## Unit 4: The Dot Product

### 1.4.1(L)

#### Comment

While we wait until the final part of this exercise to have you supply the various details, the main aim of this exercise is to emphasize that, since certain rules for numerical products are not obeyed by dot products, there is no reason to suspect that the two types of multiplication will behave exactly alike. In particular, in numerical arithmetic we may conclude that if ab = ac and a $\neq$ 0 then b = c. What we are having you show in part (a) is that if $\vec{A} \neq \vec{0}$ there are a number of (in fact, infinitely many) vectors $\vec{C}$ such that $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$. This shows that the cancellation law is different for dot products and we cannot go around blithely cancelling $\vec{A}$'s in an expression such as $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$, since it need not be true that $\vec{B} = \vec{C}$.

At any rate, tackling this exercise with the parts in the given order, we have:

a. Introducing the abbreviation (a,b,c) to denote the vector $a\vec{i} + b\vec{j} + c\vec{k}$, we have

$$\vec{A} = (1,1,1)$$
$$\vec{B} = (2,3,4)$$
$$\vec{C} = (x,y,z)$$

We desire that

$$\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$$

Since we are dealing in Cartesian coordinates, our recipe for finding dot products yields

$$\vec{A}\cdot\vec{B} = (1,1,1) \cdot (2,3,4) = (1)(2) + (1)(3) + (1)(4) = 9 \qquad (1)$$

$$\vec{A}\cdot\vec{C} = (1,1,1) \cdot (x,y,z) = (1)x + (1)y + (1)z = x + y + z \qquad (2)$$

1.4.1(L) continued

From (1) and (2) we see that $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$ if and only if

$$x + y + z = 9 \tag{3}$$

Since (3) is a linear equation in three unknowns, we have two degrees of freedom in determining $\vec{C}$. In other words, we can pick any two of its components at random and solve uniquely for its third component with equation (3).

For example, if we wish to have x = 5 and y = 7, then equation (3) tells us that z = -3. In this case, $\vec{C} = (5,7,-3)$. As a check

$$\vec{A}\cdot\vec{B} = (1,1,1)\cdot(2,3,4) = 9$$
$$\vec{A}\cdot\vec{C} = (1,1,1)\cdot(5,7,-3) = 5 + 7 - 3 = 9$$
Therefore, $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$

More generally, we could use (3) to show that z = 9 - x - y. Then letting $\vec{C} = (x,y,9-x-y)$, we have that

$$\vec{A}\cdot\vec{C} = (1,1,1)\cdot(x,y,9-x-y) = x + y + (9-x-y) = 9,$$

and comparing this with equation (1) we see that $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$ as asserted.

b. We have $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$, whence

$$\vec{A}\cdot\vec{B} - \vec{A}\cdot\vec{C} = 0 \tag{4}$$

Hence, by the distributive property of the dot product, equation (4) becomes

$$\vec{A}\cdot(\vec{B} - \vec{C}) = 0^* \tag{5}$$

*For the rigor-oriented reader, what we really are doing is

$\vec{A}\cdot\vec{B} - \vec{A}\cdot\vec{C} = \vec{A}\cdot\vec{B} + (-\vec{A})\cdot\vec{C}$
$= \vec{A}\cdot\vec{B} + (-1)\vec{A}\cdot\vec{C}$ (since $-\vec{A} = (-1)\vec{A}$)
$= \vec{A}\cdot\vec{B} + \vec{A}\cdot(-1)\vec{C}$ (since $m\vec{A}\cdot\vec{C} = \vec{A}\cdot m\vec{C}$ for any scalar m)
$= \vec{A}\cdot\vec{B} + \vec{A}\cdot(-\vec{C})$ (since $(-1)\vec{C} = -\vec{C}$)
$= \vec{A}\cdot(\vec{B} + [-\vec{C}])$ (by the "usual" distributive rule)
$= \vec{A}\cdot(\vec{B} - \vec{C})$ (by the definition of $\vec{B} - \vec{C}$)

1.4.1(L) continued

Since the dot product of two vectors implies that either one of the vectors is $\vec{0}$ or the two vectors are perpendicular, we may conclude from (5) that

(1) $\vec{A} = \vec{0}$

or

(2) $\vec{B} - \vec{C} = \vec{0}$, i.e., $\vec{B} = \vec{C}$

or

(3) $\vec{A} \perp \vec{B} - \vec{C}$

In part (a) we chose $\vec{A}$ so that $\vec{A} \neq \vec{0}$ and we required that $\vec{B} \neq \vec{C}$. Hence, it must be that $\vec{B} - \vec{C}$ is perpendicular to $\vec{A}$. As a check, notice that

$$\vec{B} - \vec{C} = (2,3,4) - (x,y,9-x-y)$$
$$= (2-x,\ 3-y,\ -5+x+y)$$

Therefore,

$$\vec{A}\cdot(\vec{B} - \vec{C}) = (1,1,1) \cdot (2-x,\ 3-y,\ -5+x+y)$$
$$= 2-x + 3-y + (-5+x+y) = 0$$

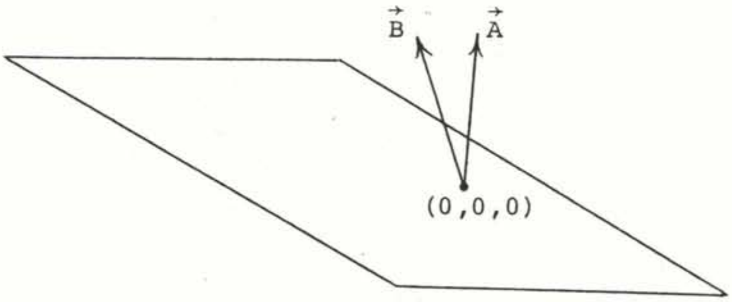
Therefore,

$$\vec{A} \perp (\vec{B} - \vec{C})$$

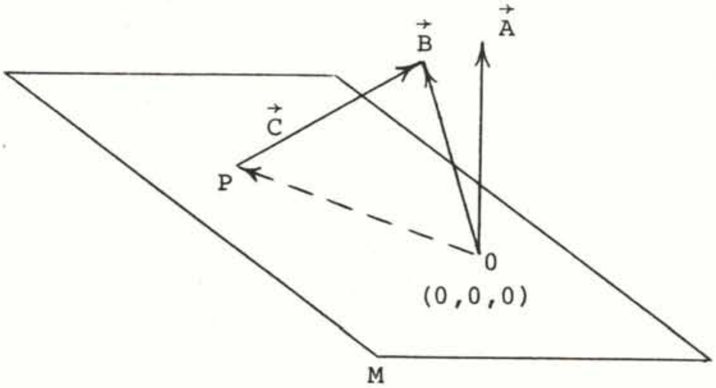
c. Since we have already agreed that vectors do not depend on their point of origin, let us assume, without loss of generality, that $\vec{A}$ and $\vec{B}$ originate at a common point, the origin.

