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**HERBERT
GROSS:**

Hi. There's a cute book called *The Point and the Line*. It was also made it into an animated cartoon that ends with the line, "To the vector belongs the spoils." And in this particular course, we're going to find that both the traditional meaning of vector, plus a modern meaning of factor plays a very fundamental role in the study of calculus of several variables.

In fact, the modern definition which we will ultimately get to later in the course, is the one that will be of utmost importance to us. But in the meantime, to better appreciate what this will be like, we will start with a more conventional treatment of vectors. Hopefully one fringe benefit of this will be to revisit what vectors really are.

And to get on with this, let me again point out that we will be giving an overview of what the unit is all about in this lecture. And the details will again be left to the supplementary notes, the text, and the exercises. I call this lecture "'Arrow' Arithmetic." And the reason for calling this "'Arrow' Arithmetic" hopefully will become clear in a moment.

For the time being, let's very quickly review what a vector is. I think most of us know this, but just to make sure there are no misinterpretations, a vector is any quantity which depends on a direction as well as a magnitude. In other words, a vector quantity is one which in order to specify uniquely, we must tell what its magnitude is - its size; we must tell what direction it's in, and finally something which I call the sense wherein I distinguish between the same direction being traversed in two different ways.

And perhaps the best way to emphasize this is in terms of a one dimensional analogy that we have already studied very, very early in our mathematical careers.

I'm thinking of the idea of visualizing signed numbers in terms of the number line. In other words, recall that the number line was essentially the x -axis. And starting at some origin, which we called say, 0, we numbered 1, 2, et cetera

if we moved in the left to right direction; and negative 1, and negative 2, et cetera, if we went from right to left. In other words, even though the direction of the x -axis is still the same, whether we go from left to right or from right to left, we distinguish between the two motions in terms of--and this what we mean by the sense of a vector.

Some people say well the direction includes the sense, I prefer to have these two words dangling around, so I can put them at my disposal in any way that I see fit. But roughly speaking then, this is all a vector is.

Now the idea is why arrows? And the answer is-- and here's a little ratio I put down here-- an arrow is to a vector as a length is to a scalar, whereby scalar, I mean a number, something which has a magnitude, but no direction.

Remember when we first started talking about numerical arithmetic, and talked about graphing, and visualizing numbers as lengths. We picked the length to represent the number, and at that particular time, since there was only one degree of freedom-- namely the x -axis-- we didn't emphasize direction.

But the idea is to carry this analogy further. What one does is when one has a quantity which is a vector, an arrow-- a traditional arrow-- a directed line segment with a given sense. The arrowhead indicating the sense is perhaps the best way of drawing what we mean by a vector.

Namely, I have just a few random examples here. Here's a collection of things which are arrows. They're line segments in a given direction, and the arrow head indicates a particular sense. Notice by the way, that the concept of a vector does not depend on an arrow, anymore than the concept of a number depends on a length.

The arrow is simply a convenient geometric representation for visualizing the vector. Now hopefully, in terms of the last unit, you now have the feeling that in

mathematics, to have an arithmetic, one needs more than a collection of objects. That to have a structure, one needs a collection of objects together with certain rules, and definitions, and what have you.

And consequently, it becomes rather mandatory that somehow or other we be able to categorize a pair of vectors as to whether they shall be called equal or not. See, what shall we mean by even saying the two vectors shall be equal? The equality of vectors.

And I claim the following-- and let me just state this cold bloodedly until we come to the end, and show what we're driving at again. To say that the vector A equals the vector B means that first of all, the magnitude of A equals the magnitude of B.

By the way, again as in the one dimensional case-- when we called the absolute value the length-- we do the same thing in terms of arrows. We abbreviate the magnitude of an arrow by writing the absolute value of the arrow. In other words, the absolute value of A must equal the absolute value of B.

Secondly, the arrow A must be parallel to the arrow B. And thirdly, A and B must have the same sense. And again to show you what this is, remember I said to you a moment ago that the arrows are just a convenient pictorial way of representing a vector.

Let's observe that these three properties here juxtaposition very nicely with these three properties here. In other words, notice that our definition of equality of arrows captures exactly what three ingredients go into the finding a vector. Since all we're talking about is magnitude, direction, and sense, how can we distinguish validly between two vectors other than in terms of magnitude, direction, and sense. OK?

By the way, let's be very careful about this. As long as the only way that we can determine the difference between two vectors-- or arrows-- is in terms of their magnitude, direction, and sense, what this means is that if you have two vectors which are parallel, have the same length. Two arrows. Same length, same direction, and the same sense, they must be called equal.

