HERBERT GROSS: Hi, our lecture today is about curve plotting. And actually, I call it 'Curve Plotting with and without Calculus' to emphasize the fact that what we're interested in is curve plotting and that calculus, in particular, differentiation, gives us a powerful tool that is not available to us, at least in an accessible form without the calculus. Let's see what I mean by this by going back to a typical high school type analytical geometry problem. For example, sketch the curve 'y' equals 'x squared'.

Now we all know that the graph of 'y' equals 'x squared' is something like this. And how'd we find that? You may recall that in the truer sense of the word, the pre-calculus approach really is curve plotting as opposed to curve sketching. And I hope I make that a little bit clearer as we go along. Namely, you look to see for certain values of 'x' what value of 'y' corresponds to that. And then we locate the corresponding point 'x' comma 'y' in the plane of the blackboard. And what we then do is somehow or other, take a French curve, or whatever it happens to be, and we sketch a smooth curve through the given points. This is the usual technique.

The question that comes up is, that as long as there are spaces between the points that you've sketched, how can you be sure that the curve that you've drawn isn't inaccurate? In other words, starting in reverse here, let's suppose that these are the points I've happened to sketch for the curve 'y' equals 'x squared'. And by the way, as is often the case in our course, don't be misled that we pick something as simple as 'y' equals 'x squared'. I simply, again, wanted to pick something that was straightforward enough so that we could concentrate our attention on what the mathematical implications were rather than to be bogged down by computation. But the idea is this.

Going back to the problem 'y' equals 'x squared', suppose we had located these points and now we said, let's sketch a smooth curve through these. What would have been wrong with say, doing something like this? Why, for example, couldn't this have been the curve?

Now again, notice that in terms of mathematical analysis, not necessarily calculus but in terms of mathematical analysis, we could immediately strike out something like this. For example, notice in terms of our input versus output, this says that for any real input since the square of a real number cannot be negative, the output can never be negative. And pictorially, this means that no portion of our diagram can be below the x-axis.

In other words, without knowing anything about calculus but knowing a little bit about arithmetic, we can supplement our knowledge of how the points go by saying look-it . Something like this can't happen because in this region over here 'y' would be negative. And we know from the relationship that 'y' equals 'x squared' that 'y' cannot be negative. Well you see, armed with this information, we could say, OK, given the same points here, why couldn't the curve go through something like this? That's what the question mark is in here for. Obviously, this is not the graph of 'y' equals 'x squared'.

But the question is, noticing that this curve doesn't go below the x-axis, what's wrong with this? And again, we can get bailed out by a pre-calculus knowledge of mathematics. Among other things, notice the rather interesting property that if we replace 'x' by 'minus x' here, we get the same curve as before.

To generalize this result, notice that in this particular case, 'f of x' gives us the same result as 'f of 'minus x''. Now this will not happen in general.

What does that mean if we know that 'f of x' equals 'f of 'minus x''? Well, look it. The relationship between 'x' and 'minus x' is clear, they are located symmetrically with respect to the y-axis. In other words, if this is 'x1', this will be 'minus x1'. Now, here's the point. Let's suppose our curve happens to be 'y' equals 'f of x'. When the input is 'x1', the output will be 'f of x1'. When the input is 'minus x1', the output will be 'f of 'minus x1''.

To say that 'f of 'minus x1'' equals 'f of x1' is the same as saying that not only are these two coordinates, 'x1' and 'minus x1', symmetric with respect to the y-axis, but the outputs also are. In other words, to say that 'f of x1' equals 'f of 'minus x1'', says that not only are these two points the same left and right displacement for the y-axis, but they're also the same height above the x-axis. In short, this point is the mirror image of this point. In terms of a graph, if 'f of x' equals 'f of 'minus x'' for all 'x', the graph of 'y' equals 'f of x' is symmetric with respect to the y-axis.

By the way, that's why we could rule out this type of possibility here. For example, here's 'x1' over here. Here's 'minus x1'. And notice that for this choice of 'x', the points on the curve are not mirror images of one another with respect to the y-axis. In short, even though we may not know what this curve is supposed to look like, the fact that 'f of x' equals 'f of 'minus x'' tells us that whatever the graph is, it should be symmetric with respect to the y-axis.

This leads us to a rather interesting aside. It's something called 'even and odd functions' that we could technically leave out here, but which I think is a good place to bring in. And the fact that there are many, many places in our advanced treatment that will come up later where it's important to understand what these things mean, that perhaps this is a nice place to bring it into play.

So I call this an aside and it's about even and odd functions. When we have the case that 'f of x' equals 'f of 'minus x'', that's the case where in terms of the graph you have symmetry with respect to the y-axis such a function is called an 'even function'. And I'll come back in a moment and show you why it's called even, though it's not really important. The counterpart to an even function as you may guess almost from the association of ideas is called an 'odd function'.