Now, the locus of all vectors $\vec{V}$ such that $\vec{A}\cdot\vec{V} = 0$ is the plane to which $\vec{A}$ is perpendicular. (Again, for the sake of uniformity, we are assuming that all vectors $\vec{V}$ start at the origin.) We then have, pictorially,

1.4.1(L) continued




If we denote this plane by M, $\vec{C}$ may be any vector which originates in the plane M and terminates at the head of B. Again, pictorially,



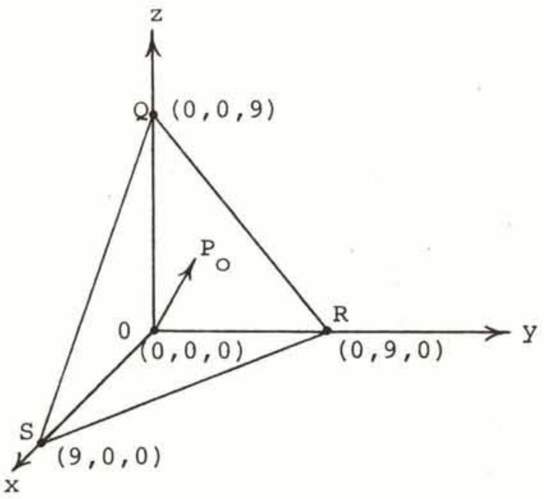

(P is any point in M
Therefore, $\vec{OP}$ lies in M
Therefore, $\vec{OP} \perp A$
But, $\vec{OP} + \vec{C} = \vec{B} \rightarrow \vec{OP} = \vec{B} - \vec{C}$. Therefore,
$\vec{A} \perp (\vec{B} - \vec{C})$ Therefore,
$\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$)

In summary, then, to find all $\vec{C}$ such that $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$, we construct the plane to which $\vec{A}$ is perpendicular. Then, any vector $\vec{C}$ which originates in this plane and terminates at the head of $\vec{B}$ satisfies the requirement.

In terms of part (a), in this particular problem, if each $\vec{C}$ which works is placed at (0,0,0) it will terminate in the <u>plane</u>, whose equation, in fact, is $x + y + z = 9$.

To see this more visually the diagram below shows the plane $x + y + z = 9$ in the first octant $(x,y,z \geqslant 0)$. Notice that to find where the plane intersects, say, the x-axis, we recall that this point is of the form (x,0,0) and hence we solve $x + y + z = 9$ with $y = z = 0$. We obtain $x = 9$.

1.4.1(L) continued




(1) $\Delta SQR$ represents the portion of the plane $x + y + z = 9$ which is in the first octant.

(2) Summarizing our preceding remarks, pick any point in $x + y + z = 9$, say, $P_o$. Then $\vec{OP_o}$ is a $\vec{C}$ which satisfies $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$ with $\vec{A}$ and $\vec{B}$ as given in the exercise.

As a more computational check, we have:

If $S(x_1,y_1,z_1)$ is in the plane perpendicular to $\vec{A} = (1,1,1)$ then $\vec{A}\cdot\vec{OS} = 0$. Since $\vec{OS} = (x_1,y_1,z_1)$ we have

$$(1,1,1) \cdot (x_1,y_1,z_1) = 0$$

or

$$x_1 + y_1 + z_1 = 0 \tag{6}$$

Equation (6) tells us that $(x_1,y_1,z_1)$ belongs to M if $x_1 + y_1 + z_1 = 0$, or $z_1 = -(x_1 + y_1)$.

So suppose $(x_1,y_1,z_1)$ does belong to M. Then $z_1 = -(x_1 + y_1)$. Hence, S has the form $(x_1,y_1,-x_1 - y_1)$. Then since B starts at 0 and terminates at (2,3,4) we have that

$$\vec{C} = \vec{SB} = (2-x_1,\ 3-y_1,\ 4+x_1 + y_1) \tag{7}$$

1.4.1(L) continued

From (7) we see that the sum of the components of $\vec{C}$ is

$$(2-x_1) + (3-y_1) + (4+x_1 + y_1) = 9$$

and this is in agreement with equation (3) with $x = 2-x_1$, $y = 3-y_1$, and $z = 4+x_1 + y_1$.

Hopefully, we now have a correlation between the arithmetic answer and the geometric construction which yields this answer.

d. One fringe benefit of the first three parts of this exercise is that not only has it supplied more drill in vector arithmetic but it has also made it seem quite natural in vector arithmetic that we can have $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$ without either $\vec{A} = \vec{0}$ or $\vec{B} = \vec{C}$.

