = 6$ with respect to the x-axis guarantee this result, but it is still important that we understand this idea.

Graphically,




5. We must not be blinded by looking at individual curves. Whenever any member of $y^2 = cx$ meets any member of $2x^2 + y^2 = c_2$, the intersection is at right angles. In the diagram below, $y^2 = cx$ and $2x^2 + y^2 = c_2$ are drawn for c and $c_2$ equal to 1/4, 1, and 4.



2.3.2 continued

Notice that as $c \to \infty$ and $x \pm 0$ $y^2 = cx$ implies $y \to \pm \infty$. I.e., the special case $x = 0$ is a "degenerate case" of $y^2 = cx$. Also notice that all points of intersection are at right angles including where $2x^2 + y^2 = c_2$ meets the y-axis.

2.3.3(L)

Here we generalize the first two exercises to show that orthogonality is not a crucial requirement - and we also hope to show how polar coordinates can be advantageous to us in the solving of certain differential equations.

To begin with, we observe that if $\phi_2 - \phi_1 = 45°$, then $\tan(\phi_2 - \phi_1) = 1$ or

$$\frac{\tan \phi_2 - \tan \phi_1}{1 + \tan \phi_1 \tan \phi_2} = 1. \qquad (1)$$

Letting $\tan \phi_1$ denote the slope of the given family, we have

$$y = mx \to \frac{dy}{dx} = m \to y = \frac{dy}{dx}\, x.$$

Hence,

$$\tan \phi_1 = \frac{dy}{dx} = \frac{y}{x} \;(x \neq 0) \qquad (2)$$

describes the members of $y = mx$.

Then letting $\tan \phi_2$ denote the slope of each member of the required family, we see from (1) and (2) that the differential equation of the given family is

$$\frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{y}{x}\frac{dy}{dx}} = 1$$

or

$$\frac{dy}{dx} - \frac{y}{x} = 1 + \frac{y}{x}\frac{dy}{dx}$$

2.3.3(L) continued

or

$$\left(1 - \frac{y}{x}\right)\frac{dy}{dx} = 1 + \frac{y}{x}.$$

Hence,

$$\frac{dy}{dx} = \frac{1 + \frac{y}{x}}{1 - \frac{y}{x}}. \tag{3}$$

We solve (3) by letting $v = y/x$, and this leads to

$$v + x\frac{dv}{dx} = \frac{1 + v}{1 - v}$$

or

$$x\frac{dv}{dx} = \frac{1 + v}{1 - v} - v = \frac{1 + v - v + v^2}{1 - v} = \frac{1 + v^2}{1 - v}.$$

Hence,

$$\frac{(1 - v)dv}{1 + v^2} = \frac{dx}{x}. \tag{4}$$

Integrating (4) yields

$$\int\frac{dv}{1 + v^2} - \int\frac{vdv}{1 + v^2} = \ln|x| + c_1$$

or

$$\arctan v - \frac{1}{2}\ln(1 + v^2) = \ln|x| + c_1.$$

Hence,

$$2\arctan v - \ln(1 + v^2) = 2\ln|x| + c_1$$

or

$$2\arctan v - \ln(1 + v^2) = \ln x^2 + c_1.$$

2.3.3(L) continued

Letting $v = y/x$, we finally obtain

$$2 \arctan \frac{y}{x} - \ln \left(1 + \frac{y^2}{x^2}\right) = \ln x^2 + c_1,$$

or

$$2 \arctan \frac{y}{x} - \ln \left(\frac{x^2 + y^2}{x^2}\right) = \ln x^2 + c_1,$$

or

$$2 \arctan \frac{y}{x} - [\ln (x^2 + y^2) - \ln x^2] = \ln x^2 + c_1,$$

or

$$2 \arctan \frac{y}{x} - \ln (x^2 + y^2) = c_1. \qquad (5)$$

Equation (5) suggests polar coordinates with $r^2 = x^2 + y^2$ and $\arctan y/x = \theta$. Thus, $2\theta - \ln r^2 = c_1$ or $\ln r^2 = 2\theta - c_1$. Hence,

$$r^2 = e^{2\theta} e^{-c_1} = k^2 e^{2\theta} \text{ (since } e^{-c_1} \geq 0). \qquad (6)$$

From (6), $r = ke^{\theta}$ or $r = -ke^{\theta}$. Thus,

$$r = ce^{\theta} \text{ (where } c = \pm k \text{ is an arbitrary constant)} \qquad (7)$$

Equation (7) suggests that polar coordinates might have yielded a better approach to this problem. Notice that in the language of polar coordinates, $y = mx$ is the ray $\theta = \pm \tan^{-1} m$. The point is that it is the radius angle $\psi$ which is 45°. Hence $\tan \psi = \pm 1$, and since

$$\tan \psi = \frac{r}{\frac{dr}{d\theta}}$$

we see at once that $dr/d\theta = \pm r$, or $dr/r = \pm d\theta$.

We thus obtain the polar form

$$\ln |r| = \pm \theta + c$$

2.3.3(L) continued

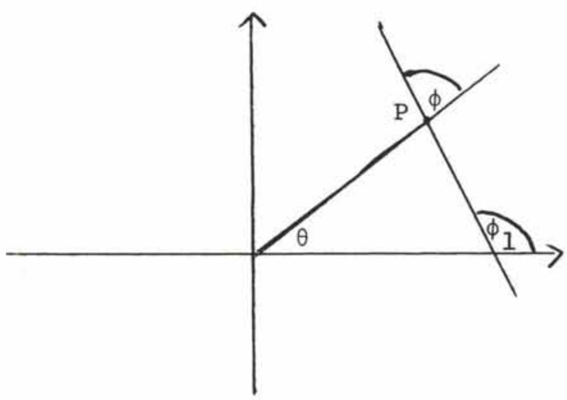
or

$r = ce^{\pm\theta}$*

very quickly using polar coordinates.

