FEMALE SPEAKER: The following content is provided under a Creative Commons License. Your support will help MIT OpenCourseWare continue to offer high quality educational resources for free. To make a donation or to view additional materials from hundreds of MIT courses, visit MIT OpenCourseWare at ocw.mit.edu.

PROFESSOR: Hi. Technically speaking, I suppose, if time permitted, we should spend, or shall we say, give equal time to integral calculus, the same time that we gave to differential calculus, starting from scratch, building ideas slowly and the like. But there are two logistic reasons for not doing this, one of which is that the basic definitions would be the same regardless of which branch we started with. The second one is, of course, that time becomes preciously tight, and we have to make the best of what we can here.

So what we're going to do now is try to show, in terms of hindsight being better than foresight, a motivation as to how one would have invented differential calculus had it been motivated by the existing integral calculus. In other words, what we want to do today is to show the beautiful interplay between differential and integral calculus. Since we have said in our last lecture that we would like to begin with a study of integral calculus, assuming that differential calculus had never been invented, I think that we're obligated to try to show how, then, differential calculus could have been invented in the environment of the studying of areas.

For this reason, I have called today's lecture "The Marriage of Differential and Integral Calculus." And essentially, the idea goes something like this. Last time, we were talking about finding areas under a curve. And to find the area under the curve, we saw that we could do this as the limit of a certain sum. In other words, an infinite sum having a particular limit.

Now, to motivate the concept of rate of change, one could ask the following question. A curve like 'y' equals 'f of x', and a starting point, 'x' equals 'a'. As I move out along the x-axis from 'a', I could study the area under the curve, say, of the curve, 'y' equals 'f of x', from 'x' equals 'a' to some value 'x1'. And the question I could then ask is, I wonder how fast this area is changing at the instant that 'x' equals 'x1'.

In other words, how fast is the area changing under the curve as 'x' moves out along the positive x-axis this way? You see? Now, the idea is this. I can then mimic the same definition that we gave for instantaneous rate of change and differential calculus. Namely, I can say, why can't we define the instantaneous rate of change to be the average rate of change with

respect to 'x'? And then we'll take the limit as 'delta x' approaches 0, et cetera.

And if I do that, watch what happens over here. You see what I do in terms of this picture, is I say, OK, here's the area under the curve when 'x' is equal to 'x1'. Now I'll let 'x1' change by some amount 'delta x', meaning this'll be the point 'x1 + delta x'. And this brings about a change in my area, which I'll call 'delta A'. See the change in area as 'x' goes from 'x1' to 'x1 + delta x'.

Now, what I would like to find is 'delta A' divided by 'delta x', taking the limit as 'delta x' approaches 0. Remember, last time we showed our three basic axioms for area. We're obliged to still use them. Let's do that here.

For example, the way I've drawn this curve by rising this way, notice that the lowest point in this interval occurs when 'x' equals 'x1'. The highest point occurs when 'x' equals 'x1 + delta x'. Consequently, the rectangle whose base is the closed interval from 'x1' to 'x1 + delta x', the rectangle whose height corresponds to 'x' equals 'x1' is too small in area to be 'delta A'. In other words, that rectangle is inscribed in 'delta A'. And in the same way, the rectangle, with base 'x1' to 'x1 + delta x', whose height is given by the x-coordinate, 'x1 + delta x', that rectangle has an area which is too large to be the exact area.

In other words, notice that the area of the rectangle, which is too small to be the exact area, is 'f of x1' times 'delta x'. The area of the rectangle which is too big to be the correct 'delta A' is 'f of 'x1 + delta x'' times 'delta x'. And the area, 'delta A', is caught between these two. In other words, we now have the inequality that 'f of x1' times 'delta x' is less than 'delta A', which in turn is less than 'f of 'x1 + delta x'' times 'delta x'.