The purpose of this part of the exercise is to once again emphasize the crucial concept of structure. Recall that, in ordinary arithmetic, if a, b, and c are numbers and if ab = ac, then either a = 0 or b = c. This result follows inescapably from the properties of multiplication of numbers. The dot product has very different properties from the "ordinary" product. To be sure the dot product is commutative ($\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$) and it obeys the distributive property of multiplication "over" addition; that is, $\vec{A}\cdot(\vec{B} + \vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$, but here the resemblance ends. For one thing, the dot product isn't even a binary operation (i.e., it is not closed) since by use of it we combine two vectors to form a scalar. For another thing, the dot product is not associative. Indeed not only is it not true that $\vec{A}\cdot(\vec{B}\cdot\vec{C}) = (\vec{A}\cdot\vec{B})\cdot\vec{C}$ but even more crucially, the expressions $\vec{A}\cdot(\vec{B}\cdot\vec{C})$ and $(\vec{A}\cdot\vec{B})\cdot\vec{C}$ are not even defined, since, for example, $\vec{A}\cdot\vec{B}$ is a number and we do not "dot" a number (scalar) with a vector. Thirdly, there is no analog of a multiplicative identity. Namely, we cannot dot a vector with any vector to obtain the original vector as the product since the dot product of two vectors is not a vector. Finally, the concept of multiplicative inverses is also lacking for dot products. Namely, not only is there no identity vector with respect to dot products, but even if there were, it would not be possible, as we have already said several times in this paragraph, to dot two vectors to obtain a vector.

1.4.1(L) continued

Among other things we now see how convenient it is when we deal with binary operations, for as soon as we combine "like things" to form "unlike things" it often becomes impossible to combine more than two like things. For example, in the present illustration we could not talk meaningfully about the dot product of three (or more) vectors even though the concept was unambiguously defined for two vectors.

In any event, then, the key thought is that the "cancellation law" for dot products does not contradict the "cancellation law" for "ordinary" products since the two types of products have different properties - and hence possess different inescapable conclusions based on these properties.* Another way of saying this is that we do not call different conclusions contradictory unless they both follow inescapably from the same set of assumptions (properties).

*A Note on Fractions Involving Dot Products

In ordinary arithmetic if $a \neq 0$ then $\frac{ab}{ac} = \frac{b}{c}$. This result was dependent on several properties of numerical arithmetic. For example, let

$$\frac{ab}{ac} = x \tag{1}$$

Then,

$$ab = (ac)x$$

But, by the associative property $(ac)x = a(cx)$. Therefore,

$$ab = a(cx),$$

and since $a \neq 0$ it follows that

$$b = cx$$

or

Note continued

$$x = \frac{b}{c} \tag{2}$$

Comparing (1) and (2) we see that $\frac{ab}{ac} = \frac{b}{c}$ if $a \neq 0$. Suppose now we have

$$\frac{\vec{A}\cdot\vec{B}}{\vec{A}\cdot\vec{C}} \tag{3}$$

Certainly we might feel tempted to cancel on $\vec{A}$ from both the numerator and denominator in (3). If we did this we would obtain

$$\frac{\vec{B}}{\vec{C}} \tag{4}$$

Well, to begin with, in terms of the properties of vector arithmetic, it is not valid to cancel since we have seen in the present exercise that the cancellation law, in its usual form, does not apply.

Secondly, we would like to observe that the expression $\frac{\vec{B}}{\vec{C}}$ is not defined!

For example, within the proper context of our remarks how should $\frac{\vec{B}}{\vec{C}}$ be interpreted? To capture the flavor of division being the inverse of multiplication and that here multiplication refers to a dot product, we would require that the definition of $\frac{\vec{B}}{\vec{C}}$ be

$$\vec{C}\cdot\left(\frac{\vec{B}}{\vec{C}}\right) = \vec{B} \tag{5}$$

if the meaning of division here is to be analogous with its meaning in numerical arithmetic.

But (5) hardly makes sense since the left side of the equality is a number and the right side is a vector.

Notice that this problem did not occur with scalar multiplication since $\frac{\vec{A}}{m}$ could be defined by

Note continued

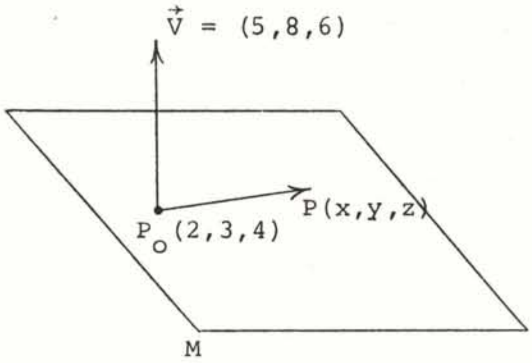
$$m\left(\frac{\vec{A}}{m}\right) = \vec{A} \qquad (6)$$

and equation (6) is meaningful since a scalar multiplied by a vector is a vector. Indeed (6) says that $\frac{\vec{A}}{m}$ is that vector which when scaled to m times its length (if m > 0) yields the vector $\vec{A}$.

Mechanically, the key point is to view $\vec{A}\cdot\vec{B}$ as a single number and thus that the $\vec{A}$ and $\vec{B}$ are inseparable in the expression. In this way we will not (hopefully) be tempted to "reduce" $\frac{\vec{A}\cdot\vec{B}}{\vec{A}\cdot\vec{C}}$. (Perhaps this is more analogous to the fact that in ordinary fractions we cannot reduce $\frac{a+b}{a+c}$ to $\frac{b}{c}$ even though a is in both numerator and denominator.)

1.4.2

a. We are given that $P_o(2,3,4)$ is in our plane M and that $\vec{V} = (5,8,6)$ is perpendicular to our plane. We place $\vec{V}$ so that it originates at $P_o$. Thus,




Now the point P(x,y,z) is in the plane if and only if $\vec{V} \perp \overrightarrow{P_oP}$, and this in turn requires that $\vec{V}\cdot\overrightarrow{P_oP} = 0$. That is

$$(5,8,6) \cdot (x-2,\ y-3,\ z-4) = 0$$

or

$$5(x-2) + 8(y-3) + 6(z-4) = 0$$

1.4.2 continued

Therefore,

$\underline{5x + 8y + 6z = 58}$ (1)

b. When x = y = 1, equation (1) yields

$13 + 6z = 58$

or

$6z = 45$

or

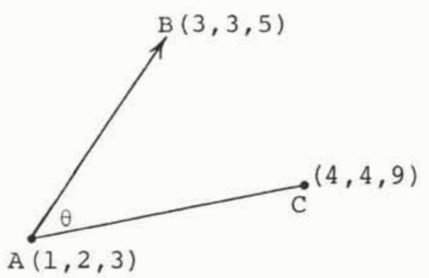
$z = 7\frac{1}{2}$

Therefore,

$(1,1,7\frac{1}{2})$

is in M. Since $7\frac{1}{2} < 8$, it follows that (1,1,8) is above the plane.

