PROFESSOR: Hi. Our lecture for today I've entitled 'Logarithms Without Exponents'. And before I try to clarify that, let me make a couple of general remarks. First of all, we have now completed the rudiments, the basics so to speak, of both differential and integral calculus. Consequently, what our next ambition will be is to take these principles and apply them to special functions which are worthy of investigation. And you see, in connection with this, the first function that I've chosen to talk about is called the logarithm.

Now the other interesting thing is that we are used to logarithms from high school where they were viewed as being a different kind of notation for talking about exponents. Now, in the same way when we treated the circular functions, we mentioned that one did not have to invent triangles to talk about trigonometry. The interesting point is that one does not have to talk about exponents to invent the logarithm function. This is why I have entitled the lesson, as I say, 'Logarithms without Exponents'.

You see, what I would like to do is try to mimic, especially from an engineering point of view, how many of these mathematical topics originated. Let's look at a rather straightforward physical principle, a principle that I'm sure we believe permeates many real life situations. It's what I call the rule of compound interest. Namely, let's suppose we have a physical situation in which the quantity that we're measuring, which we'll call 'm'. In other words, the rate of change of the quantity is proportional to the amount present, 'dm/ dt' equals 'km'.

Now notice that this is a pretty harmless statement. The rate of change is proportional to the amount present. We certainly would expect that many experiments would run this way, and there seems to be nothing at all supernatural about this kind of an assumption. At any rate, let's apply our previous principles, and see if we can't, from this differential equation, figure out what 'm' has to be. Separating the variables, we get that 'dm' over 'm' is equal to 'kdt'. And now integrating, meaning taking the inverse derivative, we have what? That the integral of 'dm' over 'm' is equal to 'kt' plus a constant.

Now you see, the interesting point is at this stage of the game, we do not know explicitly how to find a function whose derivative with respect to 'm' is '1 over m'. In fact, that's the problem that underlies today's lecture. And we will try to answer this question, showing how once we tackle this from a philosophic point of view, the rest of the details follow from material that we've already learned.

At any rate, focusing our attention on the problem, the question is this: to determine a function, which I'll call capital 'L of x'-- the 'L' sort of to forewarn us of the fact that we will somehow get a logarithm out of this-- such that 'L prime of x' is '1 over x'. And you see, what I want you to notice is that if you write '1 over x' in exponential form, in other words, if you try to write '1 over x' as 'x to the n', notice that to solve this problem, 'L of x' would be what? The integral of 'x' to the 'minus 1 dx'. In other words, this is the case of integral ''x to the n' dx', with 'n' equal to minus 1.

And you'll recall that when we studied the recipe-- let me just write that over here-- when we studied the recipe of how you integrate 'x to the n', we saw that that was 'x to the 'n + 1'' over 'n + 1' plus a constant. And we then observed that this doesn't even make sense when 'n' is equal to minus 1, because we have a 0 denominator.

From the other point of view, the way to look at this is we said what? That when you differentiate, you lower the exponent by 1. Notice that to wind up with an exponent of minus 1, we would have had to start with an exponent of 0. But when you differentiate 'x' to the 0, that being a constant, the derivative of 'x' to the 0 is 0. It's not '1 over x'. You see, in other words, this particular recipe that we're talking about, we don't have working for us.

In other words, this is an interesting point again. You say gee whiz, if the recipe for the integral ''x to the n' dx' works for everything except 'n' equals minus 1, nothing's perfect. Let's be content. Since it works for every number except that one, why worry? This is analogous to when we talked about taking derivatives and saw that we get a 0 over 0 form every time. You see, if you pick a function at random, the likelihood that the limit of 'f of x' as 'x' approaches 'a', the likelihood of that being 0 over 0 is very small, except when you form the derivative, you always get 0 over 0.

And in a similar way, granted that integral ''x' to the minus 1 dx' is one very special case. What we have seen is that this comes up every single time that we want to solve a problem of the form 'dm/ dt' equals 'km'.

In other words then, what it really boils down to is that unless we can come up with a function

'L of x' whose derivative is '1 over x', we cannot solve the problem that we started off today's lesson with. In other words, we must leave it in the form integral 'dm over m' equals 'kt' plus a constant. Well at any rate, let's see what such a function capital 'L of x' must look like. I'm going to utilize both the differential calculus approach and the integral calculus approach.

