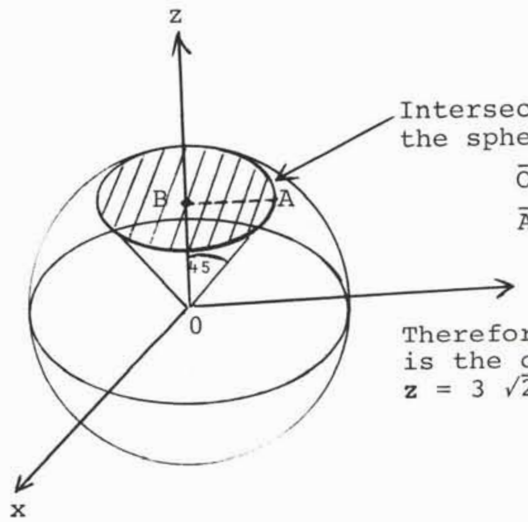
Unit 5: More on Polar Coordinates

5.5.1(L)

Knowing that $\rho = 6$ we may conclude that the point $P(\rho,\phi,\theta)$ is on the sphere centered at the origin with radius 6 (i.e., $\{(\rho,\phi,\theta): \rho = 6\}$ is the locus of all points in 3-space which are 6 units from the origin. This is why (r,ϕ,θ) are referred to as spherical coordinates; namely the sphere of radius 6 centered at the origin has the simple equation $\rho = 6$ in spherical coordinates.)

Knowing that $\phi = \frac{\pi}{4}$ * tells us that $P(\rho,\phi,\theta)$ must be the cone whose axis of symmetry is the z-axis and whose central angle is $\frac{\pi}{4}$ radians.

Thus, since $\rho = 6$ and $\phi = \frac{\pi}{4}$, $P(\rho,\phi,\theta)$ must be on both (i.e., the intersection) of the sphere and the cone; and this intersection is a circle parallel to the xy-plane. Pictorially,



Intersection of the cone and the sphere.

$\overline{OB} = 6 \cos 45° = 3\sqrt{2}$

$\overline{AB} = 6 \sin 45° = 3\sqrt{2}$

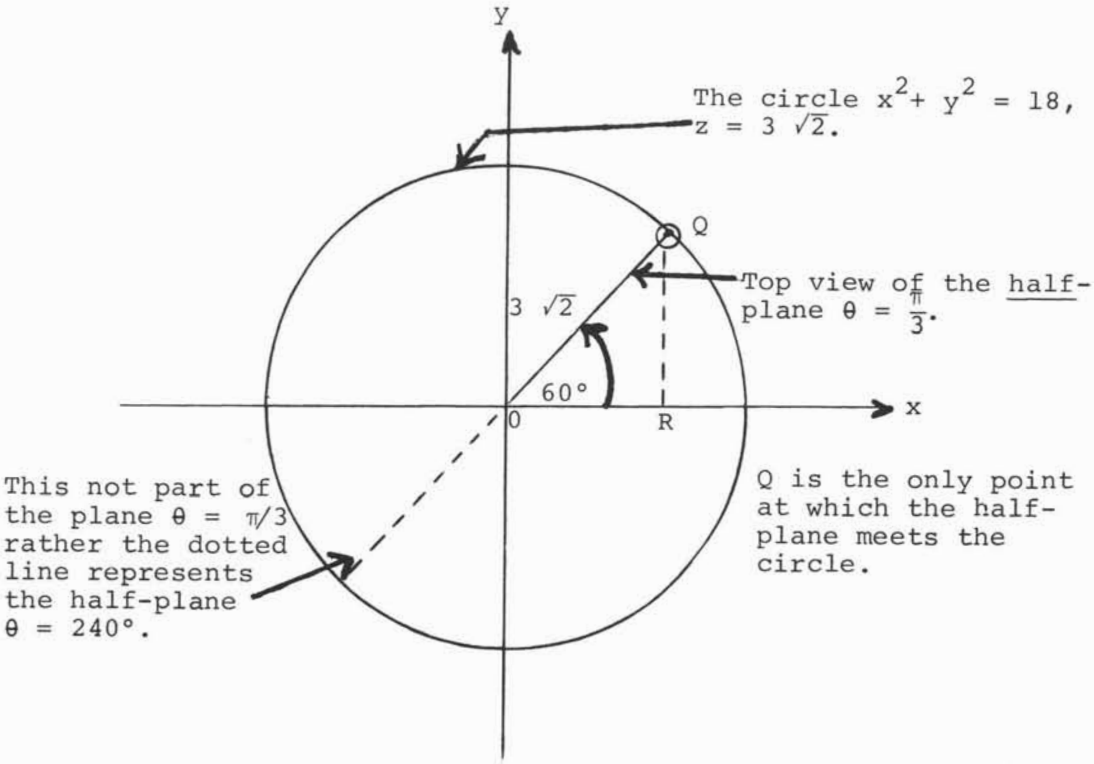
Therefore, curve of intersection is the circle $x^2 + y^2 = 18$
$z = 3\sqrt{2}$