1.4.3




$\vec{AB}\cdot\vec{AC} = |\vec{AB}|\ |\vec{AC}|\cos\theta$ (1)

1.4.3 continued

Now

$\vec{AB} = (3-1,\ 3-2,\ 5-3) = (2,1,2)$

Therefore,

$|\vec{AB}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

and

$\vec{AC} = (4-1,\ 4-2,\ 9-3) = (3,2,6)$

Therefore,

$|\vec{AC}| = \sqrt{3^2 + 2^2 + 6^2} = 7$

Putting this into (1) yields

$(2,1,2) \cdot (3,2,6) = (3)(7) \cos \theta$

$6 + 2 + 12 = 21 \cos \theta$

$20 = 21 \cos \theta$

$\cos \theta = \frac{20}{21}$

Therefore,

$\theta = \text{arc} \cos \frac{20}{21} = \cos^{-1} \frac{20}{21}$

(the angle whose cosine is $\frac{20}{21}$).

We may look this up in the tables if we wish to conclude

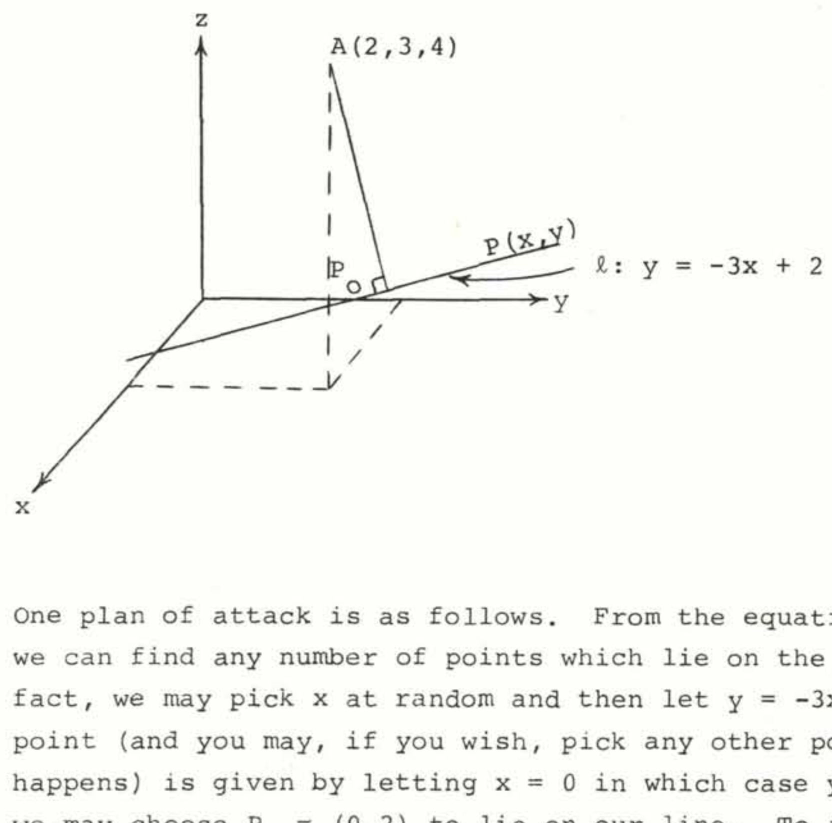
$\cos \theta \approx 0.952$

Therefore,

$\theta \approx 17.8°$

1.4.4(L)

Strictly speaking, it is our belief that from a computational point of view there is no need for this to be labeled a learning exercise. Yet, we would like to create an opportunity to emphasize once again how vector methods can be used to solve a non-vector problem. In this case, we are given a line in the xy-plane and a point not in the xy-plane (in particular, then, the point is not on the given line). From our knowledge of Euclidean geometry we know that there is one and only one line that can be drawn from the given point perpendicular to the given line. This exercise asks us to find the coordinates of the point at which the perpendicular from the point intersects the given line. Pictorially,




One plan of attack is as follows. From the equation $y = -3x + 2$ we can find any number of points which lie on the given line. In fact, we may pick x at random and then let $y = -3x + 2$. One such point (and you may, if you wish, pick any other point and see what happens) is given by letting $x = 0$ in which case $y = 2$. Therefore, we may choose $P_o = (0,2)$ to lie on our line. To indicate that we are involved in three-space we will write this as (0,2,0) rather than as (0,2). Clearly this involves no loss of generality. The point P we seek is characterized by the fact that $\vec{PA}$ must be perpendicular to $\vec{PP_o}$. If we now elect to introduce the language of vectors, we are saying that $\vec{PA}\cdot\vec{PP_o} = 0$. The computation now

1.4.4(L) continued

becomes quite simple. We first recognize that the general point P on our line is given by (x, -3x + 2, 0), whence

$$\vec{PA} = (2-x, 3 - [-3x + 2], 4-0) = (2-x, 1 + 3x, 4)$$

while

$$\vec{PP_0} = (0-x, 2 - [-3x + 2], 0-0) = (-x, 3x, 0).$$

Hence,

$$\begin{aligned} \vec{PA}\cdot\vec{PP_0} &= (2-x, 1 + 3x, 4) \cdot (-x, 3x, 0) \\ &= (2-x)(-x) + (1 + 3x)(3x) + 4(0) \\ &= -2x + x^2 + 3x + 9x^2 + 0 \\ &= 10x^2 + x = x(10x + 1) \qquad (1) \end{aligned}$$

