Hi. Our lesson today is concerned with introducing a new type of integral when we deal with calculus of several variables. And this might at first glance seem to be a rather touchy subject, the idea being that up until now we've talked about multiple integration, yet you'll notice that the title of this block of material is just called integration.

And the reason for this is that when we deal with several variables, there is more than one type of integral and what I would like to do is to motivate today's lesson in terms of a physical example and then come back and talk about in more detail what happened and why there is more than one type of integral in the calculus of several variables.

I called today's lesson Introduction to Line Integrals, and to visualize how this topic may be introduced, consider the following situation. I have a force f, say, defined throughout the xy plane here. I have two fixed points, which I'll call p0 and p1, and I pick a particular curve which joins p0 to p1. And the question that I would like to solve is what is the work done if I move a particle from p0 to p1 along the curve c by means of the force f?

And the reason that this is called a line integral is that somehow or other, you see, I am going to be dealing with the line or the curve c. Let me assume for the sake of argument that I know the equation of c in parametric form. Let's assume simply that c is given by x as some function of t and y as some function of t, and since I'm going from p0 to p1, t is defined on a specific closed interval.

Say t goes from t0 to t1. Or, if I want to write the equation of c in vector form, remember, in vector form, the i compound is x, or at least the Cartesian coordinate vector form, the x, the i component is x, the j compound is y. So I can write the same equation that the curve is r of t, where r is x of ti, plus y of tj.

As long as I'm using Cartesian coordinates here, let me assume that my force is also given specifically in terms of Cartesian coordinates. It's mi plus nj, where m and n are specific functions of x and y. And just to review briefly, the problem, I want to find the work done in moving from p0 to p1 along c under the influence of f.

Now you see, the thing is we already know how to solve work problems if the distance takes place along, if the displacement takes place along a straight line. So what we do is the following. We take the curve c, and we subdivide it into increments. We break the thing up. What we do is we look at the cased increment and join the two endpoints of that increment by a straight line, a vector which we'll call delta r sub k.

We pick a point on the curve in this interval and compute the f. After all, we know the force as a function of x and y, but we compute that force for that fixed point. We then dot that force with this displacement, and that is what? That's the work that's done if the particle had moved along the straight line connecting these two points under the constant force f of x of k* That comma y sub k*.

And then what we do is to find the approximation, we add up all of these forces, all of these works over the various increments. Notice that you're doing two things in these increments. You're first of all picking small enough segments so that you can assume that the straight line approximation is a good approximation, to the curve.

And secondly, small enough segments so that you're assuming that the force at a given point is a good approximation to the force along the entire increment. At any rate, what we then do is we sum these up over all the increments. And by the way, notice that, even though f is a function of x and y, along the curve c, x and y are functions of t.

And to emphasize this, let me rewrite this simply by dividing and multiplying by delta t sub k to indicate that if I now put this remaining part of the expression in braces, what's in here is really some function of t alone. In other words, notice that this comes very close to being how we define the ordinary definite integral when we want to add things up.

You see t is a scalar, and that the braced expression here is a function of t alone. It's a dot product obtained this way. At any rate, what we're saying now quite simply from the physical point of view is define the actual work. Why don't we define the work to simply be the limit of w sub n as the maximum delta r sub k approaches zero, which is equivalent to saying what?

Find this limit as delta t sub k, the maximum delta t sub k approaches zero. And the point is that this was the ordinary definite integral. Remember how we wrote this. You replace the sum by the integral sign. We assume that t goes from some value t0 to t1, those become our limits of integration.

Every place we see the t sub k*, we replace it simply by t, and every place we see the deltas we replace those by d's and we leave off the subscripts. So, and I can't emphasize this part strongly enough, in going from here to here, we are using nothing more than what happened in part one of our course. Namely, this is the definition of the definite integral that we arrive at this expression here.

And to show you other ways in which this is written, notice that f is mi plus nj. r is xi plus yj, so the r dt is the x dt i plus the y dj t. And so if we then mechanically just took the dot product here, notice that the integrand would be m dx t plus n dy t times dt. The conventional shortcut for writing this is to cancel dt from numerator and denominator and this becomes the standard notation for the line integral m dx plus n dy along the curve c where c joins p0 to p1.

And as I'll mention in a few minutes, it's crucial that we specify both the curve and the points being joined, and the physicist usually abbreviates this by coming back to the original notation here, and when he takes the limit, he just writes this as f dot dr. And what I'm saying is that this, this, and this are simply different ways of writing this expression here. OK.

So this is how this is then evaluated, and what I'm going to do very shortly is specifically solve problems using concrete values for f and p0 and p1, but for the time being, I would like to make a little note here to have you see what's happening, and that's the following. Notice that double integrals are concerned with regions.

In other words, we use double integrals to find area, mass, et cetera. Line integrals are concerned with curves or boundaries of regions. In other words, paths that connect two points. See, in other words, somehow or other, the double integral is associated with area, the line integral with length. And the point is that in one dimensional space, the two concepts coincide.