Finally, given that $\theta = \frac{\pi}{3}$, the point is uniquely located on the above circle of intersection. In other words $\theta = \frac{\pi}{3}$ is a (half-)

* Since by definition ϕ is an angle (not a number), $\phi = \frac{\pi}{4}$ means $\phi = \frac{\pi}{4}$ radians.

5.5.1(L) continued

plane which intersects the above circle in exactly one point. Pictorially, looking "down" along the z-axis we have,



[Since $\overline{OQ} = 3\sqrt{2}$, $\vec{OR} = 3\sqrt{2}\cos 60° = \frac{3}{2}\sqrt{2}$ and $\overline{RQ} = 3\sqrt{2}\sin 60° = \frac{3}{2}\sqrt{6}$. Hence in Cartesian coordinates Q is the point $(\frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{6}, 3\sqrt{2})$].

The main point is that $\rho=\rho_o, \phi=\phi_o$ and $\theta = \theta_o$ uniquely represents the point, given in Cartesian coordinates by $(\rho_o\sin\phi_o\cos\phi_o, \rho_o\sin\phi_o\sin\phi_o, \rho_o\cos\phi_o)$.

Moreover, if we want no two different 3-triples (ρ,ϕ,θ) to denote the same point, we may require that

$$\begin{cases} \rho \geq 0 \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{cases}$$

5.5.1(L) continued

which is the convention adopted in the text.

Notice that in this context we have a deviation from our convention concerning 2-dimensional polar coordinates where r could be positive or negative and θ could be changed by multiples of 360°. What we are really saying is that if all we want is a coordinate system for locating points in space without worrying about equations of motion then we can restrict our coordinates in the way we have done. For example, in 2-space, if we simply wnat to locate a point without worrying about the equation of the curve satisfied by the point, we can obviously restrict r to being non-negative and we can restrict θ to the range

$$0 \leq \theta < 2\pi$$

in which case each point in the plane has a unique representation in the form (r, θ).

5.5.2

r = a means either the circle of radius a centered at (0,0) or the circular cylinder with that circle as cross-section depending on whether our domain (universe of discourse) is 2-space or 3-space. That is