2.3.4

Since $\phi_1 = 2\theta$ and $\phi = \phi_1 - \theta$, pictorially,




we have that

$$\tan\phi = \tan(\phi_1 - \theta)$$
$$= \tan(2\theta - \theta)$$
$$= \tan\theta$$

and since

$$\tan\phi = \frac{r}{\frac{dr}{d\theta}},$$

this means

$$r\frac{d\theta}{dr} = \tan\theta. \qquad (1)$$

*Since $\theta$ can be positive or negative $r = ce^{\pm\theta}$ says the same as $r = ce^{\theta}$. It does, perhaps, emphasize that there are two curves $r = ce^{\theta}$ and $r = ce^{-\theta}$ rather than one.

2.3.4 continued

We may separate the variables in (1) to obtain

$$\frac{d\theta}{\tan \theta} = \frac{dr}{r}$$

or

$$\frac{\cos \theta d\theta}{\sin \theta} = \frac{dr}{r} .$$

Hence,

$$\ln |\sin \theta| = \ln|r| + c_1$$

or

$$\ln|\sin \theta| = \ln|r| + \ln c_2 = \ln(c_2|r|) \text{ (where } c_1 = \ln c_2)$$

or

$$|\sin \theta| = c_2 |r|$$

or

$$\sin \theta = c_3 r \quad (c_3 = \pm c_2).$$

Therefore,

$$r = c \sin \theta \ (c = \frac{1}{c_3}) \qquad (2)$$

is the required solution.

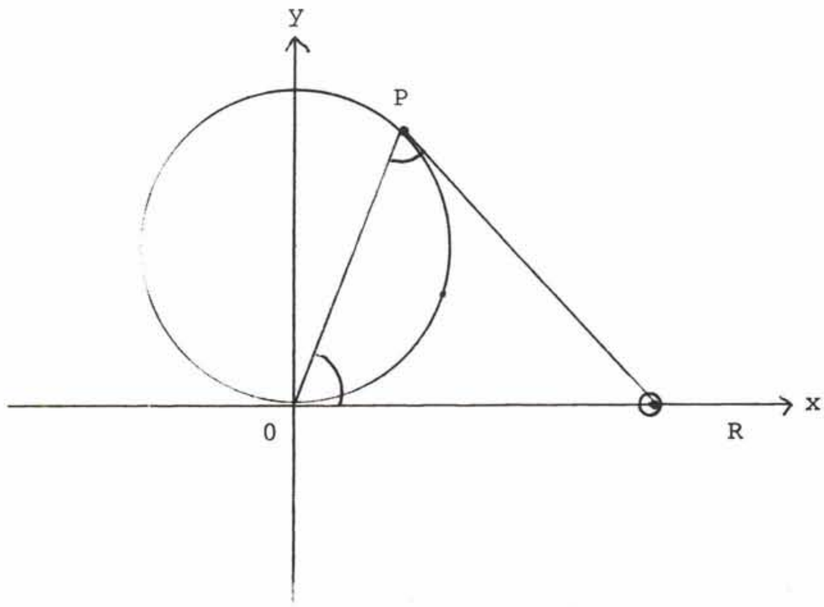
Note:
From (2) we have $r^2 = cr \sin \theta$ or $x^2 + y^2 - cy = 0$ or

$$x^2 + (y - \frac{c}{2})^2 = (\frac{c}{2})^2$$

which is a family of circles centered at $(0, \frac{c}{2})$ with radius $\frac{c}{2}$.

One such member is

2.3.4 continued




Pick any point P on this circle, let Q denote the point at which the tangent to the circle at P meets the x-axis and then, for any choice of P, $\measuredangle$ PQR is twice $\measuredangle$ POQ (i.e., $\Delta$OQP is isosceles).

2.3.5

Since

$$\frac{dy}{dx} = 3x - 4y \tag{1}$$

we have

$$\frac{dy}{dx} + 4y = 3x \tag{2}$$

which is linear with $e^{4x}$ as an integrating factor. Thus,

$$e^{4x}\frac{dy}{dx} + e^{4x}y = 3xe^{4x}$$

or

$$\frac{d(ye^{4x})}{dx} = 3xe^{4x}.$$

2.3.5 continued

Therefore,

$$ye^{4x} = \int 3xe^{4x}dx + c$$

or

$$y = 3e^{-4x}\int xe^{4x}dx + ce^{-4x}. \qquad (3)$$

Integrating by parts with $u = x$ and $dv = e^{4x}dx$ we obtain $du = dx$ and $v = 1/4\ e^{4x}$ so that

$$\int xe^{4x}dx = \frac{1}{4}\ xe^{4x} - \frac{1}{4}\int e^{4x}dx$$

$$= \frac{1}{4}\ xe^{4x} - \frac{1}{16}\ e^{4x}$$

so that (3) becomes

$$y = 3e^{-4x}\left(\frac{1}{4}\ xe^{4x} - \frac{1}{16}\ e^{4x}\right) + ce^{-4x}$$

or

$$y = \frac{3}{4}\ x - \frac{3}{16} + ce^{-4x}. \qquad (4)$$

We may observe that one and only one member of (4) passes through a given point $(x_o,y_o)$. Namely with $x = x_o$ and $y = y_o$, we find c from (4) by

$$c = \left(y_o - \frac{3}{4}\ x_o + \frac{3}{16}\right)e^{4x_o}.$$

Moreover, the right side of (1) tells us that one and only one solution of (1) passes through a given point $(x_o,y_o)$. Thus (4) yields every solution to (1), etc.

Thus, if we let $x = 0$ and $y = 13/16$ in (4), we see that

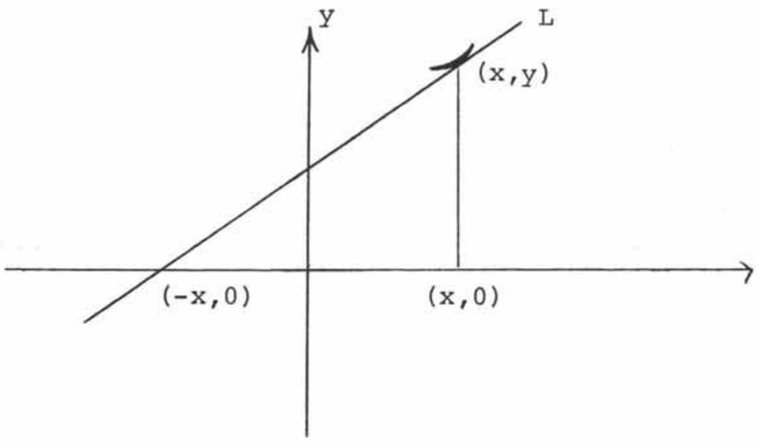
$$\frac{13}{16} = -\frac{3}{16} + ce^{o}$$

or $c = 1$. Hence, $y = 3/4\ x - 3/16 + e^{-4x}$ is the only curve which passes through (0, 13/16) and satisfies (1).