You see, in other words, vectors do not have to coincide to be equal. Unlike the case of straight lines. In other words, if I were to draw a pair of parallel lines here of equal length, these would be called different lines. As soon as I do this these are called equal arrows. Why? Because of our definition of equality. And if somebody says, "But I don't like that definition," well this is the same as the rule that three strikes is an out in baseball.

If somebody doesn't like that rule, you say to them, lookit, go and play your own game, and make as many strikes is an out as you want, but don't call that game baseball, because that will confuse it with the game that we're playing here, which is different.

In other words, we make up our rules to govern, as I said in the last lecture, what we believe to be reality. And we believe that for how we're going to use vectors this is a very, very natural definition. And this is not the first time in mathematics that we've done something like this.

Way back around the third or fourth grade, here's a little analogy, you get used to saying $1/2$ equals $3/6$. But obviously, the ordered number pair 1 comma 2, and 3 comma 6 look different. It certainly makes a difference physically whether you cut a pie in half and take one piece or cut the pie into six pieces and take three of them. The physical process is different. The two fractions don't look alike. Why do we call them equal? And it means that in terms of what we want ratios to mean, the number which you must multiply by 2 to get 1, is the same number that you must multiply by 6 to get 3, and consequently, let's just keep the equality in that sense. Who cares whether they look alike or not? Let's call them equal if they capture the important characteristic.

And that's the way this game is going to be played all the time. We will make up our definitions to fit our mood, and hopefully the mood that suits me best will suit you best, because I will try to pick the reasons that you yourselves would have liked to have seen motivate why we did the things that we did or that we're going to do here.

So let me talk about the "'Addition' of Vectors," and let me put the word "Addition" in

quotation marks here to indicate that all I mean by the addition of vectors-- and perhaps I should still say addition of arrows-- is I would like a rule that tells me how to combine two arrows to form another arrow.

Now again I can make up hundreds of rules that tell me how, for two given arrows, I'm going to form a third arrow from the given two. But since our aim is to solve problems of the real world, why not pick a definition? Why not pick a definition that has already been used in physics and in other applications? The thing called the resultant. For example, given a point O , and two forces, represented as arrows, A and B , acting at O , we have agreed-- or you may recall-- that the resultant force is the vector that forms the diagonal of the parallelogram which has a and b as consecutive edges.

In other words, this vector here was what in the old days was called the resultant. Because that has such a nice usage, I am going to call that the sum of two vectors. By the sum of two vectors I mean what physically used to be the resultant. And a very nice way of remembering the rule, is remember that this vector B is equal to this vector, because this is a parallelogram.

These are equal in magnitude, parallel in direction, and have the same sense. Notice that if I want a quick recipe that was convenient for adding two vectors A and B , you say what? Place the tail of the second next to the head of the first, and then draw the vector-- the arrow-- that goes from the tail off the first to the head of the second.

And this will be the definition of the addition of arrows. And again, this is done in much more detail in written material. By the way, notice again, a parallelism between the structure of vectors, and the structure of ordinary arithmetic. In the same way that given two numbers they have one sum, but that a given sum can be formed by different combinations of numbers, that two given vectors, which I call here x_1 and y_1 , give A as a sum, but notice also that the vector A that is sum of the vectors x_2 and y_2 .

To generalize this, if I picked any point I wanted on the blackboard here, and drew

any two lines, one going from the tail of A to this point, and one going from this point to the head of A . Notice that the resulting two vectors here-- this vector plus this vector by definition of vector addition, head to tail-- from the tail of the first to the head of the second, would still be the vector A .

In other words, in terms of the diagram that I drew originally, notice that A can be written as x_1 plus y_1 , it can also be written as x_2 plus y_2 . And notice again the similarity of the structure between arithmetic and vectors, and notice the arbitrariness with which we make up our definitions.

We make up the definitions to be our servant, rather than for us to become the servant of the subject. We model the definitions-- the truth that we call after what we believe happens in real life. By the way, a very convenient coordinate system is our old friend the X and Y Cartesian coordinate system. And the reason for this is as follows.

Given any vector A , we can always assume that the vector originates at the origin. And the reason for that is we've already seen that two vectors are equal if they have the same magnitude, direction, and sense so whenever A was in the plane of the blackboard, we can shift it parallel to itself, so that the tail starts at the origin. Let's assume that the vector A , the arrow A , which starts at the origin, terminates at the point a comma b .