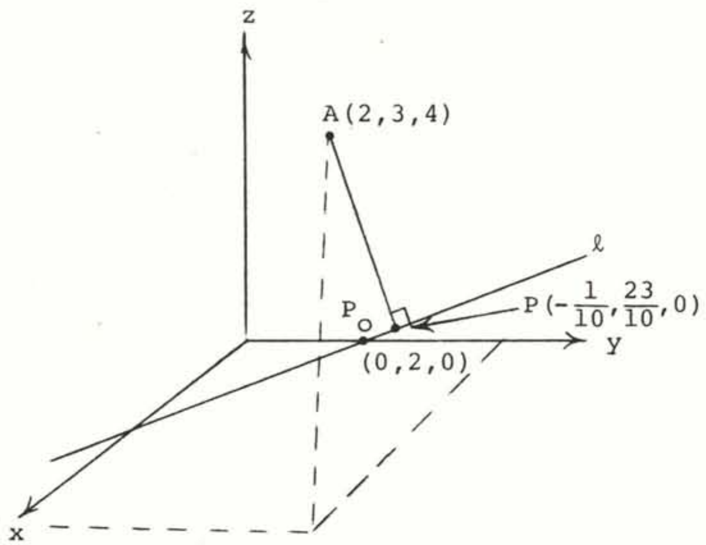
Hence, from (1), $\vec{PA}\cdot\vec{PP_0} = 0$ if and only if x = 0 or 10x + 1 = 0. If x = 0 then y = -3x + 2 = 2, and we obtain P = (0,2) which is the same as $P_0$ and in this case we do not have three distinct points, A, P, and $P_0$. (In other words the fact that $\vec{PA}\cdot\vec{PP_0} = 0$ does not preclude that $\vec{PP_0} = 0$. To be consistent with the fact that $\vec{A}\cdot\vec{B} = 0$ means $\vec{A}$ and $\vec{B}$ are orthogonal, one often <u>defines</u> the $\vec{0}$-vector to be orthogonal to every vector.)

If 10x + 1 = 0 then x = -1/10, whereupon y = -3x + 2 yields that y = 23/10. In other words, the line from A perpendicular to the given line meets that line at the point (-1/10, 23/10).

(As a final note on this technique, observe that we might have been lucky and picked our point $P_0$ in such a way that it turned out to be the point at which the perpendicular from A met our line. Had this happened, algebraically, the quadratic equation resulting from $\vec{PA}\cdot\vec{PP_0} = 0$ would have had the quadratic term vanish and there would have been only the one solution to equation (1). Anyone who is interested can work this exercise choosing $P_0$ to be (-1/10, 23/10) and see that this does indeed happen.)

1.4.4(L) continued

Pictorial Summary




1.4.5(L)

Our immediate aim in this exercise is to show structurally that carrying out the operation

$$(a_1\vec{v}_1 + a_2\vec{v}_2) \cdot (b_1\vec{v}_1 + b_2\vec{v}_2)$$

is the same as in "ordinary" numerical multiplication. That is, using our rules for vector arithmetic, we can show that

$$(a_1\vec{v}_1 + a_2\vec{v}_2) \cdot (b_1\vec{v}_1 + b_2\vec{v}_2) = a_1b_1\vec{v}_1^{\,2} + (a_2b_1 + a_1b_2)\vec{v}_1 \cdot \vec{v}_2 + a_2b_2\vec{v}_2^{\,2}$$

(where we are using $\vec{v}_1^{\,2}$ to abbreviate $\vec{v}_1 \cdot \vec{v}_1$ and $\vec{v}_2^{\,2} = \vec{v}_2 \cdot \vec{v}_2$). Without the details of a formal mathematical proof, we proceed as follows:

1.4.5(L) continued

$(a_1\vec{v}_1 + a_2\vec{v}_2) \cdot (b_1\vec{v}_1 + b_2\vec{v}_2) =$

$(a_1\vec{v}_1 + a_2\vec{v}_2) \cdot (b_1\vec{v}_1) + (a_1\vec{v}_1 + a_2\vec{v}_2) \cdot (b_2\vec{v}_2) =$

$(a_1\vec{v}_1) \cdot (b_1\vec{v}_1) + (a_2\vec{v}_2) \cdot (b_1\vec{v}_1) + (a_1\vec{v}_1) \cdot (b_2\vec{v}_2) + (a_2\vec{v}_2) \cdot (b_2\vec{v}_2) =$

$a_1b_1(\vec{v}_1 \cdot \vec{v}_1) + a_2b_1(\vec{v}_2 \cdot \vec{v}_1) + a_1b_2(\vec{v}_1 \cdot \vec{v}_2) + a_2b_2(\vec{v}_2 \cdot \vec{v}_2) =$

$a_1b_1\vec{v}_1^{\,2} + (a_2b_1 + a_1b_2)(\vec{v}_1 \cdot \vec{v}_2) + a_2b_2\vec{v}_2^{\,2}$ (1)

The point is that (1) always holds. The more convenient result that we're used to, namely

$(a_1\vec{i} + a_2\vec{j}) \cdot (b_1\vec{i} + b_2\vec{j}) = a_1b_1 + a_2b_2$

was valid because $\vec{i}\cdot\vec{j} = 0$ and $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = 1$.

More generally, a glance at equation (1) shows that this "short cut" will prevail whenever $\vec{v}_1 \cdot \vec{v}_2 = 0$ and $\vec{v}_1^{\,2} = \vec{v}_2^{\,2} = 1$, for clearly, in this case, equation (1) becomes

$(a_1\vec{v}_1 + a_2\vec{v}_2) \cdot (b_1\vec{v}_1 + b_2\vec{v}_2) = a_1b_1 + a_2b_2$ (2)

What we want to "hammer home" in this exercise is that while equation (1) always holds, equation (2) is a special case, in particular, equation (2) will yield an incorrect answer if $\vec{v}_1 \cdot \vec{v}_2 \neq 0$ or $\vec{v}_1^{\,2} \neq 1$ or $\vec{v}_2^{\,2} \neq 1$.