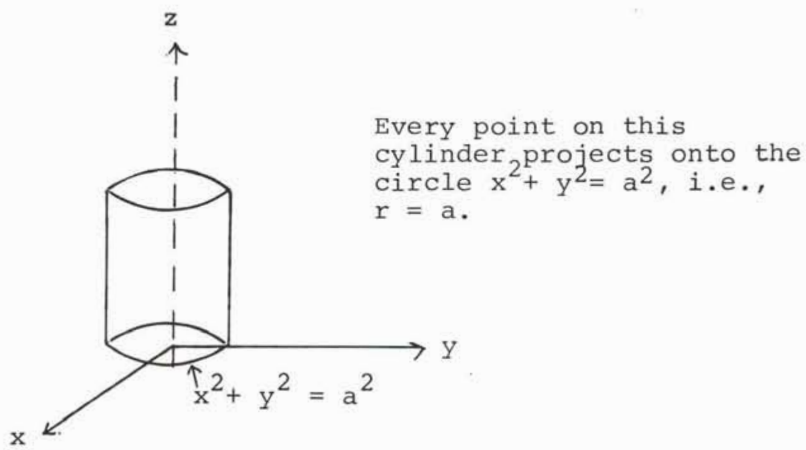
$$\{(r,\theta): r = a\}$$

is a circle, while

$$\{(r,\theta,z): r = a\}$$

is a cylinder.

Pictorially



5.5.2 continued

This is why (r,θ,z) is referred to as cylindrical coordinates. Namely, the equation of the above cylinder is simply $r = a$.

5.5.3

a. The fact that $\rho = 3$, locates us on the sphere centered at the origin with radius 3, while $\theta = \pi/4$ radians locates us on a (half-) plane (in particular the plane passing through the first quadrant portion of the line $y = x$, perpendicular to the xy-plane). Hence, if $\rho = 3$ and $\theta = \pi/4$ radians, we are on the intersection of the sphere and the plane which is the semicircle of radius 3 centered at (0,0,0), having the z-axis between (0,0,-3) and (0,0,3) as its diameter, and lying in the plane formed by the z-axis and half-line (ray) $y = x$, $x \geq 0$.

b. $\theta = \pi/4$ implies that we are in the half-plane $y = x$, $x \geq 0$. $\phi = \pi/4$ implies we are on the cone described in Exercise 5.5.1. Thus, with $\theta = \pi/4$ and $\phi = \pi/4$ we are on the intersection of the plane and the cone, which is a particular straight line through the origin.

[More specifically $\theta = \pi/4 \rightarrow \sin\theta = \cos\theta = \frac{1}{2}\sqrt{2}$ and $\phi = \pi/4$ $\sin\phi = \cos\phi = \frac{1}{2}\sqrt{2}$. Now

$$\left.\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}\right\}.$$

Hence

$$\left.\begin{aligned} x &= \frac{1}{2}\rho \\ y &= \frac{1}{2}\rho \\ z &= \frac{1}{2}\sqrt{2}\,\rho \end{aligned}\right\}$$

Therefore, $2x = 2y = 2z/\sqrt{2}$ $(=\rho)$ or $x = y = z/\sqrt{2}$. Therefore the line passes through (0,0,0) and is parallel to the vector $\vec{i} + \vec{j} + \sqrt{2}\,\vec{k}$].

5.5.4(L)

In spherical coordinates the sphere S is given by

$S = \{(\rho,\phi,\theta): 0 \leq \rho \leq 1,\ 0 \leq \phi \leq \pi,\ 0 \leq \phi \leq 2\pi\}.$

In Cartesian coordinates the required mass is given by

$$\iiint_S \sqrt{x^2+y^2+z^2}\, dV_S \qquad (1)$$

Since $\sqrt{x^2+y^2+z^2} = \rho$ we may find it helpful to convert (1) to spherical coordinates. This yields

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{1} \rho\left[\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right]d\rho \quad d\phi \quad d\theta. \qquad (2)$$

From the relations

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

we obtain that

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\phi & 0 \end{vmatrix}$$

$$= -\rho\sin\phi\sin\theta \begin{vmatrix} \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\sin\theta & -\rho\sin\phi \end{vmatrix}$$

$$-\rho\sin\phi\cos\theta \begin{vmatrix} \sin\phi\cos\theta & \cos\phi \\ \rho\cos\phi\cos\theta & -\rho\sin\phi \end{vmatrix}$$

$$= -\rho\sin\phi\sin\theta\,(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta) +$$
$$-\rho\sin\phi\cos\theta\,(-\rho\sin^2\phi\cos\theta - \rho\cos^2\phi\cos\theta)$$