Let i be a unit vector, meaning a vector that has one unit of length. In the direction of the positive x -axis, let j be a unit vector which is in the direction of the positive y -axis, and our claim is that our definition of addition-- and you've all seen this, I'm pretty sure. Is that the vector A can now be written as what?

A is equal to a_i plus b_j . Where you see, if you've seen this notation before notice that in terms of our game, we are now forced to introduce a new operation. Because we claim in our game that it's validity that's important. Validity, not truth.

If all I can use are the rules of my game, for the first time, what do I see here? I see a number multiplying the vector. And if I don't have rules of my game that tell me

what a number multiplied by a vector means, in terms of the logic machine, the logic machine can't do a thing to give me inescapable conclusions.

Without assumptions, without rules, there can be no proof. So I must make up a new definition, and this definition is called scalar multiplication. And it's going to tell me how to multiply a scalar-- a number-- by a vector. If c is a number, and v is a vector, by c times v , I mean a vector, which has-- I have to be careful, see c could be a negative number. And since you think of length as being positive, I take the absolute value of c just to get the magnitude of the number in here.

You see? In other words, this is a vector which has the magnitude of c times the magnitude of v . Meaning its c times as large as v if c is positive, and minus c times as large as v if c is negative.

The same meaning of absolute value as always. For example, 2 times v would be a vector whose magnitude was twice that of the vector v . Negative 2 times v would also be a vector whose magnitude is twice that of v . The other property is what? I've just told the magnitude, I also have to give you the direction and the sense. c times v is in the same direction as v . Minus 2 times v , and plus 2 times v both have the direction the v , and both are what? Twice the magnitude of v .

The only difference between the minus is this. That c times v has the same sense. Same sense as what? Same sense as v , if c is positive. That means if c is negative, it has the opposite sense. In other words, for example, 2 times v has the same sense as v , minus $2v$ has the opposite sense of v .

But the important point is this-- that vector arithmetic shares many structural properties of regular arithmetic. That is very crucial to understand. I have a friend of mine who doesn't like teaching students who are having a few problems with ordinary arithmetic, and likes to teach advanced courses. And he often has said that the trouble with the average high school curriculum for the average student, is that by the time the student is through, he knows three things: If you see a sign, change it; if you see a decimal point, move it; and if you see a fraction; invert it.

But one step further when the average all time engineering students saw a vector, he only knew one thing about it. And the one thing that he knew about it was, if you see a vector, break it down into components. And after that, that was the end of what he ever did with vectors in the old days. But the thing that is crucial to understand here is that we will use vector arithmetic in a far more powerful way. We will use the structure of vector arithmetic. In the same way that we formed algebra from numerical arithmetic, we will form vector algebra, and vector calculus from vector arithmetic.

Let me give you an example. We already know it that for two numbers a and b , $a + b$ equals $b + a$. My claim is that from our definition of arrows and the like, it should also be a rule that for arrows, $a + b$ should equal $b + a$. Or, for numbers, we already know that $a + b + c$ equals $a + b + c$.

That doesn't look right. By the way, what I wrote there wasn't false, it was certainly true. But it was too trivially true. I guess what I wanted to emphasize here was that the voice inflection made no difference. Now again, notice how we verify the rules of the game. I have to make up rules that I will call basic truths. Those rules-- or axioms-- will be modeled after what I believe to be true in reality.

Now, notice in reality I've formed this whole idea in terms of what? Arrows. I defined what addition meant. Let's take a look and see what $a + b$ means. If this is the vector a , and this is the vector b , then what is the vector $a + b$? By definition of how you add two vectors, this vector here would be $a + b$. On the other hand, I could take the vector a which is down here, move it parallel to itself so it originates here, mark it off the same length. This is still the vector a .

If I now draw the arrow that goes from the tail of b to the head of a , what vector is this? Well by definition, it's $b + a$. How are $b + a$ and $a + b$ related? They are opposite sides of a parallelogram, therefore they have the same magnitude, and same direction. And the picture shows us that they have the same sense. Since this is true in the picture, why not claim that one on the rules of our game of vector arithmetic, even though legally you shouldn't use pictures, but pictures do give us

the motivation.

Why not say, OK, we'll accept the rule that $a + b$ equals $b + a$. How about our second rule over here? Let's take three vectors. Let me take a over here, let me put b over here, and let me put c over here. And I hope that this will come out legible enough for all of us to see.