2.3.6(L)

The main reason for calling this a learning exercise is so that we can, as a note at the end of this exercise, list a few nice formulas and introduce a few technical terms.

We have




The diagram hinges not on whether x is positive or negative, but only that $x \neq 0$ since $x = 0$ implies $-x = x$ and L is not then determined.

From our diagram the slope of L is on the one hand dy/dx and on the other hand, $y - 0/x - (-x) = y/2x$. Hence the differential equation we must solve is

$$\frac{dy}{dx} = \frac{y}{2x} \quad x \neq 0. \tag{1}$$

We may separate the variables in (1) to obtain

$$\frac{dy}{y} = \frac{dx}{2x}$$

so that

$$\ln|y| = \frac{1}{2}\ln x + c_1 = \ln|x|^{\frac{1}{2}} + c_1$$

or

$$e^{\ln|y|} = e^{\ln|x|^{\frac{1}{2}} + c_1} = e^{c_1} e^{\ln|x|^{\frac{1}{2}}}.$$

2.3.6(L) continued

Hence,

$|y| = c_2 x^{\frac{1}{2}}$ (where $c_2 = e^{c_1} > 0$)

and squaring both sides yields

$y^2 = c|x|$ (where $c = c_2{}^2 > 0$), $x \neq 0$. (2)

Since

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

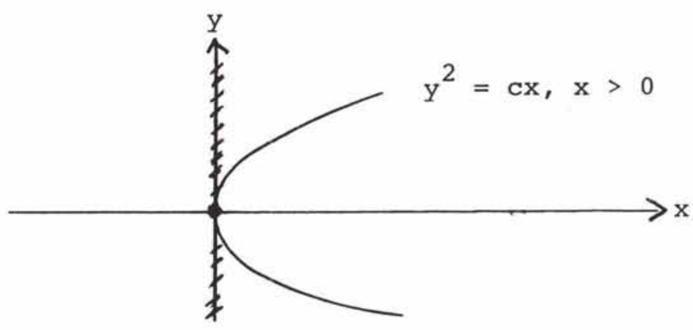
we see that equation (2) may be written as

$y^2 = cx$, if $x > 0$ (3)

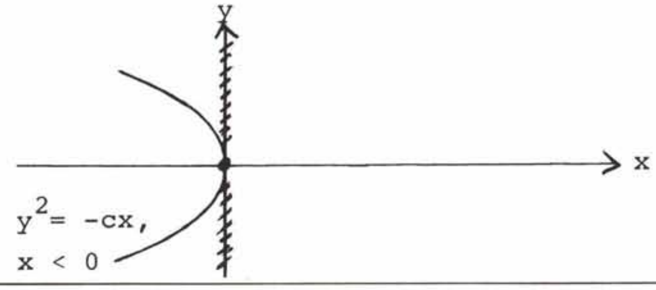
and

$y^2 = -cx$, if $x < 0$. (4)

The graph of (3) for a typical $c > 0$ is




while the graph of (4) is



2.3.6(L) continued

Thus, as long as we are restricted to a region R where $x \neq 0$, there is one and only one curve and this is a member of (2). This passes through $(x_0, y_0)$ and satisfies (1).

In still other words, if R includes no portion of the y-axis and $(x_0, y_0) \varepsilon R$, then there is one and only one curve which passes through $(x_0, y_0)$ and satisfies (1). This curve is obtained by letting $y = y_0$ and $x = x_0$ in (2) to obtain

$$c = \frac{y_0^2}{|x_0|} \tag{5}$$

so that c is a well-defined real number since $x_0 \neq 0$.

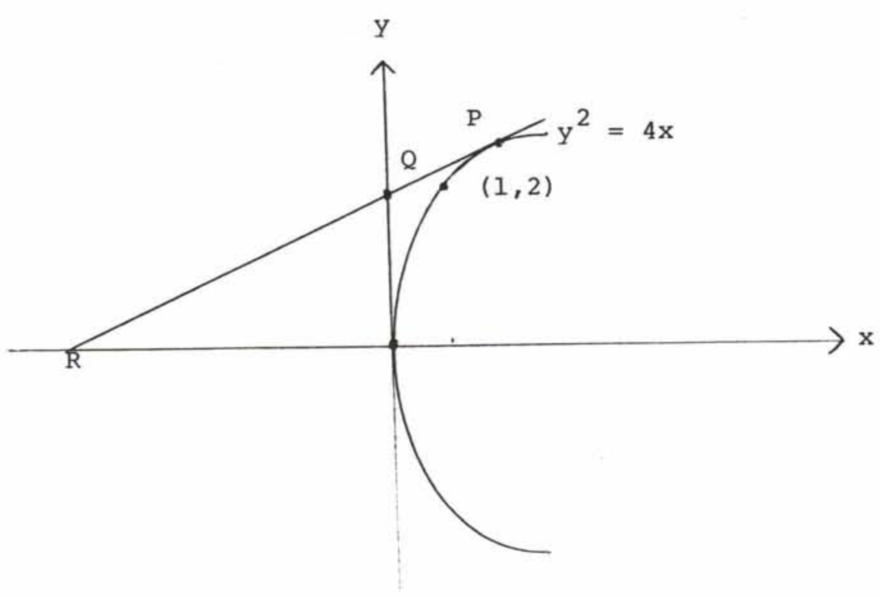
Using the value of c from (5) in (2) yields

$$y^2 = \frac{y_0^2}{|x_0|} |x| \tag{6}$$

as the required curve.

In particular, if $x_0 = 1$ and $y_0 = 2$ and $R = \{(x,y): x > 0\}$ we see from (6) that $y^2 = 4|x|$ or since $x > 0$, $y^2 = 4x$ is the only curve which passes through (1,2) and satisfies (1).