5.5.4(L) continued

$$= \rho^2 \sin\phi \sin^2\theta\ (\sin^2\phi + \cos^2\phi)$$
$$+ \rho^2 \sin\phi \cos^2\theta\ (\sin^2\phi + \cos^2\phi)$$

$$= \rho^2 \sin\phi \sin^2\theta + \rho^2 \sin\phi \cos^2\theta$$

$$= \rho^2 \sin\phi\ (\sin^2\theta + \cos^2\theta)$$

$$= \rho^2 \sin\phi$$

so that (2) becomes

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{1} \rho^3 \sin\phi\ d\rho\ d\phi\ d\theta. \qquad (3)$$

[Notice that we could have obtained (2) by the geometric argument given in Section 16.8, but our method shows the more general use of the Jacobian]

At any rate, direct ingegration now yields

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^3 \sin\phi\ d\rho\ d\phi\ d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{4}\rho^4 \sin\phi \Big|_{\phi=0}^{1} d\phi\ d\theta$$

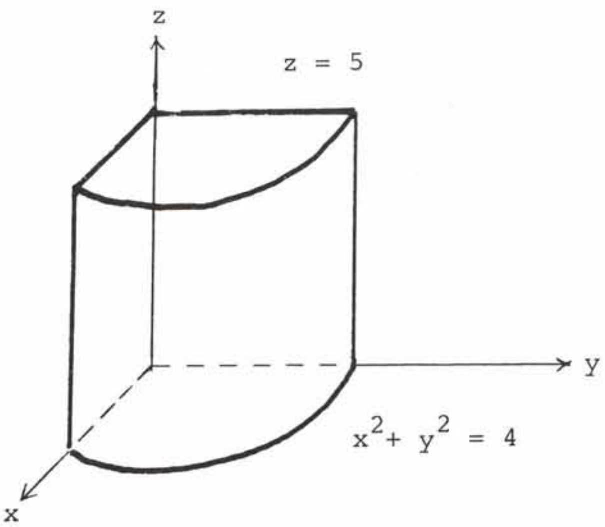
$$= \int_0^{2\pi}\left[\int_0^{\pi} \frac{1}{4} \sin\phi\ d\phi\right] d\theta$$

$$= \int_0^{2\pi} -\frac{1}{4} \cos\phi \Big|_{\phi=0}^{\pi} d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}[-1-1]\,d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi$$

5.5.5

Pictorially our region R is given by



1. For fixed r and θ z varies from 0 to 5.

2. $x^2 + y^2 = 4 \rightarrow r = 2$; therefore,

   $x^2 + y^2 \leq 4 \rightarrow 0 \leq r \leq 2$.

Therefore,

$$\iiint_R xz\,dz\,dy\,dx$$

$$= \int_0^{\frac{\pi}{2}}\int_0^2\int_0^5 (r\cos\theta)z\,\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\,dz\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^2\int_0^5 r^2\cos\theta\,z\,dz\,dr\,d\theta$$

$$= \left(\int_0^5 z\,dz\right)\left(\int_0^2 r^2\,dr\right)\left(\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\right)$$

$$= \frac{1}{2}z^2\Big|_0^5\;\frac{1}{3}r^3\Big|_0^2\;\sin\theta\Big|_0^{\frac{\pi}{2}}$$

$$= \left(\frac{25}{2}\right)\left(\frac{8}{3}\right)(1) = \frac{100}{3}.$$

5.5.6

In spherical coordinates R is defined by

$R = \{(\rho, \phi, \theta): a \le \rho \le b,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}.$