In fact, we don't even talk about area in one dimension, unless by area we meant one dimensional area or length. In other words, both arclength and area in the typical definition sense of area, the one dimensional area, coincide in one dimensional space. In two dimensional space, they are quite different.

In fact, in a nutshell to see what's happening, essentially double integrals are concerned with what's going on inside a region r, whereas the line integral somehow or other is concerned with what's happening as you move along the boundary of the region r, and we'll talk about that in more detail later in this lecture, and this will also be the topic of discussion in our next lecture.

But essentially for the remainder of this lesson, let's take a specific application of our earlier physical remarks. In the following examples, let's fix the points p0 and p1 to be the points 0, 0 and 1, 1, respectively. And let the force mi plus nj be specifically y squared i plus x squared j.

In other words, to compute the force at any point x comma y in the plane, the i compound of a force is just the square of the y-coordinate, the j component is just the square of the x-coordinate of the point at which we're competing the force.

Now what we're going to do in the next three examples is simply compute the work done along three different curves that joined the point p0 to p1 and the work done as a particle moves from p0 to p1 under the influence of this specific force, and as I say, just the curve that joined p0 and p1 will be different.

In the first example, the curve c1 is simply the straight line that join 0, 0 to 1, 1. In parametric form, that would be x equals y equals t where t goes from zero to one noticing that when t is 0, you see, x and y are both 0 so we're at the point 0, 0, and when t is 1, x and y are both 1, so we're at the point 1, 1.

And if we want to write that in vector form, remembering that the i component is just x and the j component y, since x and y are both t, the curve has the form r equals ti plus tj. The work done, writing it symbolically as we would in the physical notation integral along c1 f dot dr means what? We dot f with the r dt, and multiply that by dt and integrate that from zero to one.

That's what that infinite sum means. Notice in this particular case, f, which is y squared plus x squared, since x and y are both equal to t, f is just t squared i plus t squared j. Since r is ti plus tj, dr dt is just i plus j, and consequently, the infinite sum that we want to evaluate is just the integral from zero to one. So you're just taking the dot product here.

t squared plus t squared is 2t squared integral from zero to one, 2t squared dt, and using the fundamental theorem of integral calculus from the single variable, this is simply evaluated by this being t cubed from zero to one, 2/3t cubed from zero to one, this is just 2/3. In other words, you see what's hidden here is the fact that, when we write the definite integral--

See, this was a function of two variables in here, but along the curve, they have functions just of the parameter t. This is an ordinary integrand, which means we can either think of it as an infinite sum or the antiderivative evaluated in a specific way. Don't lose track of the fact that the work is still being defined as a limit, but we can compute it quite conveniently in terms of the fundamental theorem of integral calculus.

At any rate, the work done then in example one as we move from p0 to p1 under the force f along this curve c1 is just 2/3. Now what we're going to do is redo the same problem, only now we are going to choose a different path which connects 0, 0 to 1, 1. In fact, I will choose the parabola y equals x squared, and I'll write it parametrically this way.

I'll let x equal t, y be t squared. So that says y equals x squared. And to get to point 0, 0, t must be zero. To get to point 1,1, t must be 1. So again, t varies from zero to one continuously, or again in vector form, the equation of the curve c2 is just what? ti plus t squared j. At any rate, the work done by definition symbolically is the line integral along c2.

f dot dr. That still means f dot dr dt dt as t goes from zero to one. In this case, x is t and y is t squared. So if we come back to our definition of f and replace x by t and y by t squared, we see that our force is t to the 1/4i plus t squared j. Since r is ti plus t squared j, the r dt is just i plus 2tj. Remember, we differentiate vectors component by component.

Therefore, the integrand is simply this dot product, which turns out to be, what? t to the fourth times 1 plus t squared times 2t, or t to the fourth plus 2t cubed. We now, again, use the fundamental theorem, evaluate this integral by the antiderivative. In other words, this is 1/5t to the fifth plus 1/2t to the fourth, evaluated between zero and one, and that answer is 7/10.

In fact, this, I guess, in a way, was a bad problem from a physical point of view. An excellent problem from a mathematical point of view. You see, certainly, from a physical point of view, 7/10 looks enough like 2/3, so the change isn't that drastic. You can almost call that an experimental error.

But since we weren't doing an experiment here and we're working with exact numbers, the fact does remain that, what? 2/3 and 7/10 are different, and if nothing else, what we've proven is what? That the work done as we move from two fixed points from the fixed point p0 to the fixed point p1 under the influence of a given force f actually does depend on the path that we follow.

You see, we used two different paths, and got two different answers. In other words, the line integral m dx plus n dy along with curve c which joins two fixed points p0 and p1 depends on c as well as on p0 and p1, and that's an important point to keep track of. We're going to mention this in more detail as the course goes along.

At any rate though, let that point remain fixed in abeyance for a while, and we'll go on with our next example. And for the next example, what I'm going to do is pick the same path as an example two, only in a different parametric form. Another way of writing the parabola y equals x squared is to let x equal t squared and y be t the fourth.