I claim on the one hand that I can think of $a + b + c$ in two different ways. In other words, to use the way that says $a + b + c$ from this diagram, how do I get the vector $b + c$? Since b and c are lined up in the right position head to tail, the vector $b + c$ is the one that goes in the tail of b to the head of c . So I call this vector $b + c$.

Now, what is $a + b + c$? It's this vector plus this one. In other words, this vector here could be labeled $a + b + c$. On the other hand, what is the vector $a + b$? $a + b$ is this vector. Because it goes from the tail of a to the head of b and consequently, $a + b + c$ is the vector that goes from the tail of $a + b$ to the head of c . What vector is that? It's this vector, but what name does that have? We just saw. That's $a + b + c$. And now we see what?

We have two different names for the same arrow. In particular then, these two different names represent what? Two arrows which have the same magnitude, sense, and direction, and therefore they're equal. And that's why we make up rules like this.

And by the way, the analogy continues on, and this is done in much more detail in our notes, but I just thought I would like to say a few more words in general. For example, one talks about the zero vector. And I write that with a 0 with an arrow over it, to indicate the 0 arrow. Maybe your intuition tells you that the zero vector should somehow be connected with a 0 number, but the point that I want to drive at is an analogy to what we did in the previous lecture when we showed why b to the 0 has to be defined to be 1 .

The idea is that an ordinary arithmetic what is the beauty of the number 0 the

beauty of the numbers 0 is that with respect to addition, it does not change the identity of any number. Any number plus 0 is that number back again. In particular then, why not keep the same structure for vectors? And it's our choice to make.

We would like to say OK, let's define the zero vector to be such that if a is any other vector, a plus 0 will still be a . Well let's get an idea of what that means. Let b denote the magnitude of the zero vector. What this means is what? How do I add 0 to a ? I start with the vector a , and it originates here. And terminates at the point p .

Now how do I add on the zero vector? The zero vector, all I know is it's going to start at p , and b length b . So the surest thing I can say is, lookit, let me draw a circle centered at p , with the radius equal to b . In other words the radius of this circle here, the radius of this circle is b . All I know is that whatever the zero vector is, if its magnitude is b , the zero vector must originate at p , and terminate someplace on this circle. Because that's the locus of all vectors of length b which originate at p .

For the sake of argument, let's say that the zero vector happens to be pq . Then a plus 0 is the vector that goes from this point to q . But a plus 0 must be a . And since a and a plus 0 originate at the same point, the only way they can have the same magnitude, direction, and sense, is if they must coincide. In other words, in particular, p must equal q . And as soon as p equals q , the magnitude of the zero vector must be the number 0.

In other words, in terms of our game structure again, if we want this structural to be obeyed, and again I must emphasize this: The choice is ours to make. But if we want the structural rule to be obeyed, we must define the zero vector to be the vector which has 0 magnitude.

As natural as it seems, that's not the real reason for doing it. We do it because it gives us the structure that we want. The same is true, for example, about the additive inverse. Remember in ordinary arithmetic, the additive inverse meant what? The number that must be added to the given number to give the 0 number. Or the number 0.

In vectors, we should define negative a to be the vector which when added to a gives the zero vector. Well, lookit, any vector that I add to a -- let me call that vector b . Any vector that I add to a -- which I call a plus b -- looks like this. Well what is the only way that the vector a plus b can be the zero vector?

Well the zero vector has no length. That means the only way that's possible is if the tail of a plus b coincides with the head of a plus b . And the only way that can happen, is if b terminates right here. In other words, the vector that we call negative a is going to be the vector which has the same magnitude and direction as a , but the opposite sense.

And we're not saying it's self evident. All we're saying is look-- if you want this structural property to be true, you have no choice but to define negative a to be the vector which has the same magnitude, and direction as a , but the opposite sense. And by the way, as soon as you do that, there are no more guessing games involved in how you subtract two vectors.

For just in the same way that we did it in ordinary arithmetic, we define the difference of two vectors a minus b to mean the vector a added to the vector negative b . And what is negative b ? The vector which just has the opposite sense of the vector b . And again, these computational details are left to the notes, and to the exercises. But the main overview that I want you to get from this lecture is to see that there are more to vectors than dividing them up into components.

That there is an arithmetic imposed on them, and that arithmetic that's imposed on vectors is just as real as the arithmetic of the ordinary real numbers. And to reinforce this, in our next lesson, we will talk about vectors in three dimensional space, where the geometry becomes tougher, but the structure remains exactly the same. But until next time, good bye.

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