Now an odd function is one for which 'f of x' is the negative of 'f of 'minus x''. See for 'f' to be odd, 'f of x' has to be the negative of 'f of 'minus x''.

Now what does this mean pictorially? Again, 'x1' and 'minus x1' are symmetrically located with respect to the y-axis. This height would be called 'f of x1' and this height here would be called 'f of 'minus x1''. And to say that these two heights are equal in magnitude but opposite in sign means what? That these two lengths are equal but on opposite sides of the x-axis. In other words, here's 'f of x1'. 'f of 'minus x1'' is the height corresponding to this point. And the fact that these must be equal in magnitude but opposite in sign says that one of the heights must be above the axis, the other one must be below the axis. And if that's the case from some very elementary geometry, if 'f' is odd, what it means with respect to its graph is this.

If you put a ruler connecting the origin with any point on the curve, if you extend that line that line will meet the curve again, such that these distances here will be equal. This is called symmetry with respect to the origin.

By the way, an example of this kind of a curve is 'y' equals 'x cubed'. You see, if I replace 'x' by 'minus x', this becomes what? 'Minus x cubed'. And 'minus 'minus x cubed'' is 'x cubed'.

By the way, all I'm saying now is what? If I were to take a ruler and place it here and let this line go from curve to curve. all I'm saying is that this portion would equal this portion.

And by the way, maybe you can already guess where the words even and odd come from. Notice that when I dealt with 'y' equals 'x squared' I had an even function. The exponent was even. When I dealt with 'y' equals 'x cubed', I had an odd function. The exponent was odd.

And by the way, there are other examples. But for the time being, notice something like say, 'y' equals 'x to the fourth' plus 'x squared' say. If I replace 'x' by 'minus x', I get back the same thing. In other words, here's a case where if 'y' equals 'f of x', 'f of x' is the same as 'f of 'minus x''.

An example of an odd function might be 'y' equals 'x cubed' say, plus 'x'. See first power over here. If I replace 'x' by 'minus x', I have 'minus x cubed' plus 'minus x'. That of course, is just 'minus x cubed' minus 'x'. And that's minus the quantity 'x cubed + x'. That when I replace 'x' by 'minus x' all I do is change the sign here. In other words, that would be an example for which 'f of x' is minus 'f of 'minus x''.

By the way, while we're dealing with examples like this, unlike the case with whole numbers where a whole number is either even or odd but not both, it's rather important to notice that a function need not be either even or odd.

And by the way, if you think of our geometric definition, that's not too hard to see. Namely, there's no reason why a curve drawn at random should be symmetric either with respect to the y-axis or with respect to the origin. In fact, maybe the quickest way to see this is to put an odd power of 'x' in with an even power of 'x' in the same diagram.

If I now replace 'x' by 'minus x' you see 'minus x' cubed is of course, 'minus x cubed'. But 'minus x' quantity squared is just 'x squared'. And now you'll notice that if I compare these two, I don't get the same thing nor do I get the same thing with a sign change. You see, in other words, this is not either equal to this or to the negative of this. Obviously you understand that to be the negative of this, there would also have to be a minus sign over here.

Or if we wanted to go into more detail about this, it is perhaps exciting to know that whereas it's not true that every function is either even or odd, every function can be written as the sum of two functions, one of which is even, one of which is odd. Just to give you an idea of what that means let me just write down something here. This will come back to be more important later on, but for the time being, just to show you a connecting thread here as long as we're talking about even and odd functions, all I want to see is the following identity.

If you write ''f of x' plus 'f of 'minus x'' over 2. And don't worry about what motivates this, I just wanted to show you this to keep this fairly complete. Now suppose I add on to that ''f of x' minus 'f of 'minus x'' over 2. You see, notice that the expression on the right-hand side is just writing 'f of x' the hard way. See, here's half of 'f of x', here's half of 'f of x'. The sum is 'f of x'. And here's half of 'f of 'minus x''. And I'm subtracting half of 'f of 'minus x''. That drops out. The point is that this is always an even function and this is always an odd function. And just to review the definition so that you see what happens here, all I'm saying is if I replace 'x' by 'minus x' here, what happens?

If I replace 'x' by 'minus x' this becomes 'f of 'minus x''. And if I replace 'x' by 'minus x' here since minus minus is plus, this becomes 'f of x'. Now notice that when you add two terms, the sum is independent of the order in which you add them. 'f of x' plus 'f of 'minus x'', therefore, is the same as 'f of 'minus x'' plus 'f of x'. In other words, when I replace 'x' by 'minus x' in this bracketed function, I do not change the value of what's in the brackets.