Now, we'll assume that 'delta x' is positive, the way we've drawn it in this diagram. Adjustments have to be made if 'delta x' is negative, in other words, if we let 'delta x' approach 0 through negative values. This is taken care of in the textbook, as well as mentioned in our notes. But for the sake of just getting the main idea here, we'll assume that 'delta x' is positive. We divide through by 'delta x'. That gives us 'f of x1' is less than 'delta a' over 'delta x', which in turn is less than 'f of 'x1 + delta x''.

In other words, had 'delta x' been negative, we would have had to reverse the inequality signs, et cetera, but we're not going to worry about that right now. Remember, what is 'delta x'? It's a non-zero number which is going to be made to become arbitrarily close to 0. Well, here's the idea. We now have 'delta a' over 'delta x' squeezed between these two values. And we say,

OK, let's let 'delta x' approach 0.

Now, here's the key point. As 'delta x' approaches 0, certainly, 'x1 + delta x' approaches 'x1'. And here's where we use the fact that 'f' is continuous. In other words, notice that the mere fact that 'x1 + delta x' approaches 'x1' is not enough to say, therefore, that 'f of 'x1 + delta x'' approaches 'f of x1'. As we saw before, that's only true if 'f' is continuous. In other words, by continuity, if 'x1 + delta x' approaches 'x1', then 'f of 'x1 + delta x'' approaches 'f of x1'.

So assuming, then, that 'f' is continuous, what do we get we put the squeeze on? 'x1' is what's picked to be a fixed point. Consequently, as 'delta x' approaches 0, 'f of x1' stays 'f of x1'. 'f of 'x1 + delta x'' approaches 'f of x1'. In other words, 'delta A' divided by 'delta x' is caught between two numbers, two sequences, two sets of bounds, whichever way you want to say this, both of which converge to the common limit, 'f of x1'.

In other words, since 'delta A' over 'delta x' is squeezed between these two, the limit of 'delta a' over 'delta x', as 'delta x' approaches 0, must equal this common limit. In other words, 'dA/dx', the limit of 'delta A' over 'delta x', as 'delta x' approaches 0, evaluated at 'x' equals 'x1', is 'f of x1'. Or another way of saying this, 'dA/dx' is 'f of x'.

Now, that result is partly intuitive and partly remarkable. What it says is, roughly speaking, is that the area under the curve is changing at a rate equal instantaneously to the height corresponding to the x-coordinate at that point. In other words, to define how fast the area is changing as 'x' moves, all you have to do is measure the height of the curve corresponding to that particular value of 'x'.

Now, what does this have to do, then, with relating integral and differential calculus? Notice already we begin to get some sort of a hint. What we've shown is now that the area is going to be related somehow to the inverse derivative of 'f of x'. In fact, this is precisely what is meant by a rather important result. In fact, it's important enough so it's called the first fundamental theorem of integral calculus.

And it simply says this. Suppose we know explicitly a function, capital 'G', such that the derivative of capital 'G' is 'f', where the 'f' now refers to the 'f' that we're talking about in our problem. Now, look it, what we already know is that 'A', the area function, has its derivative equal to 'f'. Consequently, since the area function and G have the same derivative, they must differ by, at most, a constant. In other words, the area as a function of 'x' is this function, 'G of

x' plus 'c'.

And what is this 'G of x'? It's not any old function. It's what? An inverse derivative of 'f of x'. Since the area evaluated at 'x' equals 'a' is 0, see, namely, the area under the curve, as 'x' goes from 'A' to 'a', see, that's just a line. A line has no thickness, has no area. This is 0. We have, by plugging back in to here, that 'A' of little 'a' is 0. That in turn is 'G of a' plus 'c'. And solving this for 'c', we find that 'c' is equal to minus 'G of a'.