In our present exercise

(1) $\vec{v}_1^{\,2} \equiv \vec{v}_1 \cdot \vec{v}_1 = (3,2,5) \cdot (3.2,5) = 3^2 + 2^2 + 5^2 = 38$

(2) $\vec{v}_2^{\,2} \equiv \vec{v}_2 \cdot \vec{v}_2 = (2,7,3) \cdot (2,7,3) = 2^2 + 7^2 + 3^2 = 62$

(3) $\vec{v}_1 \cdot \vec{v}_2 = (3,2,5) \cdot (2,7,3) = 3(2) + 2(7) + 5(3) = 35$

1.4.5(L) continued

By equation (1), therefore,

$$(3\vec{v}_1 + 4\vec{v}_2) \cdot (4\vec{v}_1 - 3\vec{v}_2) = 12\vec{v}_1^{\,2} + 7\vec{v}_1 \cdot \vec{v}_2 - 12\vec{v}_2^{\,2*}$$

$$= 12(38) + 7(35) - 12(62)$$

$$= -43$$

(The answer $(3\vec{v}_1 + 4\vec{v}_2) \cdot (4\vec{v}_1 - 3\vec{v}_2) = 0$ would have been true had $\vec{v}_1 \cdot \vec{v}_2 = 0$ and $\vec{v}_1^{\,2} = \vec{v}_2^{\,2} = 1$, but this was not the case here.)

An alternative method would have been to write all vectors in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$. Then

$$\vec{A} = 3\vec{v}_1 + 4\vec{v}_2 = 3(3,2,5) + 4(2,7,3) = (17,34,27) = 17\vec{i} + 34\vec{j} + 27\vec{k}$$

$$\vec{B} = 4\vec{v}_1 - 3\vec{v}_2 = 4(3,2,5) - 3(2,7,3) = (6,-13,11) = 6\vec{i} - 13\vec{j} + 11\vec{k}$$

In this form (since $\vec{i}^{\,2} = \vec{j}^{\,2} = \vec{k}^{\,2} = 1$ and $\vec{i}\cdot\vec{j} = \vec{i}\cdot\vec{k} = \vec{k}\cdot\vec{j} = 0$) we have

$$\vec{A}\cdot\vec{B} = (17,34,27) \cdot (6,-13,11)$$

$$= 17(6) + 34(-13) + 27(11)$$

$$= 102 - 442 + 297$$

$$= -43$$

and this agrees with our answer obtained by the former method. One often chooses vectors which have this so-called orthonormal property where the magnitude of each vector is 1 and the dot product of all pairs of different vectors is 0. Such a choice allows us to compute using equation (2) rather than equation (1).

As a final note, notice that whether (1) or (2) holds, $\vec{A}\cdot\vec{B} = 0$ means that $\vec{A}$ and $\vec{B}$ are orthogonal regardless of how we elect to

*See note at the end of this exercise.

1.4.5(L) continued

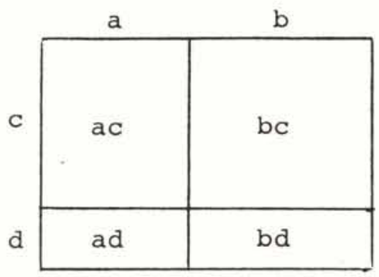
represent $\vec{A}$ and $\vec{B}$. The representation only affects how we compute $\vec{A}\cdot\vec{B}$, for example, whether equation (1) must be used or whether we can use (2).

## A Note on Exercise 1.4.5(L)

In high school algebra we learned the so-called "rule of foil" (first, outer, inner, last) which asserted

$$(a + b)(c + d) = ac + ad + bc + bd \tag{1}$$

One rather neat way of demonstrating (1) was by the geometric observation that $(a + b)(c + d)$ was the area of a rectangle whose dimensions were $(a + b)$ by $(c + d)$. That is,




On the one hand, the rectangle has area $(a + b)(c + d)$; on the other hand, it is $ac + ad + bc + bd$.

There is, however, a way to demonstrate (1) so that we illustrate the structure of our game.

Namely, since $a + b$ is a number, say $e$, we have

$$(a + b)(c + d) = e(c + d)$$

But, by the distributive property $e(c + d) = ec + ed$. Hence,

$$\begin{aligned}(a + b)(c + d) &= ec + ed\\ &= (a + b)c + (a + b)d\end{aligned}$$

By the commutative property $(a + b)c = c(a + b)$ and $(a + b)d = d(a + b)$.

Note continued

Therefore,

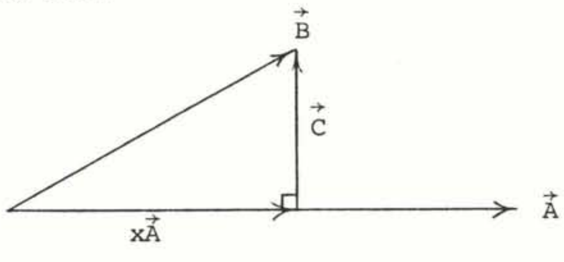
$$(a + b)(c + d) = c(a + b) + d(a + b) = ca + cb + da + db$$
$$= ac + bc + ad + bd$$
$$= ac + ad + bc + bd \quad (\text{since } bc + ad = ad + bc)$$

The crucial point is that every property of numerical arithmetic which was needed to prove this result also happens to be true in our vector case. Namely, $\vec{A} + \vec{B} = \vec{B} + \vec{A}$, $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$, and $\vec{A}\cdot(\vec{B} + \vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$. Hence, the "rule of foil" applies equally well to dot products. Indeed, merely by recopying our earlier proof we have

$$(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{e}\cdot(\vec{c} + \vec{d}) \quad (\text{where } \vec{e} = \vec{a} + \vec{b})$$
$$= \vec{e}\cdot\vec{c} + \vec{e}\cdot\vec{d}$$
$$= \vec{c}\cdot\vec{e} + \vec{d}\cdot\vec{e}$$
$$= \vec{c}\cdot(\vec{a} + \vec{b}) + \vec{d}\cdot(\vec{a} + \vec{b})$$
$$= \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} + \vec{d}\cdot\vec{a} + \vec{d}\cdot\vec{b}$$
$$= \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} + \vec{a}\cdot\vec{d} + \vec{b}\cdot\vec{d}$$
$$= \vec{a}\cdot\vec{c} + \vec{a}\cdot\vec{d} + \vec{b}\cdot\vec{c} + \vec{b}\cdot\vec{d}$$

