Unit 4: Linear Differential Equations

1. Overview

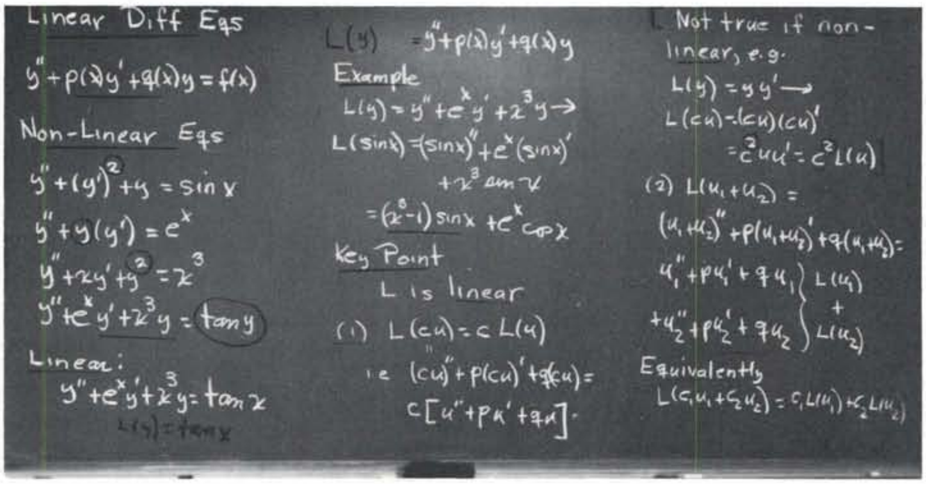
Unless a first order differential equation has a rather special form, we often have to resort to numerical methods in order to solve the equation. Even when the equation has the special form, it seems that the proper technique is a "trick." This problem is compounded in the case of higher order differential equations.

There is, however, a special category of nth-order differential equations about which a great deal is known. In fact, so much is known that in most "real-life" situations in which this equation occurs, we can always construct the general solution to the equation. This type of equation is known as the linear differential equation and it is the type of equation to which the remainder of this block is dedicated.

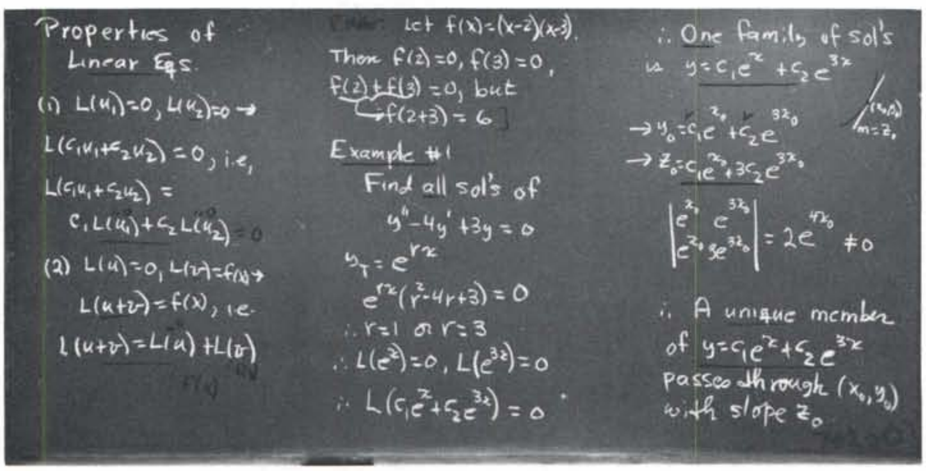
Not only does the linear differential equation lend itself nicely to solution, but it has the added feature that in many practical situations in which we have to come to grips with differential equations, the equation is linear. In other cases in which the equation is not linear, we may be able to replace the given equation by a reasonable approximation which is linear. In still other cases, it is possible to make a change of variables which converts a nonlinear equation into a linear equation. (See, for example, Exercise 2.4.11.)