In other words, the area under the curve 'y' equals 'f of x', as a function of 'x', is simply what? It's 'G of x' minus 'G of a', where 'G' is any function whose derivative is 'f'. If we now let 'x' equal 'b', we're back to the situation of finding the usual region that we're talking about here, namely, what is 'A sub R'? It's the region bounded above-- let's come back here and take a look at this-- it's the region bounded above by the curve, 'y' equals 'f of x', bounded on the left by 'x' equals 'a', on the right by 'x' equals 'b', and below by the x-axis. According to this notation, the area of the region 'R' is just the area as a function of 'x' when 'x1' is equal to 'b'. In other words, the area of our region 'R' is just 'a' of 'b', which is 'G of b' minus 'G of a', where 'G' is any function whose derivative is 'f'.

Now remember-- and here's the key point-- remember that the area of the region 'R' did not necessitate us having to know a function whose derivative was 'f'. In other words, remember, in our last lecture, how did we find areas of regions like 'R'? What we did was is we formed these 'U sub n's, these 'L sub n's. We inscribed and circumscribed networks of rectangles, put the squeeze on by evaluating this particular limit. And whatever that limit was, that was the area of the region 'R'. In other words, if I had never heard of the inverse derivative, I could still find the area of the region 'R' this way.

However, if I just happen to know a function 'G', whose derivative is 'f', I have a much easier way of doing this. In fact, let me summarize that. You see, we can compute the limit as 'n' approaches infinity. Summation 'k' goes from 1 to 'n', 'f of 'c sub k'' 'delta x', by use of inverse derivatives. See, again, the highlight being what? You can still work with this the same way as we did in our last lecture. There is absolutely no need to have had to ever heard of a derivative to solve this type of problem, even though it may be messy.

Now, what's the best proof I have of this? Well, I guess one of the best proofs is to specifically refer to one of the exercises in the previous unit. It was a tough exercise. I did it as a learning exercise because I felt it was something that was messy and that you had to be guided

through in order not to become hopelessly lost. By way of review, the problem was this.

We took as our region 'R' the region bounded above by 'y' equals 'sine x', below by the x-axis, on the right by the line 'x' equals pi/2. And the problem was, define the area of the region 'R'. And we solved this problem in the last unit without recourse to derivatives because, as of the time that we were in the last unit, we had no results relating derivatives to limits of sums.

How did we do this? We partitioned this into n parts, and we formed the sum limit as 'n' approaches infinity. 'Sigma k' goes from 1 to 'n'. We broke this thing up into 'n' equal parts, so the size of each piece was what? pi/2 divided by 'n'. That's 'pi/2n'. We computed the sine of each of these n-points, et cetera. And whatever that limit was, that was the area of the region 'R'.

And as you recall, that homework problem, we found that the answer to that problem was that the area of the region 'R' was 1. Now again, as messy as that was, it proved that we could at least solve the problem with no knowledge of derivatives. How does the first fundamental theorem apply here? The way the first fundamental theorem applies is the following.

The first fundamental theorem says this. Look it, to evaluate this particular sum, all we have to do is find a function whose derivative is 'sine x'. See, notice in this problem, the general 'a' and 'b' of the above is now played by 'a' equals 0 and 'b' equals pi/2. All we have to do is what? Find the function 'G' whose derivative is 'sine x', and the answer should then be 'G of pi/2' minus 'G of 0'.

Well, you see this happens to be one that, with our knowledge of differential calculus, we can do rather easily. In other words, do we, at the tip of our tongues, have a function whose derivative is 'sine x'? And the answer is yes. Since the derivative of 'cosine x' is 'minus sine x', the derivative of 'minus cosine x' is 'sine x'. In other words, we can choose for our 'G of x' here 'minus cosine x'.

In other words, that the area of the region 'R' should be what? 'G of pi/2' minus 'G of 0', where 'G of x' is 'minus cosine x'. Well, you see, cosine pi/2 is 0. Cosine of 0 is 1, so 0 minus minus 1 is equal to 1. And notice that, first of all, we get the same answer. And secondly, notice this two-line job over here. We not only get the correct answer, but we get the answer very much more rapidly than we did by the so-called limit process.

Now, I'll come back to this in a minute, because you may be a little bit angry at me now.