1.4.6(L)

a. We have




$$\vec{B} = x\vec{A} + \vec{C} \qquad (1)$$

Since $\vec{A}\cdot\vec{C} = 0$, we may dot both sides of (1) with $\vec{A}$ to obtain

1.4.6(L) continued

$$\begin{aligned}\vec{A}\cdot\vec{B} &= \vec{A}\cdot(x\vec{A} + \vec{C})\\ &= \vec{A}\cdot(x\vec{A}) + \vec{A}\cdot\vec{C}\\ &= x\vec{A}^2 + 0\\ &= x\vec{A}^2 \qquad (2)\end{aligned}$$

Since $\vec{A}\cdot\vec{B}$, $\vec{A}^2$, and x are scalars, equation (2) may be solved to yield

$$x = \frac{\vec{A}\cdot\vec{B}}{\vec{A}^2} = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|^2} \qquad (3)$$

Referring to our diagram the vector projection of $\vec{B}$ onto $\vec{A}$ is $x\vec{A}$; hence, from (3), this is

$$\left(\frac{\vec{A}\cdot\vec{B}}{|\vec{A}|^2}\right)\vec{A} \qquad (4)$$

Finally, from (1), $\vec{C} = \vec{B} - x\vec{A}$ (notice how we continue to use vector algebra in a rather natural way), whereupon (3) yields

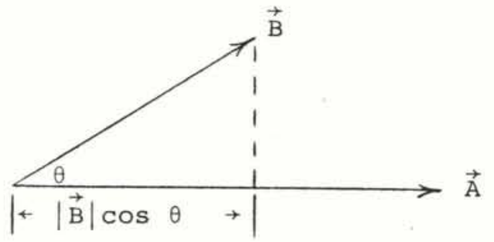
$$\vec{C} = \vec{B} - \left(\frac{\vec{A}\cdot\vec{B}}{|\vec{A}|^2}\right)\vec{A} \qquad (5)$$

b. In part (a) we emphasized algebraic vector techniques in order to show again the structure of vector arithmetic. In this particular exercise, this was equivalent to solving a geometric problem non-geometrically.

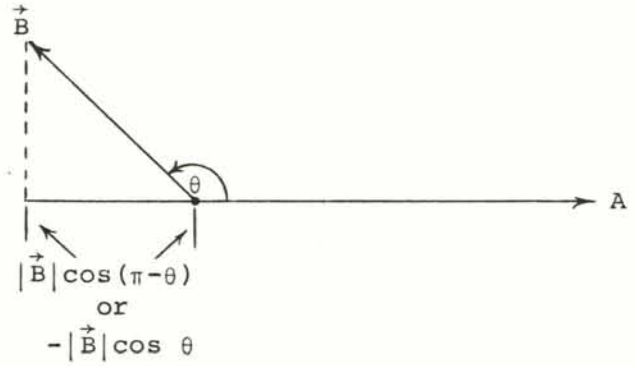
The purpose of part (b) is to emphasize the geometric fact that if $\vec{u}_A$ denotes the unit vector in the direction of $\vec{A}$ then $\vec{B}\cdot\vec{u}_A$ denotes the length of the projection of $\vec{B}$ onto $\vec{A}$. (If $\vec{B}\cdot\vec{u}_A$ is positive then the angle between $\vec{A}$ and $\vec{B}$ is acute, while if $\vec{B}\cdot\vec{u}_A$ is negative, the angle is obtuse - as we shall see below.)

To see this result without reference to dot products, we have

1.4.6(L) continued




If $\theta > 90°$ then $|\vec{B}|\cos\theta$ is negative or the projection is measured in the opposite sense of $\vec{A}$. That is

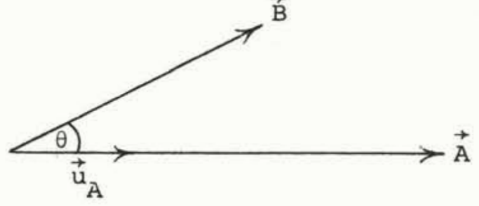



Now θ is clearly the angle between $\vec{A}$ and $\vec{B}$. Thus

$$|\vec{B}|\cos\theta = |\vec{B}|\cos\angle(\vec{A},\vec{B}) \tag{6}$$

Equation (6) starts to look like $\vec{A}\cdot\vec{B}$ (= $\vec{B}\cdot\vec{A}$) except that the factor $|\vec{A}|$ is missing. We then notice that if $|\vec{A}| = 1$ the missing factor causes no change. But $|\vec{A}| = 1$ is the same as saying that $\vec{A}$ is a unit vector and this certainly applies to $\vec{u}_A$. Moreover, the fact that $\vec{u}_A$ is defined to have the same sense as $\vec{A}$ (that is, since $|\vec{A}| > 0$, $\frac{\vec{A}}{|\vec{A}|}$ is a positive scalar multiple of $\vec{A}$) means that the angle between $\vec{A}$ and $\vec{B}$ is the same angle as between $\vec{u}_A$ and $\vec{B}$. Pictorially,

1.4.6(L) continued




In other words,

$$|\vec{B}|\cos\angle(\vec{A},\vec{B}) = |\vec{B}|\ |\vec{u}_A|\cos\angle(\vec{A},\vec{B}) = |\vec{B}|\ |\vec{u}_A|\cos\angle(\vec{u}_A,\vec{B}) \tag{7}$$

But, by definition of the dot product,

$$\vec{B}\cdot\vec{u}_A = |\vec{B}|\ |\vec{u}_A|\cos\angle(\vec{u}_A,\vec{B}) \tag{8}$$

Comparing (7) and (8) yields the result that the (directed) length of the projection of $\vec{B}$ onto $\vec{A}$ is $\vec{B}\cdot\vec{u}_A$.