By differential calculus, assuming that 'y' equals 'L of x', what do we know about the function 'y'? By definition, we've defined it to be that function whose derivative is '1 over x'. So we know its derivative is '1 over x'. The next point is that knowing that 'y prime' is '1 over x', and even though we can't integrate '1 over x', we can certainly differentiate '1 over x'. We can now differentiate '1 over x'. The derivative of '1 over x' is minus '1 over 'x squared'', and we find that 'y double prime' is minus '1 over 'x squared''.

By the way, we should observe that we must shy away from 'x' equaling 0, because you see, we would have a 0 denominator in that case. If we think of most physical situations, notice that we talk about the amount-- the rate of change is proportional to the amount present. From a physical point of view, the amount present can't be negative, so let's put the restriction on here that the domain of 'L' will be all positive numbers.

Now here's the interesting thing. Intuitively, I can now visualize how the function 'L of x' must look. Namely, since its first derivative is '1 over x' and 'x' is greater than 0, that tells me that the curve is always rising. And secondly, since '1 over 'x squared'' is always positive, minus '1 over 'x squared'' is always negative, that tells me that my curve is always spilling water.

In essence then, what must the curve do? The curve must be rising but spilling water. In other words, the curve 'L of x' belongs to this particular family. Notice that once we have one curve which we call 'y' equals 'L of x', by any displacement parallel-- vertical displacement, we get what? A member called 'y' equals 'L of x' plus 'c'. All that changes is what? The point at which the curve crosses the x-axis, but whichever member of the family we pick, what typifies 'L of x'? Its derivative is '1 over x'. In other words, somehow or other, we visualize that 'L of x' exists, and its graph is something like this.

Now suppose we want a more tangible form, namely how do you compute 'L of x' for a given 'x'? I thought what might be a nice review now is to see how we can use our integral calculus approach, and in particular the second fundamental theorem of interval calculus.

By the integral calculus approach, we do is is we pick any positive number 'a', and once picked, we fix it. You see, we have a great deal of freedom in how we choose the positive number a. But let's pick one and leave it here. Now what we'll do is in the 'y-t' plane, we'll draw the curve 'y' equals '1 over t'. And what we'll do now, we'll study the area of the region 'R' where 'R' is bounded above by 'y' equals '1 over t', on the left by 't' equals 'a', on the right by 't' equals 'x', and below by the t-axis.

Now we've already seen that the function that we get by taking the definite integral from 'a' 'to x', 'dt over t', has the property that its derivative is '1 over x'. In other words, recall from the fundamental theorem, this is just a generalization of the fact that if 'f of t' is a continuous function, then the integral from 'a' to 'x', ''f of t' dt', is a function of 'x'. And its derivative with respect to 'x' is just 'f of x'. You see, in other words, I can now view capital 'L of x' as being an area under the curve 'y' equals '1 over t'.

Notice that I have a degree of freedom here, namely what that area is depends on where I choose 'a'. Notice that if I choose a differently, changing 'a'-- let's call this 'a prime', say-- changing 'a' changes the area under the curve, but notice it changes it by a fixed amount. In other words, notice that shifting 'a' just changes 'L of x' by a constant, just like the ordinary in definite integral.

At any rate, these are the two approaches that we have. And if we superimpose them, all we're saying is what? If you start with the differential calculus approach, 'y' equals capital 'L of x' must be a member of this family, crossing the axis at some point (a,0) . And from the integral calculus point of view, if you take the curve 'y' equals '1 over t' and compute the area under that curve from 'a' to 'x', that area function is 'L of x'.

Now we'll let that rest for the time being, and now have a brief digression. You see, sooner or later, I've got to get to what a logarithm is, and we might just as well do that now. So let's now take a look and see what we mean by a logarithmic function. You see, quite frequently in mathematics, what one does is one deals with a particular special case in which one is interested. And having deduced very nice results, he looks at this thing and says, you know, all of these results came from a particular recipe, a particular structure that this system obeyed.

What he then does is he takes this recipe or structure, gives it a general name, and now is able to extend this property to a larger class of objects. For example, let's look at this way. What is the nice thing about logarithms in the traditional sense of the word? Remember when we learned logarithms in high school, by use of logarithms, we were able to replace multiplication problems by addition problems, et cetera. In other words, remember the key structural property was that the logarithm of a product was the sum of the logarithms. That was the structural thing that we used over and over and over again.