Namely, you may be asking, after having to do this problem the hard way, why did I have to do the usual teacher's trick here and show you the hard way of doing this and then waiting until the next unit before I show you the easy way? Well, there happens to be a catch here that I'll come back to in just a moment. But before I do that, I'd like to make an aside, an aside that's rather important.

Namely, you may recall, as I started to work over here, that this reminded you of a notation that we were using back when we introduced the concept of the inverse of differentiation. Namely, we earlier used a notation, integral from 'a' to 'b', 'f of x' 'dx' to denote 'G of b' minus 'G of a', where 'G prime' equals 'f'. What we have shown by the first fundamental theorem, then, using this notation, is that the area of the region 'R' is integral from 'a' to 'b', 'f of x' 'dx'. In other words, this means what? It means 'G of b' minus 'G of a', where 'G prime' equals 'f'. OK, let's pause here for a moment.

See, as we developed our course, this is how this would evolve. The interesting thing is that, historically, this notation was not used to define 'G of b' minus 'G of a', where 'G prime' equaled 'f'. Historically, what happened was that this notation, called the definite integral, the integral from 'a' to 'b', 'f of x' 'dx', was-- I don't know if it's proper to say invented, but let me just say it in quotation marks to play it safe-- it was "invented" to denote the limit of this sum.

In fact, notice how much more meaningful the symbol looks in this connotation. In other words, this is the limit of a sum. You can think of sum as beginning with 's', and the integral of sine as an elongated 's'. In other words, when one first wrote this symbol, the definite integral, it was meant to denote this limit. The important point is that, by the first fundamental theorem, the definite integral, whether it's a limit or not, turns out to be 'G of b' minus 'G of a', where 'G prime' equals 'f'.

In other words, it really makes no difference in terms of what answer you get, whether you think of this definite integral as meaning this, or whether you think of the definite integral as meaning this. See, numerically, they'll be the same. But conceptually, they're quite different. And that's precisely the point that leads to the most, I think, probably the most difficult part all the course, at least up until now.

It's something that more people cause and have misinterpretation over. And so I'd like to present this in a very, very gradual way. See, the result will ultimately be known as the second fundamental theorem of integral calculus. I've taken the liberty of writing that over here. But it's

going to be quite a while before I actually say what the theorem is explicitly.

What I'd like to do is to pick up where I left off when I made my aside about the notation of a definite integral. And what I'd like to mention now is, for the person who says, look it, why did you make us find the areas the hard way? Why didn't you just say let us compute 'G of b' minus 'G of a', where 'G prime' equals 'f', and the heck with all this summation process? Now, I think the best way to answer that question-- and this is probably the ultimate in teaching technique, when somebody thinks that he has an easier way than the way that he was taught-- give them a problem that can't be done by that easier way.

In fact, as a sarcastic aside, if you can't invent that kind of a problem, maybe the so-called easier way is the better way of doing a thing. But let me just show you what I'm driving at over here. Suppose I say to you, let's find the area of the region 'R' where 'R' is now the following. It's bounded above by the curve, 'y' equals '1/x'. It's bounded below by the x-axis. It's bounded on the left by the line, 'x' equals 1, and on the right by the line, 'x' equals 2.

I would like to find the area of this region 'R'. Now look it, from a purely conceptual point of view, what difference is there between this problem and the problem where the upper curve was 'y' equals 'x squared'? Conceptually, what's going on is the same thing. We have a bounded region and we want to compute the area. To this end, notice that, if I use the precise definition of a definite integral, the area of the region 'R' is what? The definite integral from 1 to 2, 'dx/x'.

What does that mean? It means a particular limit. What limit? Well, you partition this integral from 1 to 2 into n equal parts, pick a point, 'c sub k' in each partition, then 'f of 'c sub k'' in this problem is just '1/'c sub k''. In other words, what we want to do is to compute this limit. 'Sigma k' goes from 1 to 'n', '1/'c sub k'', times 'delta x'.