Hence,

$$\iiint_R z^2 dz\ dy\ dx$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_a^b (\rho \cos \phi)^2\ \rho^2 \sin \phi\ d\rho\ \ d\phi\ \ d\theta$$

$$= \left(\int_a^b \rho^4 d\rho\right)\left(\int_0^{\pi} \cos^2\phi\ \sin \phi\ d\phi\right)\ \left(\int_0^{2\pi} d\theta\right)$$

$$= [\tfrac{1}{5}\ \rho^5 \Big|_0^b\ ][\ -\ \tfrac{1}{3}\cos^3\phi \Big|_0^{\pi}\ ][\ 2\pi\ ]$$

$$= \frac{1}{5}\ (b^5 - a^5)\ (\tfrac{2}{3})\ (2\pi)$$

$$= \frac{4\pi}{15}(b^5 - a^5).$$

5.5.7(L)

Technically speaking, this exercise could have been given in Unit 3 of this Block, but we elect to do it here as a preliminary to a 3-dimensional change of variable problem which makes up the next exercise. Our main aim here is to show how we may often use more than one change of variables in the same exercise. In this exercise, the first "trick" is to "straighten out" the ellipse R and shape it as a circle. This is readily accomplished by the change of variables

$u = \frac{x}{a}$ and $v = \frac{y}{b}$. (1)

5.5.7(L) continued

That is, we map R onto $\underline{f}(R) = S$, where

$\underline{f}(x,y) = (u,v)$,

with u and v as in (1).

We also see from (1) that

$$\frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1} = \begin{vmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{vmatrix}^{-1} = \left(\frac{1}{ab}\right)^{-1} = ab$$

so that

$$\iint_R x^2y^2 dy\ dx = \iint_{\underline{f}(R)} (au)^2(bv)^2 \frac{\partial(x,y)}{\partial(u,v)}\ dv\ du$$

$$= \iint_S (ab)^3\ u^2v^2\ dv\ du \qquad (2)$$

where

$S = \{(u,v): u^2 + v^2 \leq 1\}$.

[That is,

$R = \{(x,y): \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\} = \{(au, bv): u^2 + v^2 \leq 1\}$,

therefore,

$\underline{f}(R) = \{\underline{f}(au,bv): u^2 + v^2 \leq 1\} = \{(\frac{au}{a}, \frac{bv}{b}): u^2 + v^2 \leq 1\}$

$= \{(u,v): u^2 + v^2 \leq 1\}$].

Now, since S (in the uv-plane) is a unit disc centered at the origin with radius 1, a switch to polar coordinates is desirable.

5.5.7 (L) continued

That is, we now map the uv-plane into the rθ-plane by the mapping

$$\underline{g}(u,v) = (r,\theta)$$

where

$$\left.\begin{aligned} u &= r\cos\theta \\ v &= r\sin\theta \end{aligned}\right\}. \qquad (3)$$

From (3) it follows that

$$\frac{\partial(u,v)}{\partial(r,\theta)} = r,$$

which checks with the geometrical interpretation that dv du = rdrdθ*.

At any rate, then,

$$\iint_S (ab)^3u^2v^2\,dv\,du = \iint_{\underline{g}(S)} (ab)^3(r\cos\theta)^2(r\sin\theta)^2\,r\,dr\,d\theta$$

$$= (ab)^3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^5\sin^2\theta\cos^2\theta\,dr\,d\theta$$

$$= a^3b^3\int_0^1 r^5dr\int_0^{2\pi}\sin^2\theta\cos^2\theta\,d\theta \qquad (4)$$

$$= a^3b^3\ \frac{1}{6}r^6\Big|_{r=0}^{1}\left[\int_0^{2\pi}\frac{1}{4}\sin^2 2\theta\,d\theta\right]$$

*Do not be confused by our being in the uv-plane rather than the xy-plane. After all, the uv-plane is a replica of the xy-plane (i.e., the planes are the same but the mapping "scrambles" the points). Thus, switching to polar coordinates in the uv-plane is structurally the same as it was when we dealt with the xy-plane.