At any rate, in this unit we introduce the notion of an nth-order linear differential equation and present a general overview of the technique for solving such equations. Then, in the remaining units of this block, we look at the individual components of the overall approach in more detail.

2. Lecture 2.020



a.



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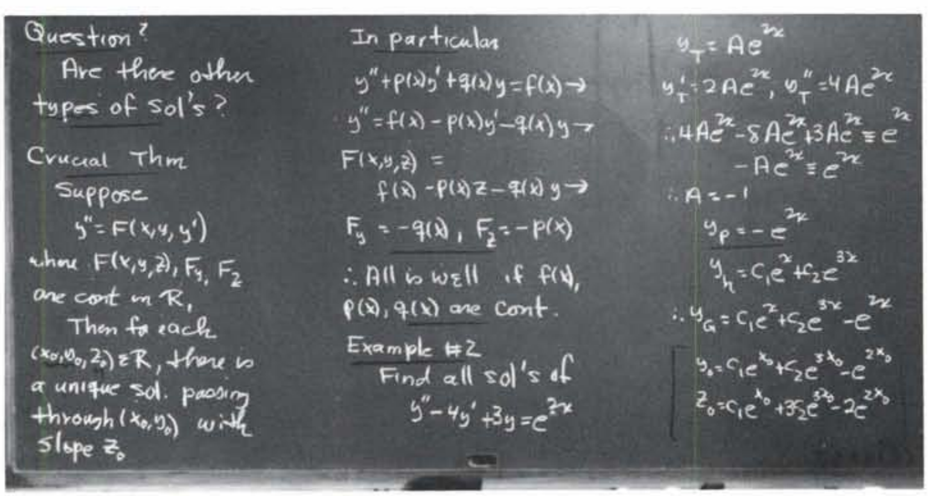
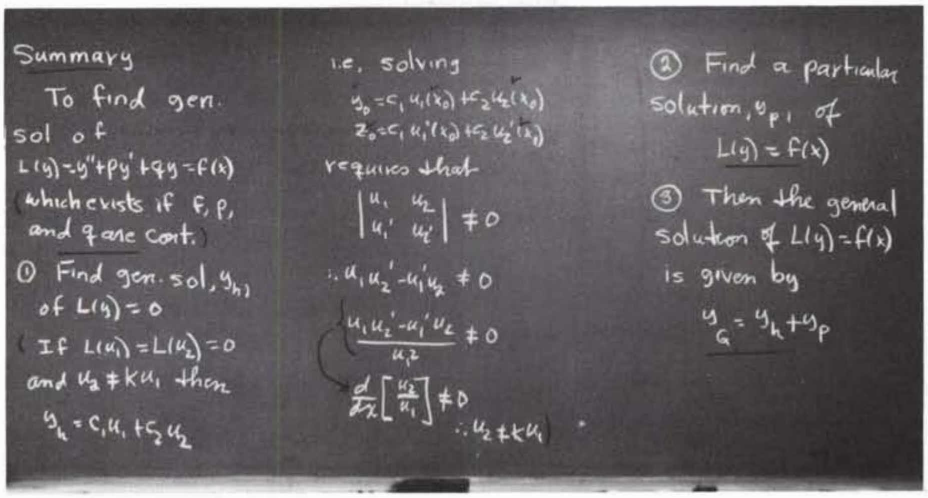


c.

Lecture 2.020 continued

d. 

3. Do Exercises 2.4.1 through 2.4.5.

4. After doing Exercise 2.4.5, read Supplementary Notes, Chapter 3, Sections A and B.

5. Do the rest of the exercises.

6. Exercises:

2.4.1(L)

Let L(y) be defined by

$L(y) = x^2y'' - 3xy' + 3y.$

a. Compute L(y) in each of the following cases:

(i) $y = \sin x$, (ii) $y = e^x$, (iii) $y = x$,

(iv) $y = x^2$, (v) $y = x^3$,

b. Use the results of part (a) to compute L(y) if

(vi) $y = e^x + \sin x$, (vii) $y = 6 \sin x$, (viii) $7e^x + 6 \sin x$

(ix) $y = xe^x$.