On the other hand, if I interchange 'x' with 'minus x' here, notice that since I'm subtracting look what happens. I replace 'x' by 'minus x'. This gives me an 'f of 'minus x''. Now I make the same replacement here. Minus minus is positive. But now notice that if I look at this expression here, I've changed the order. See 'f of x' minus 'f of 'minus x''. Here, 'f of 'minus x'' minus 'f of x'. And when you change the order you change the sign. In other words then, all I'm saying is that when we talk about even and odd functions, they play a very important role in calculus and in other mathematical analysis topics. That not every function is either even or odd, but every function that's defined on the appropriate domain is the sum of both an even and an odd function.

But in terms of curve plotting, the main point is not so much these extra remarks as much as what? An even function is symmetric with respect to the y-axis and an odd function is symmetric with respect to the origin.

At any rate, if we now go back to our curve 'y' equals 'x squared' the fact that 'f of x' equals 'x squared' is even means that whatever our graph looks like, it must be symmetric with respect to the y-axis. Now give or take how I've drawn this, this should be symmetric with respect to the y-axis. If it doesn't look that way, imagine that it is that way. And so the idea is what? Well, any knowledge of calculus whatsoever, what I was able to do here is show that whatever the graph of 'y' equals 'x squared' is, it must never dip below the x-axis. And whatever the curve looks like to the right of the y-axis, it must be the mirror image of that to the left of the y-axis. Again, this is how much one can do without calculus. And most of you who are practicing engineers, I'm sure not only understand this type of technique as far as the pre-calculus is concerned, but can probably draw curves much better than I can. In fact, even if you're not practicing engineers you can probably draw curves much better than I can. But that part is irrelevant.

What I wanted to show up to now-- and this is what's important to stress-- is that to get as far as I've gotten so far, I did not have to have any knowledge of calculus. The way calculus comes in, as I say again, is a supplement to our previous techniques.

For example, let's suppose we did have the curve drawn this way. From this, we certainly aren't contradicting the fact that 'x' can't be negative. We're not contradicting the fact that 'f' is an even function. But how do we know that this is the wrong picture? Well, given that 'y' equals 'x squared', we can easily verify that 'dy dx' is '2x.' Knowing that 'dy dx' is '2x', that tells us among other things that 'dy dx' and 'x' have the same sign. In other words, 'dy dx' is positive if 'x' is positive, 'dy dx' is negative if 'x' is negative.

In terms of geometry, that means what? That the curve is always rising for positive values of 'x' and always falling for negative values of 'x'. Well, you see with that as a hint, I say ah-ha. This can't happen. Because look what's happening over here. Or for that matter, over here. Here the curve is falling for positive values of 'x'. And that contradicts the fact that the curve must always be rising when 'x' is positive.

In a similar way, we know that the curve can't be rising when 'x' is negative, yet get over here and here too, we've drawn the curve to be rising. That again is contradicted by this diagram. So you see the knowledge of the first derivative does what? It tells us where the curve is rising or falling, and that gives us another way of checking whether the graph we've drawn is accurate or not.

By the way, it's not all quite that simple. And notice again, subtlety how step by step we strengthen our procedures each time. For example, now knowing that the curve must always be rising when 'x' is positive and always falling when 'x' is negative, how about this possibility for the graph of 'y' equals 'x squared'. You see this curve is always fallen for negative values of 'x', it's always rising for positive values of 'x'.

Let's take a look now at what the second derivative means. If 'y' equals 'x squared', obviously as we saw before, 'dy dx' is '2x' and the second derivative of 'y' with respect to 'x' is 2. And 2 is a constant, which is always positive. This says what? That the second derivative is always positive.

Now what is the second derivative? The second derivative is the first derivative of the first derivative. That means the rate of change of the rate of change. Well, the rate of change of the rate of change is called acceleration. So if the rate of change of the rate of change is positive that means that the curve must be accelerating, or the function is accelerating. And if it's negative, function is decelerating. What does that mean in terms of a picture? And the author of the text uses a very descriptive phrase here. He talks about 'holding water' and 'spilling water'.

Notice, for example, here the curve would tend to collect water, whereas here if water were poured on it, the curve would tend to spill water. Holding water represents acceleration. You see that not only is the curve rising here, but it's rising at a faster and faster rate. Again, more primitively, in terms of slopes, notice that what? Not only is the slope positive, but as you move along this portion of the curve, the slope increases as you move along. And what typifies this portion? That even though the slope is always positive, it decreases as you move along the curve. In other words, 'holding water' corresponds to the second derivative being positive, 'spilling water' corresponds to the second derivative being negative.