In fact, it's rather interesting to point out that in many cases where we used the properties of logarithms, we never really had to know what a logarithm was. All we had to know was what? What properties did it obey, and did we have a book of tables so we could look up the logarithm when we had to. Well at any rate, using this as motivation, we now define a logarithmic function, specifically a function f is called logarithmic if for all 'x1', 'x2' in the domain of 'f', 'f' of the product of 'x1' and 'x2' is 'f of x1' plus 'f of x2'. In other words, since we know what nice properties a logarithm has, let's define any function 'f' to be logarithmic if 'f' of a product is equal to the sum of the 'f's.

Now you see, this may sound like a pretty simple statement, but it's quite demanding. In other words, notice-- by the way, somebody says, I wonder if there are any logarithmic functions? You see, notice that by the way we began, there must be at least one logarithmic function, namely the usual logarithm. We know there's at least one function which has these properties. So the set of all logarithmic functions is certainly not the empty set.

At any rate, let's see what we can deduce about any logarithmic function. Well for example, I claim that if 'f' is logarithmic, 'f of 1' must be 0. Why is that? Well, notice that 1 can be written as 1 times 1. And since 'f of 1' times 1 is 'f of 1' plus 'f of 1', we have what? That 'f of 1' is equal to 'f of 1' plus 'f of 1', therefore twice 'f of 1' is 'f of 1', and therefore by algebraic manipulation, 'f of 1' is 0. Which, by the way, does check out with the usual logarithm, the logarithm of 1 to any base 'b' is 0.

Secondly, if 'f of '1 over x'' is defined, you see in other words, I'm not sure whether '1 over x' is in the domain of 'f' just because 'x' is, but suppose '1 over x' is in the domain of 'f'. Then 'f of '1 over x'' is equal to minus 'f of x'. In other words, 'f' of a number is minus 'f' of the reciprocal of that number. Again, a familiar logarithmic property, why does it follow from our basic definition? Well notice that we already know that 'f of 1' is 0. We also know that 1 is equal to 'x' times '1 over x', provided 'x' is not 0.

By the way, if 'x' were 0, 1 over 0 wouldn't be defined, so we wouldn't be worrying about that anyway. But notice now we have what? If 'x' is not 0, 'f of 'x times '1 over x'' is 0, but by the logarithmic property, that's also 'f of x' plus 'f of '1 over x''. Since these two together add up to

0, 'f of '1 over x' must just be minus 'f of x'. And we'll just take a few more properties like this just to make sure that you see what properties logarithmic functions have.

Remember again the ordinary logarithm, the logarithm of a quotient was the difference of the logarithms. Again, 'f of 'x over y'' is equal to 'f of x' minus 'f of y'. Why is that the case? Well, look at 'f of 'x over y''. That says 'f of x' times '1 over y'. But again, by logarithmic properties, 'f' of a product is the sum of the 'f's. Therefore 'f of 'x times '1 over y''' is 'f of x' plus 'f of '1 over y''. But we just saw that 'f of '1 over y' is minus 'f of y'. So we have this familiar result now.

And finally, by mathematical induction, if 'n' is any positive integer, 'f of x' to the n-th power is just 'n' times 'f of x'. And the proof is rather clear. Namely, 'f of x' to the n-th power when 'n' is a positive integer means 'f of 'x' times 'x' times 'x' times 'x'', 'n' times. And since the logarithmic property says that 'f' of a product is a sum of the 'f's, that's equal to 'f of x' plus 'f of x' plus et cetera plus 'f of x', 'n' times, and that precisely is just 'n' times 'f of x'.

In other words, notice that our definition of a logarithmic function, simply that 'f of 'x1 times x2'' is equal to 'f of x1' plus 'f of x2' allows us to deduce all of the familiar properties that were known to us in terms of the traditional meaning of logarithm. And now we come to the most important question of today's lecture, and that is what does all of this discussion have to do with the function that we've called capital 'L of x'? And that's what will tackle next.