2.4.2(L)

Let L(y) be defined by

$L(y) = y'' - 3yy' + 3y.$

Compute $L(x^3)$ and $L(2x^3)$.

2.4.3(L)

Use parts (iii) and (v) of Exercise 2.4.1(L) to find the general solution of the equation

$x^2y'' - 3xy' + 3y = 0 \quad (x \neq 0).$

2.4.4

Find the curve which passes through the point (1,4) with slope 2 and which satisfies the differential equation,

$$x^2y'' - 3xy' + 3y = 0.$$

2.4.5(L)

The family $y = c_1x + c_2x$ contains two arbitrary constants c_1 and c_2, and each member of the family satisfies the equation $x^2y'' - 3xy' + 3y = 0$. Explain why the given family is not the general solution of the given equation.

2.4.6

Find the general solution of $y'' + 7y' - 8y = 0$. (Verify that your solution is the general solution.)

2.4.7

Find the curve which passes through the point (0,1) with slope equal to -3, and which satisfies the equation $y'' - 8y' + 15y = 0$.

2.4.8(L)

Consider the equation

$$y''' - y' = 0.$$

a. Find the general solution.

b. Find the particular solution for which $y = 1$, $y' = 3$, and $y'' = 5$ when $x = 0$.

2.4.9(L)

With $L(y) = x^2y'' - 3xy' + 3y$, compute $L(e^x)$ to find a particular solution of $x^2y'' - 3xy' + 3y = e^x(x^2 - 3x + 3)$. Then use this result together with the result of Exercise 2.4.3 to find the general solution of this equation.

2.4.10 (Optional)

a. If a and b are given constants, show that the substitution $z = \ln x$ transforms the linear equation

$$x^2y'' + axy' + by = 0 \quad (x > 0)^*$$

into a linear equation (in which y is still the dependent variable, but z is the new independent variable) which has constant coefficients.

b. Use the result of part (a) to find the general solution of

$$x^2y'' - 3xy' + 3y = 0.$$

[Exercise 2.4.10 is optional in the sense that its content will be covered in other units in this block. It may, therefore, be omitted at this time without harm by the student who feels he has received enough of the message of this unit and wishes to continue with the next unit. On the other hand, for the student who has the time, this exercise introduces a very special class of linear equations which do not have constant coefficients but which can be reduced to equivalent equations with constant coefficients. The solution of this exercise also supplies additional drill and insight to the meaning of solving differential equations. Finally, for the student who likes things to be self-contained without the necessity for introducing cute "tricks," this exercise supplies a <u>direct</u> technique for solving Exercise 2.4.3. If you elect to omit this exercise at the present time, then do it as an additional exercise at the end of the next unit.]

The following optional exercise is designed to show how one may make a suitable change of variable which reduces a non-linear equation to a linear equation. In this way, an equation which we might not have been able to solve in its original form is replaced by one which we may know how to solve. The exercise is optional in the sense that it has no direct bearing on the technique of solving linear equations (which is the purpose of the present unit)

*The key point is that $x \neq 0$. If we allow $x < 0$, then the substitution $z = \ln x$ must be replaced by $z = \ln |x|$ otherwise our substitution is non-real.

but rather on applying the knowledge of linear equations toward solving a category of nonlinear equations. Moreover, as one will notice in going through the solution of this exercise, the application shown in this exercise is a bit on the "trick" side. Consequently, unless the student desires to see how a nonlinear equation may be reduced to a linear equation, there is nothing in this exercise which is at all necessary to any of the remaining discussion of linear differential equations.

2.4.11 (Optional)

a. Show that the substitution $z = \frac{y'}{y}$ replaces the equation $y'' + p(x)y' + q(x)y = 0$ by the equation $z^2 + z' + p(x)z + q(x) = 0$.

b. Use the result of part (a) to solve the first order nonlinear differential equation

$$z^2 + \frac{dz}{dx} - \frac{3}{x}z + \frac{3}{x^2} = 0 \quad (x \neq 0).$$