Now, the thing to notice is, this thing may be a mess but it's computable. We could use specially designed graph paper or measuring devices to count the units of area under here. We can find all sorts of ways of getting estimates, even the long hard way of the 'U sub n's and the 'L sub n's to pinpoint the area of the region 'R' to as close a degree of accuracy as we want, et cetera.

The point is that, if we have never heard of a derivative, the area of the region 'R' is given precisely by this sum. And admittedly, the sum is a mess. So let's try to do it the so-called easier way. The skeptic looks at this thing and says, who needs this? The area of the region

'R', we just saw by the first fundamental theorem, is 'G of 2', 'G of 2', minus 'G of 1', where 'G' is any function of 'x' whose derivative with respect to 'x' is '1/x'.

And the answer to that is yes. So far, so good. But do we know a function 'G' whose derivative with respect to 'x' is '1/x'? Remember, this is calculus revisited. For those of you who remember calculus from the first time around-- and I'll talk about this later-- it's going to turn out that the required 'G' is a logarithmic function.

For those of you don't remember that, there's no harm in not remembering that. All I'm trying to bring out is that, whether you remember it or you don't, as far as we're concerned so far in this course, as far as what we've developed in this course, we do not know a function 'G' whose derivative with respect to 'x' is '1/x'.

That's what we mean by saying we can't exhibit 'G' explicitly. What do I mean by that? Somebody says, I am thinking of a function whose derivative is '1/x'. I say, well, that's simple. It's 'G of x'. He says, well, what's 'G of x'? I say, that's the function whose derivative with respect to 'x' is '1/x'. Well, you see, that implicitly tells me what 'G' is like.

But in terms of concrete measurements, I don't know anything about 'G'. I can't express it in terms of well-known, familiar functions. You see, I'm hung up now. Namely, this is precise. The answer to this problem will be 'G of 2' minus 'G of 1', where 'G prime' is '1/x'. But I don't know explicitly-- yikes-- I don't know, explicitly such a 'G'.

And I put the exclamation point out here to emphasize that particular fact. That's the hang-up. The statement that says that the area is 'G of b' minus 'G of a', where 'G prime' equals 'f' hinges on the fact that you can explicitly exhibit such as 'G'. Certainly, if you can't exhibit that 'G', you can still compute this area as a limit. It would be tragic to say, oh, the area doesn't exist because I don't know a 'G' whose derivative is '1/x'. Certainly, this region 'R' has an area.

In fact, because we can pinpoint the area of the region 'R', we can actually construct the 'G' such that 'G prime of x' equals '1/x'. And by the way, there's plenty of drill on this. This is a hard concept. And as a result, you'll notice that the exercises in this section hammer home on this point, because it's a point that I'm positive that, if you're having trouble at all with integral calculus, this is certainly the most sophisticated part of what we're doing right now.

You see, look it, suppose I want to construct a function 'G' whose derivative is '1/x'. The idea looks something like this. I'm on the interval from 1 to 2 here. What I do is, I plot the curve, 'y'

equals '1/x'. And I compute the area of the region formed by the curve, 'y' equals '1/x'. 'x' equals 1. The x-axis and the line, 'x' equals 'x1'. This is my region 'R'. And what I say is, look it, what do I know about this area?

Remember, what I started this lecture with was the knowledge that a prime of 'x' is 'f of x'. 'f of x', in this case, is '1/x'. In other words, what property does this area function have? It has the property that its derivative with respect to 'x' is going to be '1/x'. In other words, if I now say, look it, boys, I can compute this area, if not exactly, at least to as many decimal place accuracy as I wish, so that the area function is something I can construct.

Let me define 'G of x1' simply to be the area under this curve when 'x' is equal to 'x1'. In other words, written now as a limit-- see how this definite integral comes in here-- written as integral from 1 to 'x sub 1', 'dx/x', which I now can compute as closely as I want as a limit.