Let's take a look and see whether capital 'L of x' is indeed a logarithmic function. Well, if 'L' is to be a logarithmic function, if 'b' is any constant, in particular 'L of 'b times x'' had better be equal to 'L of b' plus 'L of x'. That's the definition of logarithmic. Now we don't know if that's true. What do we have at our disposal to be able to see whether we can solve this problem or not? How do we, using calculus, check to see whether two functions are equal? Remember the standard approach is what? Take the derivative of both sides. If the derivatives are equal, then the functions differ by a constant. And if we can show that that constant is 0, then the two functions are equal. So let's take a look and see how that works over here.

First of all, let's see what the derivative of 'L of bx' is with respect to 'x'. And by the way, here again is the beauty of what we mean when we say that all of the study that we're making allows us to utilize every fundamental result that we've learned before. We've learned that the basic definition of 'L' is what? That if you differentiate 'L' with respect to the given variable, you get 1 over that variable. We want the derivative of 'L of bx' with respect to 'x'.

Now, you see the idea is the derivative of 'L of u' with respect to 'u' is '1 over u'. You see by the chain rule, if we were differentiating 'L of bx' with respect to 'bx', that would be '1 over bx'. I shouldn't say by the chain rule yet. What I'm saying is if we differentiate 'L of bx' with respect to 'bx', we would have '1 over bx'. But we're not differentiating with respect to 'bx', we're differentiating with respect to 'x'. And this is where the chain rule comes in. Namely, we rewrite the derivative of 'L of bx' with respect to 'x' as the derivative 'L of bx' with respect to 'bx' times the derivative of 'bx' with respect to 'b'. And in this way we get what? The first factor is '1 over bx'. The second factor is 'b' itself-- remember 'b' is a constant-- and therefore the derivative of 'L of bx' is '1 over x'.

On the other hand, what is the derivative of 'L of b' plus 'L of x'? Since 'b' is a constant, the derivative 'L of b' is 0. And by definition, the derivative of 'L of x' with respect to 'x' is '1 over x'. And now you see, comparing these two results, we see that 'L of bx' and 'L of b' plus 'L of x have the same derivative, hence they differ by a constant which I'll call 'c'. In other words, notice that capital 'L' is what I call almost logarithmic. If it weren't for this factor constant 'c' in here, it would be a logarithmic function.

Well at any rate, what do we have? We know that capital 'L of bx' is equal to capital 'L of b' plus capital 'L of x' plus some constant 'c'. To evaluate the constant, we only have to evaluate it at one element in the domain, let's pick 'x' equal to 1. The motif being that if 'x' is 1, notice that on the left hand side you have an 'L of b' term, on the right hand side you have an 'L of b' term, and they will cancel. In other words, if 'x' is 1, this equation becomes 'L of b' equals 'L of b' plus 'L of 1' plus 'c', from which it follows that 'c' is minus 'L of 1'. 'c' is minus 'L of 1'.

Now what do we want to 'c' to be if 'c' were equal to 0? If 'c' where equal to 0, this would be logarithmic, and all we need to make 'c' equal to 0 is to set 'L of 1' equal to 0. In other words, summarizing this result, capital 'L of x' is logarithmic if capital 'L of 1' is 0. Now remember, we have a whole family of 'L's that work. What we're saying now is let's pick the member of the family of 'L's that passes through the point (1,0) . And because that's logarithmic, let's give that a special name.

In essence, we will define this symbol. It's written 'ln of x'. It will later be called the natural logarithm of 'x'. I'm trying to avoid the word logarithm here as much as possible, because I want you to see that we haven't had to use exponents at all in making this kind of a definition. But let's define 'ln of x' to be the member of the family capital 'L of x' plus 'c' which passes through the point (1,0). In essence then, what is 'ln of x'? I'll call it natural log, it's easier for me to say. The natural 'log of x' is that function whose derivative with respect to 'x' is '1 over x', and characterized by the fact that what? If the input is 1, the output is 0. In other words, the graph passes through the point (1 , 0).

Again, notice that in terms of what we said earlier in today's lesson, we talked about the functions 'L of x', and talked about, in the differential calculus approach, the curve could go through any point on the x-axis. What we're saying is now if the point on the x-axis that the curve crosses at is (1 , 0), that's what 'lnx' will mean, the natural log function.