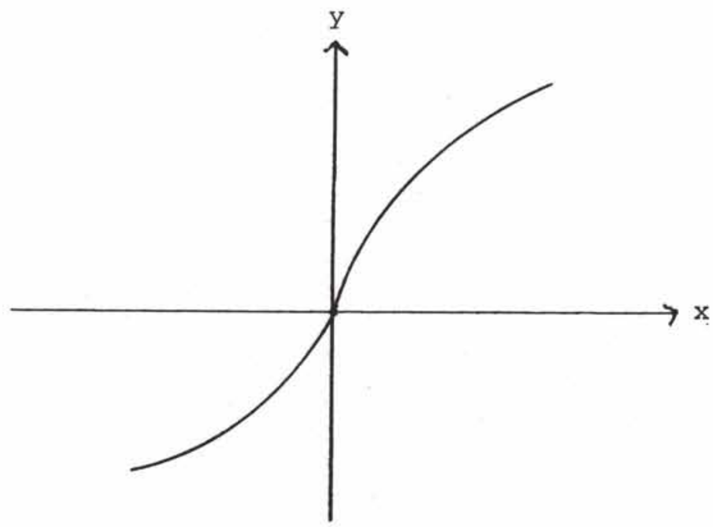
Geometrically,



2.3.6(L) continued

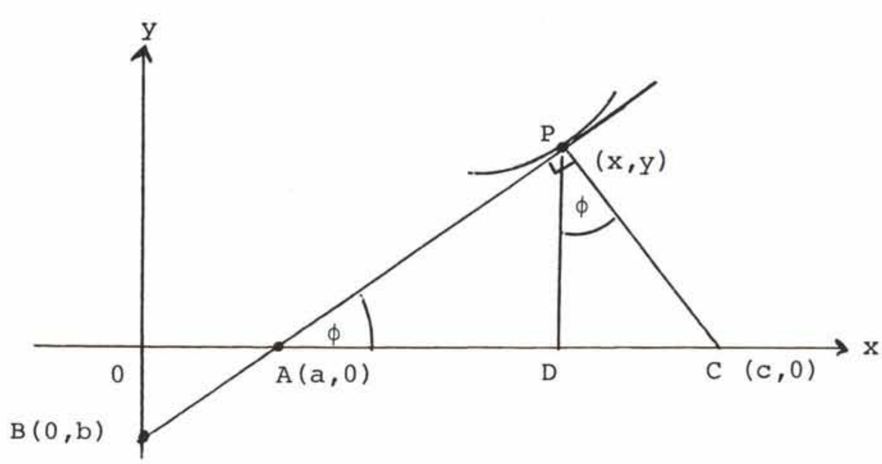
Pick any point P on $y^2 = 4x$ and draw the tangent to $y^2 = 4x$ at P. If Q denotes the point at which this line meets the y-axis and R where it meets the x-axis, then $\overline{RQ} = \overline{QP}$.

Notice that if we allow x to be 0 we may combine the upper half of $y^2 = 4x$ $(x > 0)$ with the lower half of $y^2 = -4x$ $(x \leq 0)$ to obtain




and this curve also passes through (1,2) and satisfies (1). Again, the extra solution stems from allowing R to include part of x = 0.

A few helpful facts that are useful in geometry related problems and which may be read from the diagram are given below:



2.3.6(L) continued

1. Equation of AP is

$$\frac{y}{x - a} = \frac{dy}{dx} = y'$$

so

$$\frac{x - a}{y} = \frac{1}{y'},$$

or

$$x - a = \frac{y}{y'},$$

or

$$a = x - \frac{y}{y'} = \text{x-intercept of tangent line.}$$

1'. $\frac{y - b}{x} = y'$

or

$$y - b = xy'.$$

Hence, $b = y - xy'$ = y-intercept of tangent line.

2. $\frac{y}{x - c} = -\frac{dx}{dy}$

$$\frac{x - c}{y} = -\frac{dy}{dx} = -y'$$

$$x - c = -yy'$$

or

$$c = x + yy' = \text{x-intercept of normal line.}$$

2'. $\frac{y - e}{x} = -\frac{dx}{dy}$

$$y - e = -x\frac{dx}{dy}$$

$$e = y + x\frac{dy}{dx} = y + \frac{x}{y'} = \text{y-intercept of normal line.}$$

2.3.6(L) continued

3. $\overline{AP} = \sqrt{y^2 + (x - a)^2}$

and since $a = x - y/y'$, $x - a = x - x + y/y' = y/y'$. Hence,

$$\overline{AP} = + \sqrt{y^2 + \left(\frac{y}{y'}\right)^2}$$

$$= \left|\frac{y}{y'} \sqrt{(y')^2 + 1}\right| = \text{length of tangent line from P to x-axis.}$$

3'. $\overline{BP} = \sqrt{x^2 + (y - b)^2}$

and $b = y - xy'$. Hence, $y - b = y - y + xy' = xy'$. Therefore,

$$\overline{BP} = \sqrt{x^2 + x^2 y'^2}$$

$$= \left|x \sqrt{1 + (y')^2}\right| = \text{length of tangent from P to y-axis.}$$

4. $\overline{PC} = \sqrt{(x - c)^2 + y^2}$ and $c = x + yy'$. Hence $x - c = x - x - yy'$, and $(x - c)^2 = y^2(y')^2$. Thus,

$$\overline{PC} = \sqrt{y^2(y')^2 + y^2} = \left|y \sqrt{1 + (y')^2}\right|$$

= length of normal line from P to x-axis.

4'. $\overline{PE} = \sqrt{x^2 + (y - e)^2}$ and $e = y + \frac{x}{y'}$. Hence, $y - e = y - y - x/y'$, so $(y - e)^2 = x^2/(y')^2$. Therefore

$$PE = \sqrt{x^2 + \frac{x^2}{(y')^2}}$$

$$= \left|\frac{x}{y'} \sqrt{(y')^2 + 1}\right|$$

= length of normal line from P to y-axis.