The beauty is what? That 'G prime of x1' is, by definition, 'A prime of x1'. And we have already seen that the derivative of 'A' is 'f', where 'f' is the top curve. In this case, the derivative of 'A' is '1/x'. In other words, the area under the curve explicitly can be computed as a function of 'x'. And that function has the property, but its derivative with respect to 'x', in this case, is '1/x'.

And by the way, let me make just a quick aside over here. I happened to throw in the word "logarithms" before. It's going to turn out, as we'll talk about in our next block of material, that 'G of x' turns out to be the natural log of 'x', which means that the area of the region 'R' is precisely natural log 2. Now, the idea is-- and this is exactly what's going to happen in our last block in our course-- that this is exactly how the log tables are constructed, in fact.

I mean, how do you think they find the log of 2 to eight decimal places? Did they strand somebody on a desert island and tell them to compute these things, measure it with laser beams, what? No. The way we do it is, we know it's an area, and knowing how to find the area as a limit to as many decimal place accuracy as we wish, give or take a few tricks of the trade, this is how we find logs, and later on, trigonometric tables, and the like. But I just mention this as an aside to show you how important knowing area under a curve is.

At any rate, to summarize what we're saying over here, we have the following. In general, if 'f' is any continuous function on the closed interval from 'a' to 'b', we define a function 'G' as follows. 'G of x1' is the definite integral from 'a' to 'x1', 'f of x' 'dx', where 'x1' is any point in the closed interval for 'a' to 'b'. What does this thing mean, geometrically? It's the area under the curve, 'y' equals 'f of x' on top. 'x' equals the x-axis on the bottom, the line 'x' equals 'a' on the

left, the line 'x' equals 'x1' on the right.

And that area is what? It's the limit as 'n' approaches infinity. 'Sigma k' goes from 1 to 'n', 'f of 'c sub k'' 'delta x', et cetera, et cetera, et cetera, meaning we can go through this the same way as we did in our previous lecture. This function, 'G of x1'-- which we can now compute explicitly as an area, because we have the curve 'f' drawn, we can approximate it to as close a degree of accuracy as we want-- whatever that function 'G' is, what property does it have? It has the property that 'G prime' is equal to 'f'.

And that's exactly what the second fundamental theorem really means. The second fundamental theorem says, look it, if you can compute the area, you can now reverse this procedure that we're talking about in the first fundamental theorem. See, in the first fundamental theorem, what did we do? In the first fundamental theorem, we found a quick way of computing the area as a limit by knowing the appropriate inverse derivative.

The second fundamental theorem, in a sense, is the inverse of the first fundamental theorem. Namely, it switches the roles. Namely, with the second fundamental theorem, what we say is this. If we know how to find the area under the curve, 'y' equals 'f of x', that gives us a way of computing a function 'G' whose derivative is 'f'.

In fact, to summarize this-- and I think this is very important. Let me just summarize this, then. First of all, the first fundamental theorem allows us to compute this particular limit, provided we can find a 'G', such that 'G prime' equals 'f'. In fact, in this case, the limit, as 'n' approaches infinity, summation 'k' goes from 1 to 'n', 'f of 'c sub k'' 'delta x', is given very, very ingeniously, beautifully, and concisely by 'G of b' minus 'G of a'.

Secondly, the second fundamental theorem allows us, given 'f', to construct 'G', such that 'G prime' equals 'f'. Namely, the required 'G' is just the definite integral from 'a' to 'x1', 'f of x' 'dx', which in turn is a limit of a network of rectangles, namely the limit as 'n' approaches infinity, summation 'k' goes from 1 to 'n', 'f of 'c sub k'' times 'delta x'.

This then is perhaps the most beautiful lecture in calculus, not necessarily the way I've given it, but in terms of what? Relating two apparently diverse branches of calculus, integral and differential calculus. This is a difficult subject. I have taken great pains to try to write this clearly in the supplementary notes.

I have tried to pick typical exercises that bring home these highlights. And in addition, our next

two lectures will emphasize the relationship between derivatives and integrals as we study volume and arc length. At any rate, until next time, goodbye.