5.5.7(L) continued

$$= \frac{a^3b^3}{24} \int_0^{2\pi} \frac{1 + \cos 4\theta}{2}\, d\theta$$

$$= \frac{a^3b^3}{48} \left[\theta + \frac{1}{4} \sin 4\theta \Big|_{\theta=0}^{2\pi} \right]$$

$$= \frac{\pi a^3b^3}{24}.$$

[Note: Had the problem been $\iint_R xy\, dy\, dx$, in place of equation (4) we would have had

$$a^2b^2 \int_0^1 r^3 dr \int_0^{2\pi} \sin\theta \cos\theta\, d\theta$$

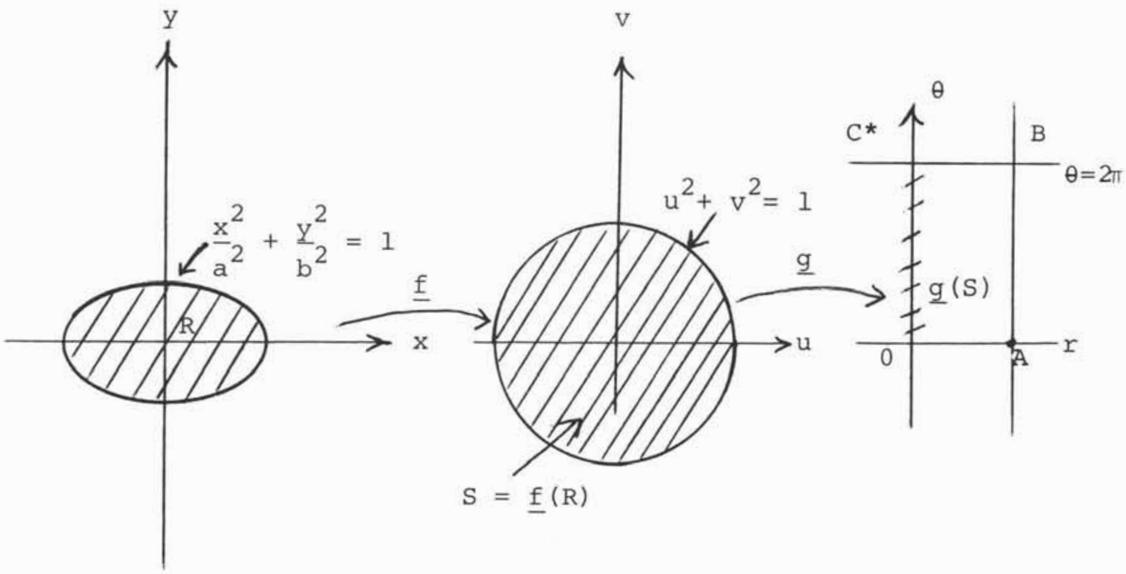
(details are left to you). Notice that

$$\int_0^{2\pi} \sin\theta \cos\theta\, d\theta = \frac{1}{2} \sin^2\theta \Big|_{\theta=0}^{2\pi} = 0.$$

In other words $\iint_R xy\, dy\, dx = 0$. The reason for this is that the sign of xy is positive in the first and third quadrants, and negative in the second and fourth quadrants so that by symmetry the integrals cancel. In still other words $\iint_R f(x,y)\, dA_R$ is a "net volume" if f(x,y) is not always positive.]

5.5.7(L) continued

Pictorially, what we have done is:



Aside:
We clearly can view the composite mapping shown in our diagram as a single mapping from the xy-plane into the rθ-plane. Analytically this corresponds to the fact that we may substitute

$$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$$

into

$$\begin{cases} x = au \\ y = bv \end{cases}$$

*Again notice that OC is not part of g(S) if g is 1-1. However, the area of g(S) is not affected by the inclusion (or exclusion) of OC. In this same vein, one often reformulates this entire exercise by saying that R is the interior of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

the point being that the boundary of the ellipse contributes nothing to $\iint_R x^2y^2\,dy\,dx$.

5.5.7(L) continued

to obtain

$x = ar \cos \theta$
$y = br \sin \theta$

(and this is a parametric equation for the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$).