5. The projection of AP on the x-axis is called the subtangent. That is the subtangent is $\overline{AD}$. Now, $\overline{AD} = \overline{AP} \cos \phi$

$$= \overline{AP} \ \frac{\overline{PD}}{\overline{PC}}$$

2.3.6(L) continued

$$= \left|\frac{y}{y'}\sqrt{(y')^2 + 1}\right| \frac{|y|}{\left|y\sqrt{1 + (y')^2}\right|}$$

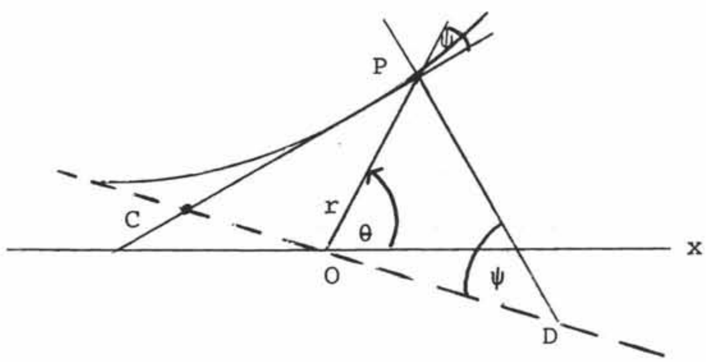
$$= \left|\frac{y}{y'}\right| = \text{length of subtangent.}$$

6. Similarly the projection of $\overline{PC}$ on the x-axis, $\overline{DC}$, is called the subnormal. Hence,

$$\overline{DC} = \sqrt{\overline{PC}^2 - \overline{PD}^2} = \sqrt{y^2(1 + [y']^2 - y^2} = |y|\sqrt{1 + (y')^2 - 1}$$

$$= |yy'| = \text{length of subnormal.}$$

If we prefer to use polar coordinates, then




We let COD be perpendicular to $\overline{OP}$ at 0. Then $\overline{CP}$ is defined to be the polar tangent; $\overline{DP}$ , the polar normal; $\overline{CO}$ , the polar subtangent; and $\overline{OD}$, the polar subnormal

$$\tan\psi = \frac{r}{r'}, \text{ (r' means } \frac{dr}{d\theta}\text{ )}$$

but

$$\tan\psi = \frac{\overline{OC}}{\overline{OP}} = \frac{\overline{OC}}{r} .$$

Hence,

$$\overline{OC} = |r\tan\psi| = \frac{r^2}{|r'|} = \text{length of polar subtangent.}$$

2.3.6(L) continued

Similarly

$$\frac{\overline{OD}}{r} = \cos\psi = \frac{r'}{r}$$

so that

$\overline{OD} = |r'|$ = length of polar subnormal, etc.

2.3.7

a. The y-intercept of the tangent line is given by

$$y - xy' \qquad (1)$$

and since the slope of the tangent line is y', we see from (1) that our differential equation is given by

$$y' = y - xy' \qquad (2)$$

or in differential form

$$dy = ydx - xdy$$

or

$$ydx - (x + 1)dy = 0. \qquad (3)$$

The variables are separable in (3), and we have

$$\frac{dx}{x+1} - \frac{dy}{y} = 0 \quad (x \neq -1,\ y \neq 0). \qquad (3')$$

Hence,

$$\ln|x + 1| - \ln|y| = c_1$$

or

$$\ln\left|\frac{x+1}{y}\right| = c. \qquad (4)$$

2.3.7 continued

Hence,

$$\frac{x+1}{y} = e^{c_1},$$

and removing the absolute value sign

$$\frac{x+1}{y} = \pm e^{c_1} = c_2 \qquad (5)$$

where $c_2$ is an arbitrary non-zero constant since $e^{c_1}$ is an arbitrary positive constant. Taking reciprocals in (5) we obtain

$$\frac{y}{x+1} = c \text{ (where } c = \frac{1}{c_2}\text{)}.$$

That is

$$y = c(x + 1),\ c \neq 0. \qquad (6)$$

The restriction $c \neq 0$ may be removed by observing that from (6), $c = 0$ implies $y = 0$. Certainly the x-axis has the property that at each point, its slope (=0) equals y-intercept of its tangent (=0, since the x-axis is its own tangent).

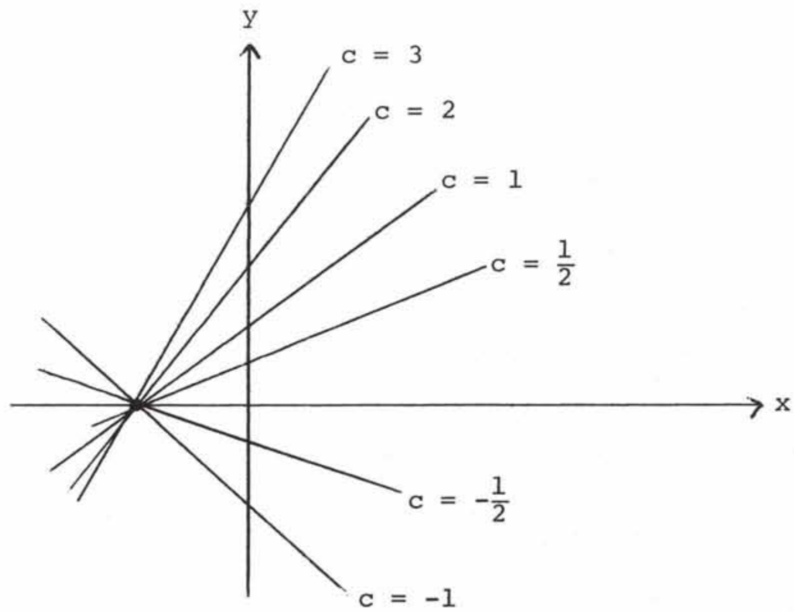
Thus, (6) may be rewritten as

$$y = c(x + 1),\ c \text{ an arbitrary constant.} \qquad (7)$$

Note:

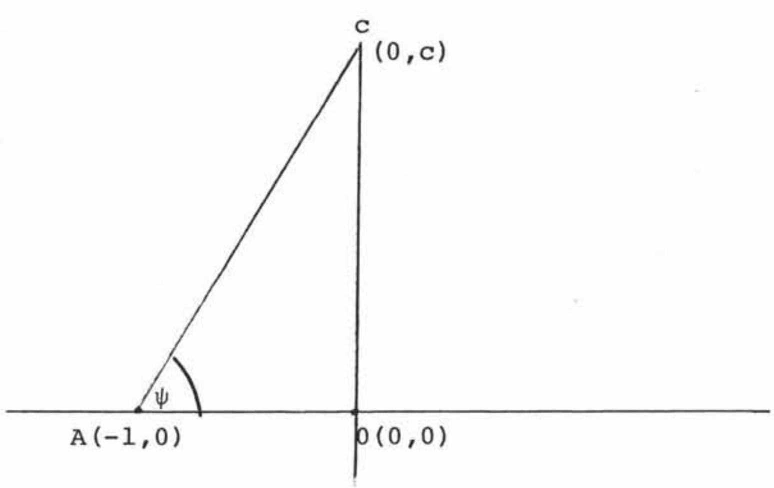
For those who may be a bit curious, notice that we made the restriction $x \neq -1$ in (3'). What does this mean? Well, notice that when $x = -1$, equation (7) shows that $y = 0$ for each choice of c. In other words the x-intercept of each member of c is -1. Pictorially,

2.3.7 continued




(Figure 1)

Notice that once we have (7), it is trivial to vary geometrically that each member of (7) is a solution of (2). Namely,




$$\tan \psi = \frac{y}{x} = \frac{\overline{OC}}{\overline{AO}} = \frac{c - 0}{0-(-1)} = c,$$

i.e., the slope of AC is equal to its y-intercept.

2.3.7 continued

b. Letting $x = x_o$ and $y = y_o$ in (7) we obtain

$$y_o = c(x_o + 1)$$

so that

$$c = \frac{y_o}{x_o + 1} \tag{8}$$

so c is uniquely determined and exists unless $x_o = -1$.

Thus, if $x_o \neq 0$ , the member of (7) which passes through $(x_o, y_o)$ is

$$y = \frac{y_o}{x_o + 1}(x + 1). \tag{9}$$

In particular with $x_o = 2$ and $y_o = 9$, we see from (9) that

$$y = 3(x + 1) \tag{10}$$

is the only member of (7) which passes through (2,9) with the desired property.

If $x_o = -1$, then every member of (7) passes through (-1,0) but no member of (7) passes through $(-1, y_o)$ if $y_o \neq 0$.

Note:
In terms of the basic existence theorem, we see from (2) that

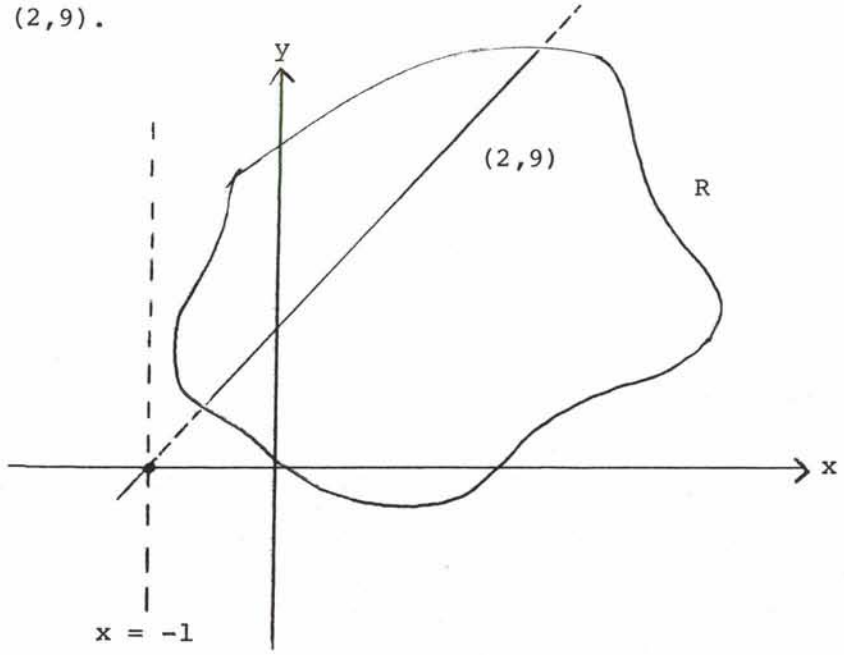
$$\frac{dy}{dx} = \frac{y}{x + 1} \tag{11}$$

so that the only thing we have to worry about is when $x = -1$ since then the denominator vanishes. The line $x = -1$ is a "degenerate" solution of our problem (as in any vertical line) since then both the y-intercept and the slope both "equal $\infty$ ".

With respect to (10) notice that since (2,9) is not "near" $x = -1$, then no matter what other solutions might exist when $x = -1$, none of these can pass through (2,9). Namely, we can pick a region R which includes (2,9) but which doesn't

2.3.7 continued

intersect the line $x = -1$ and from (11) we can conclude that in R (10) is the only solution of our problem which passes through (2,9).

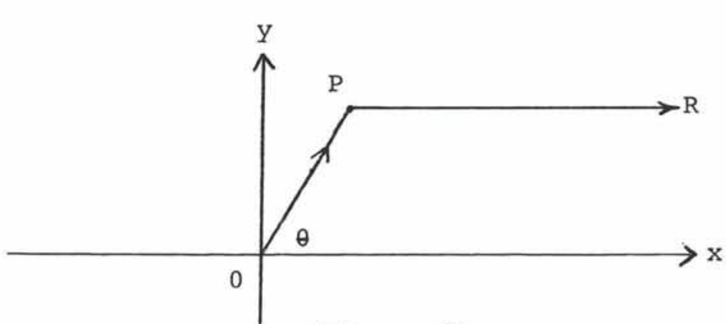



Any solution of (2) which passes through (2,9) must coincide with $y = 3(x + 1)$, at least throughout R.

2.3.8 (optional)

Here, in addition to the usual rules of geometry, we also need the physical fact that "the angle of incidence equals the angle of reflection". Otherwise, this problem is no different from any other exercise we have tackled in this unit.