You see, in that case, y is still equal to x squared, and to connect 0, 0 to 1, 1, t still goes from zero to one. So the work done along the curve c3 is given by this particular symbol. To evaluate this, remember now x is t squared. So x squared will be t to the fourth. Y is t to the fourth, so y squared would be t to the eighth.

Remembering that our forces y squared i plus x squared j written in two tuple notation our forces t to the eighth comma t to the fourth. Our dr dt vector is what? It has components 2t and 4tq, and we're going to multiply that by dt and integrate that from zero to one. That's simply 2t to the ninth plus 4t to the seventh. dt from zero to one.

That's t to the 10 over 5 plus t to the eighth over two evaluated between zero and one, and that's equal to 7/10, which agrees with this result. By the way, that's not a proof. All I've shown is that two parametric forms led to the same amount of work. But at any rate, it does seem to indicate that when you can compute the work over along the curve c, that that work done seems to be independent of the equation for c.

And hopefully this should be the case, because we would be in terrible trouble, I guess is the best word to put this as, if it turned out that the work done in moving along a specific curve joining two points under a specific force depended on the equation by which you represented the curve. But I simply wanted to give you an example to show at least that the mathematical definition appears to be consistent.

In other words, along the curve c1, we got a different amount of work done then along the curves c2 c c3. But c2 and c3 were actually what? The same curve given by two different equations. So at any rate, the work does seem to depend on the curve, but it seems to be independent of the equation by which the curve is represented.

I think we're in position now to start an overall summary going right back to the very crux of the problem as to why line integrals and multiple integrals exist. Namely in particular, in the case of one independent variable, the integral from a to b f of x dx has two equivalent interpretations. And by equivalent, I mean numerically equivalent. You get the same answer.

They are not equivalent philosophically. You know, this is one of the reasons I have this hangup about trying to spell out the philosophy for you. In many cases, you will say we're interested more in the application than in the theory. As I may have said to you before, in fact on many different occasions, the easy part of mathematics in most cases is actually applying the computational recipes.

The validity of the recipes is what is usually the difficult thing, and the biggest mistakes are not made by a person using the right formula and making an arithmetic mistake. It's that he uses the wrong formula and doesn't make an arithmetic mistake. So I want this to be very clear to you why a certain problem exists in two variables that didn't exist in one.

And the point is, what? That conceptually, there were two different ways to interpret the meaning of the symbol integral from a to b f of x dx. Numerically, they came out to be the same, but what were the two different interpretations? One was as the limit of a particular infinite sum. That was the definite integral notation.

The other was, we could visualize the interval from a to b as being a curve that could be written parametrically say as x is some function of t where t goes from t0 to t1, and we then make the usual change of variables. The same change of variables that we were talking about in the previous lecture. Namely, every place we see x, we replace it by x of t, and then the correction factor is the x dt and the new limits are from t0 to t1.

In other words, we get what? That the integral from a to b f of x dx is the integral from t0 to t1 f of x of t dx dt times dt. Again, observing that this is a bona fide function of t alone here so this is just a change of variable type thing. And what am I saying? That in the one dimensional case, the case of one independent variable, these two concepts, while different, give rise to the same numerical answer.

In two space, however, notice that our first interpretation as an infinite sum translates into the double or multiple integral. I keep saying double because we're dealing with two independent variables. Obviously by now, I hope it's clear to you that the results generalize to end variables. But the idea is, what?

That the first interpretation of the integral in terms of two independent variables gives rise to the multiple integral. In fact, I think I have the-- Yeah, that's right. And the second interpretation gives rise to this expression here. Noticing again that if you have a one dimensional vector, the dot product is the ordinary product.

In other words, f of x dx is f of x dot dx if you want to think of x as being a one dimensional vector. But why is this interpretation? Noticed the change of variables again, that we think of the curve c as being given by the equation r equals some function of t as t goes from t0 to t1. So the substitution is what?

This means the particular definite integral in terms of t as t goes from t0 to t1, f of r of t dotted with the r dt, that again is what? That again is some scalar function of t, you see? Times dt. That's what that second interpretation says, and that's of course what we mean by this particular notation.

Two entirely different things. But at any rate now, getting back to what we saw in our examples, the line integral c f of r dot the r often depends on x, but not on the equation which represents c. By the way, we're going to show a very interesting thing in the next lecture as to when the line integral will depend on c and went it wont, but the next lecture actually is more general than just answering that question.

The next lecture centers around this question. Is there any interesting relationship between line integrals and double integrals? You see, in the case of one independent variable, there was a very interesting relationship between line integrals and so-called double integrals or integrals of infinite sums. Namely, they were exactly the same thing. You couldn't tell them apart.

That's what we meant in our summary when we said we had two different interpretations of the same numerical result. Now the question is, in two independent variables, these two concepts are different. Are they related, and if so, how? And that will be the body of our next lecture.

At any rate then, until next time, goodbye.

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