In terms of the integral calculus approach, if the particular 'a' value that we choose, the fixed value that we're going to study the area under, notice that all we're saying is pick 'a' to be 1. And by the way, as a quick check, all we're saying is what? The 'natural log of x' is defined to be the area under this curve as a function of 'x'. In terms of the definite integral, that's the integral from '1 to x,' 'dt over t'. And notice that if you pick 'x' to be 1, the natural log of 1 is the integral from 1 to 1 'dt over t', and that's 0, just as it should be.

So at any rate now, that tells me how to define the natural log. Am I doing this in terms of exponents? No. I am trying to do what? Find a function whose derivative with respect to 'x' is '1 over x', and I'd also like that function, since it's that close to being a logarithmic function, to be a logarithmic function. In other words, this is how I invented the function 'ln of x', the 'natural log of x'. It's derivative with respect to 'x' is '1 over x', and the natural log of a product is the sum of the natural logs.

And by the way, that's exactly how we use this material. Let me just take a few minutes and go over something that we already had an answer for, just to show you how this works. Let me try to rederive the product rule using logarithms. Suppose 'u' and 'v' are differentiable functions of 'x', and I want to find the derivative of 'u' times 'v' with respect to 'x'. In other words, I'd like to find 'dy dx'. OK, I take the natural log of both sides. In other words, I say if 'y' equals 'u' times 'v', the natural log 'y' equals natural log 'u times v'.

Now what is the property that the natural log function has? I deliberately chose it to be that member of the l family that was logarithmic. The natural log of 'u times v' is 'natural log u' plus 'natural log v'. Now, can I differentiate this implicitly with respect to 'x'? You see how all of our old stuff keeps coming up in new context. I take this equation and I differentiate it implicitly with respect to 'x'. The derivative of the left hand side is '1 over y', 'dy dx'. The derivative of the right hand side is what? Well, the derivative of 'log u' with respect to 'u' is '1 over u', but I'm differentiating it with respect to 'x', so I must use the correction factor by the chain rule 'du dx'.

And similarly, the derivative of 'log v' with respect to 'x' is '1 over v', 'dv dx'.

And now multiplying through by 'y', I have obtained that 'dy dx' is ''y over u' du dx' plus ''y over v' dv dx'. But remembering that 'y' is equal to 'u' times 'v', this becomes 'v 'du dx'' plus 'u 'dv dx', and we have arrived at the familiar product rule using logarithmic differentiation.

You see the point that's really important to stress here it that it wasn't important whether the natural log was an exponent or not. The important thing that we used about logarithms, whether it was our high school course, whether it's going to be in our college calculus course, the important thing was what? That when we wanted to write the log of a product, it was the sum of the logs. And now the only additional fact that we have from the calculus approach is that the derivative of 'log x' with respect to 'x' was defined to be '1 over x'.

By the way, since there is a tendency to think of traditional logarithms when one uses the word logarithm, if we so wanted-- and there's no reason why we have to do this-- if we wanted to associate a base with the natural log system-- this is just a little aside, and we'll talk about this more perhaps in the learning exercises or in the supplementary notes as need be-- but the idea is this. Notice that in a traditional logarithm system, if the base is 'b', the base is characterized by the fact that the 'log of 'b to the base b'' is 1.

Thus, if we use 'e' to denote the base for the natural log, whatever 'e' is, it must be characterized by the fact that the 'natural log of e' is 1. And the reason that I put the word base in quotation marks here is that I would like you to observe that again, if I have never heard of the word exponent, it still makes sense to say find the number e such that the 'natural log of e' is 1. In fact, I can give you two interpretations for that number 'e', and as a byproduct, even show you why the second fundamental theorem of integral calculus is as powerful as it really is.

See, the idea is this. To find what 'e' is, all we're saying is take the curve 'y' equals 'natural log x'. Look to see where the y-coordinate is 1. In other words, draw the line 'y' equals 1. Where that line intercepts the curve 'y' equals 'log x', that x-coordinate is 'e'. In other words, the 'natural log of e' must be 1, so this is geometrically how I would locate 'e' using differential calculus.

If I wanted to use integral calculus, notice that the 'log of e' by definition is what? The integral from 1 to 'e', 'dt over t'. That's the area of this region 'R'. Now what do I want 'e' to be? I want 'e' to be that number that makes this area 1. In other words, I want the 'natural log of e' to be

1. So again, in the same way that one can think of pi as being geometrically constructed, notice I can construct 'e' geometrically, namely again what?