To begin with, we have

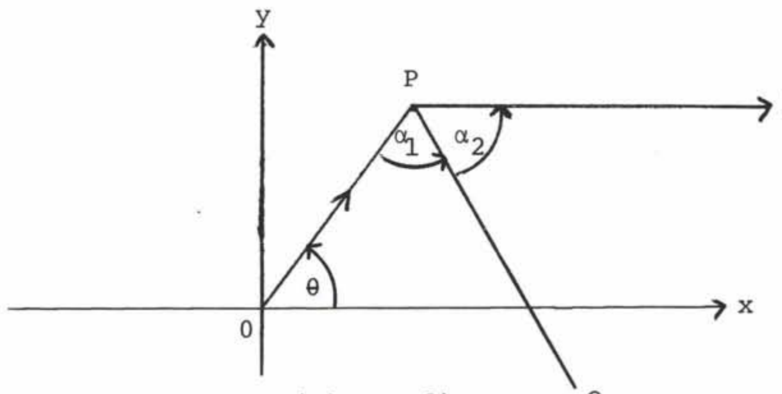



(Figure 1)

2.3.8 continued

1. Figure 1 emphasizes the local nature of our eventual differential equation. We assume that P is a fixed, but arbitrary, point on our mirror. By the given data, the light ray which emanates from 0 and strikes our mirror at P is reflected parallel to the x-axis.

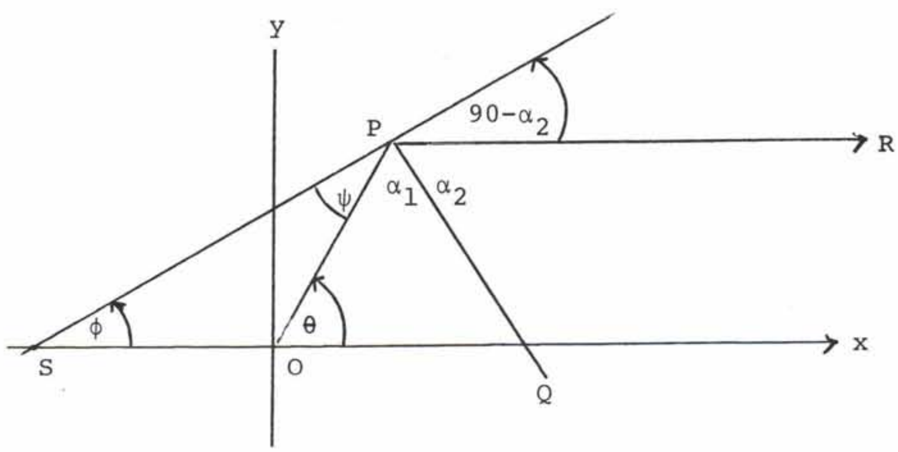
2. The angles of incidence and reflection, being measured from the normal, tell us that the normal to our mirror at P must be the angle bisector of ∢ OPR (since these two angles are equal). Thus, Figure 1 becomes




(Figure 2)

Angle of incidence equals angle of reflection means $\alpha_1 = \alpha_2$ where PQ is the normal to the mirror at P.

3. By definition of the normal line, the tangent line to our mirror at P is perpendicular to PQ at P. This leads to Figure 3.




(Figure 3)

2.3.8 continued

SP is tangent to the required mirror at P. Notice that $\theta, \phi$, and $\psi$ have their usual meanings (i.e., $\tan\phi = dy/dx$, $\tan\psi = r/\ dr/d\theta$, etc.).

From this point, the problem requires no further physical knowledge, only geometry. First of all, since $\alpha_1 = \alpha_2$ and $QP \perp SP$, it follows that

$$90 - \alpha_2 = \psi. \qquad (1)$$

But, since PR is parallel to the x-axis,

$$90 - \alpha_2 = \phi\ . \qquad (2)$$

Comparing (1) and (2), we conclude that

$$\phi = \psi\ . \qquad (3)$$

If we now elect to use Cartesian coordinates, we have that $\tan\phi = dy/dx$ and $\tan\theta = y/x$.

From $\Delta$ SOP, $\theta = \phi + \psi$, or by (3),

$$\theta = 2\phi\ . \qquad (4)$$

Hence,

$$\tan\theta = \tan 2\phi$$

$$= \frac{2\tan\phi}{1 - \tan^2\phi},$$

or

$$\frac{y}{x} = \frac{2y'}{1 - y'^2}\ . \qquad (5)$$

*Perhaps Figure 3 is a good place fo us to emphasize the local nature of differential equations. At a particular moment, Figure 3 is interested only in how our mirror behaves at a particular point P [which may be labeled $P(x,y)$ or $P(r,\theta)$, depending on whether we elect to use Cartesian or polar coordinates in our analysis].

2.3.8 continued

Clearing (5) of fractions and solving the resulting quadratic equation for y', we obtain

$$y' = -\frac{x \pm \sqrt{x^2 + y^2}}{y} \quad . \tag{6}$$

Equation (6) has the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right),$$

if we divide numerator and denominator on the right side of (6) by x, but it also happens to be exact. Namely, (6) may be written as

$$\frac{xdx + ydy}{\pm\sqrt{x^2 + y^2}} = dx$$

or

$$\pm\, d\left(\sqrt{x^2 + y^2}\right) = dx.$$

Hence,

$$\pm\sqrt{x^2 + y^2} = x + c. \tag{7}$$

Squaring both sides of (7) we obtain

$$x^2 + y^2 = x^2 + 2cx + c^2$$

or

$$y^2 = 2cx + c^2 \tag{8}$$

which is a family of parabolas with focus at the origin.

Aside:

Equation (8) is double-valued and occurs because equation (6) is double-valued. Had we written (6) as the two single-valued

2.3.8 continued

differential equations:

$$y' = \frac{x + \sqrt{x^2 + y^2}}{y} \tag{6'}$$

and

$$y' = \frac{x - \sqrt{x^2 - y^2}}{y} \tag{6"}$$

Then our fundamental theorem would tell us that unless $y_o = 0$, there is one and only one solution of (6') or (6") which passes through $(x_o, y_o)$. Otherwise notice that unless $x_o = y_o = 0$, equation (8) allows 2 values of c for each $(x_o, y_o)$. Namely

$$y_o^2 = 2cx_o + c^2 \rightarrow$$

$$c^2 + 2x_o c - y_o^2 = 0 \rightarrow$$

$$c = -\frac{2x_o \pm \sqrt{4x_o^2 + 4y_o^2}}{2} \rightarrow$$

$$c = -x_o \pm \sqrt{x_o^2 + y_o^2} \rightarrow$$

c has two distinct real values unless $x_o^2 + y_o^2 = 0$ (i.e., $x_o = y_o = 0$). The point is that one of these values satisfies (6') and the other (6").