Thus, the <u>vector</u> projection of $\vec{B}$ onto $\vec{A}$ is simply the directed length of the projection of $\vec{B}$ onto $\vec{A}$ multiplied by the unit vector in the direction of A. That is,

$$(\vec{B}\cdot\vec{u}_A)\vec{u}_A \tag{9}$$

Since $\vec{u}_A = \dfrac{\vec{A}}{|\vec{A}|}$, (9) becomes

$$\left(\vec{B}\cdot\frac{\vec{A}}{|\vec{A}|}\right)\frac{\vec{A}}{|\vec{A}|} =$$

$$\left[\frac{1}{|\vec{A}|}(\vec{B}\cdot\vec{A})\right]\left[\frac{1}{|\vec{A}|}\vec{A}\right] =$$

$$\left[\frac{1}{|\vec{A}|^2}(\vec{B}\cdot\vec{A})\right]\vec{A} =$$

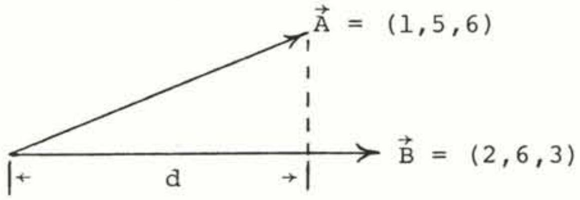
$$\left(\frac{\vec{A}\cdot\vec{B}}{|\vec{A}|^2}\right)\vec{A} \tag{10}$$

1.4.6(L) continued

Notice that (10) and (4) are identical, as they should be.

1.4.7

Pictorially, we have




$$|d| = |\vec{A}\cdot\vec{u}_B|$$

$$= \left|\vec{A} \cdot \frac{\vec{B}}{|\vec{B}|}\right|$$

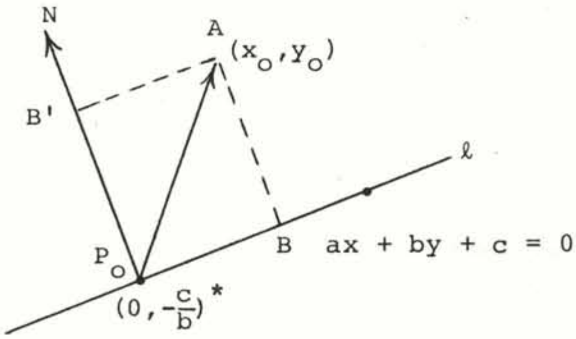
But

$$|\vec{B}| = \sqrt{2^2 + 6^2 + 3^2} = 7$$

Therefore,

$$|d| = \left|\frac{(1,5,6) \cdot (2,6,3)}{7}\right| = \frac{1(2) + 5(6) + 6(3)}{7} = \frac{50}{7}$$

1.4.8(L)




The slope of ℓ is $-\frac{a}{b}$ since $ax + by + c = 0$, $b \neq 0 \rightarrow y = -\frac{a}{b}x - \frac{c}{b}$. Hence, the line perpendicular to ℓ has slope $\frac{b}{a}$. Therefore, the vector $\vec{V} = a\vec{i} + b\vec{j}$ is perpendicular to ℓ.

The distance we seek is equal to the length of the projection of $\vec{P_oA}$ onto $\vec{N}$, and this is merely the dot product of $\vec{P_oA}$ with the unit vector in the direction of $\vec{N}$.

Now,

$$\vec{u}_N = \text{unit vector in direction of } \vec{N} = \frac{a\vec{i} + b\vec{j}}{\sqrt{a^2 + b^2}}$$

while

$$\vec{P_oA} = \left(x_o - 0,\ y_o - \left(-\frac{c}{b}\right)\right) = \left(x_o,\ y_o + \frac{c}{b}\right)$$

Therefore,

*This point was obtained by letting $x = 0$ in $ax + by + c = 0$ (other points would do as well). The only time we are in trouble is when $b = 0$, but in this case $ax + by + c = 0$ reduces to $ax + c = 0$ which is the line $x = -\frac{c}{a}$ parallel to the y-axis. In this case, it is trivial to find the required distance. So, without loss of generality, we may as well assume that $b \neq 0$.

1.4.8(L) continued

$$\vec{P_oA} \cdot \vec{u}_N = (x_o, y_o + \frac{c}{b}) \cdot \frac{(a,b)}{\sqrt{a^2 + b^2}}$$

$$= \frac{ax_o + b(y_o + \frac{c}{b})}{\sqrt{a^2 + b^2}} = \frac{ax_o + by_o + c}{\sqrt{a^2 + b^2}}$$

Finally, since distance is defined to be non-negative and $ax_o + by_o + c$ might be negative, we put our final answer in the form

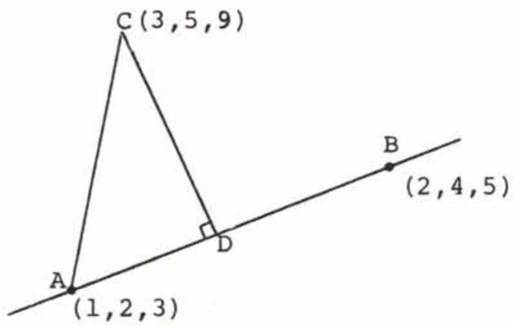
$$\frac{|ax_o + by_o + c|}{\sqrt{a^2 + b^2}} \qquad (1)$$

(An interesting aside occurs if $\sqrt{a^2 + b^2} = 1$, for in this case (1) becomes

$$|ax_o + by_o + c|$$

which means that we find the distance from $(x_o, y_o)$ to $ax + by + c = 0$ simply by replacing x and y by $x_o$ and $y_o$ in the left side of the equation for the line.)

1.4.9




$\vec{AC} = (2,3,6)$

$\vec{AB} = (1,2,2)$

$|\vec{AC}| = 7$

$|\vec{AB}| = 3$

1.4.9 continued

$\overline{AD} = \vec{AC} \cdot \vec{u}_{AB} = (2,3,6) \cdot (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) = \frac{2}{3} + 2 + 4 = \frac{20}{3}$

$\overline{AC} = 7$

Hence, by the Pythagorean Theorem,

$$\overline{CD} = \sqrt{\overline{AC}^2 - \overline{AD}^2} = \sqrt{49 - \frac{400}{9}}$$

$$= \frac{1}{3}\sqrt{41}$$