I take the curve 'y' equals '1 over t' from 't' equals 1, bounded below by the t-axis, and I keep shifting over to the right until the area of this region 'R' is exactly 1. The 't' value that makes this area 1 is called 'e'. And by the way, notice again the power of what I mean by the area approach. Notice I can start now to get estimates on what 'e' must look like. Let me show you what I'm driving at over here.

Let's suppose we take the curve 'y' equals '1 over t' from 1 to 2. Now what do we mean by natural log 2? Natural log 2 is by definition the area of this region here. Now notice that the smallest height of this region, since the curve is 'y' equals '1 over t', the smallest height of this region is 1/2, and the tallest height, the highest height in this region is 1. Consequently, whatever the area of this region is, it's less than the area of the inscribed rectangle-- it's greater than the area of the inscribed rectangle, and less than the area of the circumscribed rectangle.

Now notice that both of these rectangles have base 1, and we're going from 1 to 2. The height of the inscribed rectangle is 1/2, therefore the area of the small rectangle is 1/2. The area of the big rectangle, it's a 1 by 1 rectangle, is 1. And now what we have is that whatever the natural log of 2 is, it being the area of this region, it must be what? Less than 1 but greater than 1/2. We also know that whatever 'e' is, the 'log of e' is equal to 1.

Well, let's go on a little bit further. What about the log of 4? The log of 4, natural log of 4, is the log of 2 squared. But we've already seen that one of the properties of a logarithmic function is that you can bring the exponent down as a multiplier. See, 'f of 'x to the n'' is 'n 'f of x''. This becomes 2 log 2, and because the natural log of 2 is greater than 1/2, twice the natural log of 2 is more than 1. In other words, if log 2 is more than 1/2, twice log 2 is more than 1.

In other words, if we put these three lines together, we have that the natural log of 2 is less than the 'natural log of e', which in turn is less than the natural log of 4. By the way, notice that since the derivative of 'log x' with respect to 'x' is '1 over x', and that's positive, 'log x' is a one to one function. Therefore, if the log of 2 is less than the 'log of e' is less than the log of 4, it follows that 2 must be less than 'e', which in turn must be less than 4.

In other words, even with this crude approximation of just inscribing and circumscribing rectangles, I can show again without reference to exponents that whatever the number 'e' is, the number e defined by the fact that 'ln of e' is 1, that number 'e' is some number between 2 and 4. But again, we won't dwell on this too long. What I want to do now is to come back to a summary point, and at the same time, lead into the lecture for next time.

Recall that we began this lecture with the problem of solving the situation where the rate of change was proportional to the amount present. Given 'dm/ dt' equals 'km', we separated variables and got down to the stage where the integral of 'dm over m' was 'kt' plus a constant. And all we did in the rest of today's lecture that was at all different, was we invented and constructed the particular function whose derivative would be '1 over m', and which had the logarithmic property. In other words, we can now say that this is the answer to the problem. This problem is explicitly solved because the natural log function can be viewed as an area under a curve. We can construct it for each 'm'. This is then what we managed to do.

And by the way, all I'm saying now is that since the log is a one to one function-- see, its derivative is always positive-- can't we talk about the inverse function? In other words, notice that another way of writing this is that id 'natural log m' is 'kt' plus 'c', 'm' itself must be the 'inverse log of kt' plus 'c'. And you see, next time what we're going to do is to explore what we mean by the inverse log.

At any rate, in summarizing today's lecture, to make sure that we didn't fall victim to all the computational details, notice that all we did was physically motivated the necessity for inventing a function whose derivative with respect to 'x' was '1 over x'. And from that point on, everything else that we did followed as applications of material that came before. It's in this sense that the course now picks up in tempo. That as we go on now, we're going to be able to cover larger globs of material in one sitting, because we will find, at least for the next several lectures, that every new topic is basically one new idea together with all of the old recipes. At any rate, until next time, goodbye.

ANNOUNCER: Funding for the publication of this video was provided by the Gabriella and Paul Rosenbaum Foundation. Help OCW continue to provide free and open access to MIT courses by making a donation at ocw.mit.edu/donate.