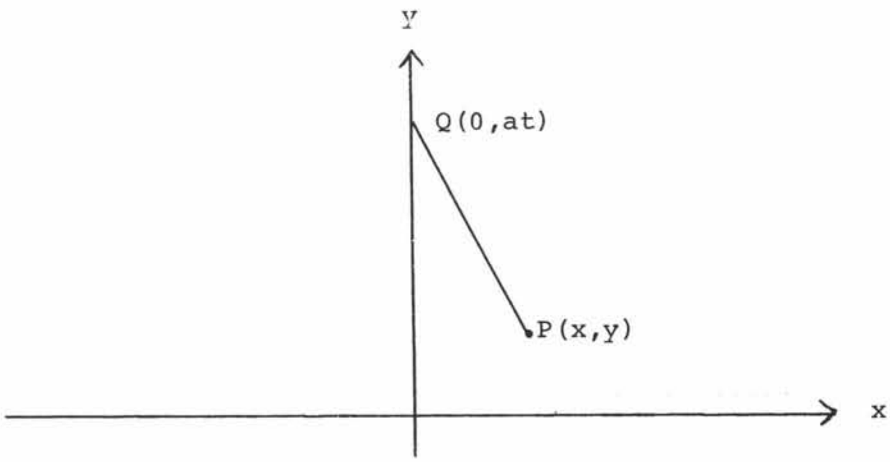
2.3.9 (optional)

Here we have an elementary illustration of a famous applied type of differential equation known as pursuit problems. Again, once the problem is set up it becomes another geometric example, but in this case the computation does beccme a bit more overwhelming.

Let us use time, t, as our parameter, beginning with t = 0 when A is at (0,0). What we know for sure then is that at any time t, A is at (0, at) while B is at some Point P(x,y). [It is the relationship between x and y which we seek in this exercise].

2.3.9 continued

Pictorially, we have




(Figure 1)

Since the direction of motion always has B pointed toward A and since at a given moment, the direction of a curve is measured by the direction of its tangent line, we have from Figure 1 that the slope of PQ is $(y - dt)(x - 0)$, but since this line is also tangent to the path traced out by P, the slope of PQ is also the slope of our required "present curve", and this slope is denoted by dy/dx.

Equating these two expressions for the slope of PQ we obtain

$$\frac{y - dt}{x} = \frac{dy}{dx}$$

or

$$y - at = xy'. \qquad (1)$$

To eliminate t from (1) we may differentiate (1) with respect to x (remembering that x and t are dependent) to obtain

$$y' - at' = xy'' + y'$$

or

2.3.9 continued

$$t' = \frac{xy''}{-a} . \tag{2}$$

We can also compute $t' = dt/dx$ another way. Namely,

$$\frac{dt}{dx} = \frac{dt}{ds}\frac{ds}{dx}$$

$$= \frac{\frac{ds}{dx}}{\frac{ds}{dt}} . \tag{3}$$

Now,

$$\frac{ds}{dx} = \sqrt{1 + (y')^2}$$

and $ds/dt = b$ since $ds/dt$ is the speed along the desired curve (which is the speed of B) or b. Thus (3) becomes

$$t' = \frac{1}{b} \sqrt{1 + (y')^2} \tag{4}$$

and equating the values of $t'$ as given by (2) and (4) we obtain

$$\frac{1}{b} \sqrt{1 + (y')^2} = \frac{xy''}{-a}$$

or

$$xy'' + \frac{a}{b}\sqrt{1 + (y')^2} = 0. \tag{5}$$

Since the y term is missing from (5) we use the substitution $p = y'$ to obtain from (5) that

$$x\frac{dp}{dx} + \frac{a}{b}\sqrt{1 + p^2} = 0. \tag{6}$$

We may then separate variables in (6) to obtain

$$\frac{dp}{\sqrt{1 + p^2}} = -\frac{a}{b}\frac{dx}{x}$$

or

2.3.9 continued

$\sinh^{-1} p = -\frac{a}{b} \ln |x| + c$

or

$$p = \sinh(-\frac{a}{b} \ln |x| + c) \qquad (7)$$

or

$$\frac{dy}{dx} = \sinh(-\frac{a}{b} \ln |x| + c_1). \qquad (7')$$

We may simplify (7') by letting $r = a/b$ and $c_1 = \ln c_2$. Then,

$-\frac{a}{b} \ln |x| + c_1$

$= - r \ln |x| + \ln c_2$

$= \ln |x|^{-r} + \ln c_2$

$= \ln c_2 |x|^{-r}$ .

Hence,

$\sinh(-\frac{a}{b} \ln |x| + c_1)$

$= \sinh(\ln c_2 |x|^{-r})$

$= \dfrac{e^{\ln c_2 |x|^{-r}} - e^{-\ln c_2 |x|^{-r}}}{2}$

$= \frac{1}{2} \left\{ c_2 |x|^{-r} - \dfrac{1}{c_2 |x|^{-r}} \right\}$

$= \dfrac{c_3}{x^r} - \dfrac{x^r}{4c_3}$

*Where $c_3 = \frac{1}{2} c_2$ and $|x| = x$ [since we are assuming that the present curve is in the right half plane (i.e., $x > 0$).]

2.3.9 continued

Thus, our final differential equation is given by

$$\frac{dy}{dx} = c_3x^{-r} - \frac{x^r}{4c_3}$$

Hence,

$$y = \begin{cases} \dfrac{c_3\, x^{1-r}}{1-r} - \dfrac{x^{1+r}}{4c_3(1+r)} + c_4, & \text{if } r \pm 1 \text{ (i.e., if A and B have unequal speeds)} \\[2ex] c_3 \ln x - \dfrac{x^2}{8c_3} + c_4, & \text{if } r = 1 \end{cases}$$