As a final note on this exercise, notice that we could have evaluated the given integral without resorting to any change of variables, but certainly the change of variables converted the given double integral into a form that was much easier to evaluate than was the given integral. As the number of variables increases, it becomes even more important from a computational point of view to make the type of change of variables discussed in this exercise. We shall see this in more detail in the next exercise.

5.5.8

$R = \{(x,y,z): \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1;\ x,y,z \leq 0\}.$

We first map R onto $\underline{f}(R) = S$ by mapping $\underline{f}(x,y,z) = (u,v,w)$ where

$$u = \frac{x}{a},\ v = \frac{y}{b},\ w = \frac{z}{c} \qquad (1)$$

[i.e., we make the change of variables given by (1)]

Therefore $x = au$, $y = bv$, $z = cw$ and

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

Hence,

$$\iiint_R xyz\ dz\ dy\ dx = \iiint_{\underline{f}(R)} (au)(bv)(cw)\left[\frac{\partial(x,y,z)}{\partial(u,v,w)}\right] dw\ dv\ du$$

5.5.8 continued

$$= \iiint_S (abc)^2 uvw\ dw\ dv\ du \qquad (2)$$

where $S = \{(u,v,w): u^2 + v^2 + w^2 \leq 1;\ u,v,w, \geq 0\}$.

Thus, in uvw-space S is the solid <u>sphere</u> of radius 1 centered at the origin, and this suggests spherical coordinates. In other words, in the language of mappings, we map S onto $\underline{g}(S)$ by

$$\underline{g}(u,v,w) = (\rho,\phi,\theta)$$

where

$$\left.\begin{aligned} u &= \rho \sin\phi \cos\theta \\ v &= \rho \sin\phi \sin\theta \\ w &= \rho \cos\phi \end{aligned}\right\}$$

(and $0 \leq \phi \leq \frac{\pi}{2}$, $0 \leq \theta \leq \frac{\pi}{2}$ since u,v, and w are all non-negative) so that

$$\frac{\partial(u,v,w)}{\partial(\rho,\phi,\theta)} = \rho^2 \sin\phi.$$

This leads to

$$\iiint_S (abc)^2\ uvw\ dw\ dv\ du$$

$$= \iiint_{\underline{g}(S)} (abc)^2(\rho \sin\phi \cos\theta)(\rho\sin\phi \sin\theta)(\rho\cos\phi)[\rho^2\sin\phi]d\rho\ d\phi\ d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_{\rho=0}^{1} (abc)^2\rho^5\sin^3\phi\ \cos\phi\ \sin\theta\ \cos\theta\ d\rho\ \ d\phi\ \ d\theta$$

5.5.8 continued

$$= (abc)^2 \int_0^1 \rho^5 d\rho \int_0^{\frac{\pi}{2}} \sin^3\phi \ \cos\phi \ d\phi \int_0^{\frac{\pi}{2}} \sin\theta \cos\theta \ d\theta$$

$$= (abc)^2 \ \frac{1}{6} \rho^6 \Big|_{\rho=0}^{1} \frac{1}{4} \sin^4\phi \Big|_{\phi=0}^{\frac{\pi}{2}} \frac{1}{2} \sin^2\theta \Big|_{\theta=0}^{\frac{\pi}{2}}$$

$$= \frac{a^2b^2c^2}{48}$$

Comparing (2) with (3) we see that

$$\iiint_R xyz \ dz \ dy \ dx = \frac{a^2b^2c^2}{48} \qquad (3)$$

where $R = \{(x,y,z): \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$

[Notice that if $\overline{R}$ is the entire ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ then xyz is negative in four octants and positive in the other four octants. Consequently $\iiint_R xyz \ dzdydx = 0$, by symmetry. However, by the same symmetry, we can use equation (3) to deduce that

$$\iiint_{\overline{R}} |xyz| dz \ dy \ dx = 8 \iiint_R |xyz| dz \ dy \ dx = 8 \iiint_R xyz \ dz \ dy \ dx$$

$$= \frac{a^2b^2c^2}{6} \ ]$$