Returning then to our original problem, the fact that this thing here is greater than 0 for all 'x' says that this curve could never spill water. And that rules out this portion in here. In other words, now we put everything together, we know what? The curve can never go below the x-axis. It's symmetric with respect to the y-axis. It's always rising when 'x' is positive and always holding water.

Now you see this is what I call curve sketching versus curve plotting. With the information that I have from calculus, I know what's going on for each point, not just for the isolated points that I happened to have plotted in the data for. See, the calculus fills in the missing data very, very nicely. Now you see this does not mean I'm going to replace this by-- my previous analysis by calculus. It means I'm going to add calculus as one of my bags of tools here.

Notice again that there is a very nice relationship between pictures and analysis. And I'm not going to belabor that point. All I'm saying is that if you add to our previous identifications-- what identifications? Like increase means rising. I mean this, if a derivative is positive, the curve is rising. If a derivative is negative, the curve is falling. If the second derivative is positive, the curve is holding water. If the second derivative is negative, the curve is spilling water. And again, we have ample exercises and portions of this in the reading material to illustrate the computational aspects. But again, all I want you to get from this lecture is what's happening here conceptually.

Let's look at this, a few more applications. We're going to find in subsequent lectures as well as other portions of the course, we're going to be interested, for example, in things called stationary points. A 'stationary point' is a point at which the curve is neither rising nor falling. In other words, if the curve happens to be smooth, it's characterized by the fact that 'dy dx' is 0 it such a point. In terms of the language of functions to say that the curve is smooth means that the function is differentiable and what we're saying is to be stationary at 'x' equals 'x1', it must be that 'f prime of x1' equals 0. And the importance of stationary points can be seen in terms of a physical interpretation. We haven't used our freely falling body for quite a while, let's go back to such an example.

Suppose a particle is projected vertically upward in the absence of air resistance, et cetera, with an initial speed of 160 feet per second. It can then be shown that the height 's' to which the ball rises in feet at time 't' is given by '160t - 16't squared''. A very natural question to ask is, when will the ball be at its maximum height? And I'm sure you can see in terms of this physical example, the ball will be at its maximum height when the velocity is 0. In other words, since this is a smooth type of motion, if the velocity is not 0, the ball is still rising. If the velocity is positive, the ball is still rising. If the velocity is negative, the ball is already falling. Consequently, if the speed is smooth, which it is in this case, the only way it can go from rising to falling is to first level off and the velocity must be 0.

But you see, what is the velocity? The velocity is the derivative of the displacement. In other words, without solving this problem, which is not important here, to find the time at which you have the maximum height, you simply do what? Set the derivative equal to 0. In other words, stationary points tell us where we have high and low points for functions. And knowing where we have high and low points is a very important portion in curve plotting. We'll talk about this in a future lecture very shortly in more detail.

But coming back here to what we were talking about before, what we're saying in terms of curve plotting is that where the derivative is 0 gives us a good candidate to have either a low point on the curve or to have a high point on the curve. However, we should not read more into this than what's already there. Namely, it's possible that you have what? A situation like this, in which the derivative is 0 here, but the curve is rising every place.

And secondly, there is the possibility that if the derivative doesn't exist, for example, if there's a sharp corner. Where you have a sharp corner notice that-- see this is a straight line, this is a straight line. In this special case, the derivative is the slope of this straight line. Derivative here is the slope of this straight line. Therefore, the derivative is positive on this line, negative on this line. Yet the point is what? At their junction, there's a discontinuity. The function is continuous, but the derivative isn't. And at that particular point notice that you have a high point even though the derivative doesn't exist at that particular point.

All we're saying is what? That for a smooth curve if there is to be a high point or a low point, a maximum or a minimum, and we'll talk about this as I say, in a future lecture, it must be that at particular point the derivative is 0. On the other hand, conversely, if the derivative is 0, you may not have a high or low point. It may be what we call a saddle point, the curve just levels off after rising and then rises again. And secondly, if the function isn't differentiable, or the curve isn't smooth at that particular point, you can have a high or a low point regardless of what the knowledge about the derivative is. Just a little buckshot here to give you an idea of how we're going to use this material.

A very related topic that's also quite important here is something called points of inflection. Points of inflection are, in a way, to the second derivative what stationary points are to the first derivative. In many cases, we are interested in knowing, where does the curve change its concavity? Where does it go, in other words, from holding water to spelling water? And by the way, again, in terms of a geometrical interpretation, there's a very what I call exciting answer to this question. It almost results in what looks like an optical illusion.

You see, if a curve is holding water, the tangent line lies below the curve. If the curve is spilling water, the tangent line to the curve at a point lies above the curve. Consequently, at a 'point of inflection', meaning where the curve changes concavity, the tangent line on one side must be above the curve, on the other side below the curve. And all I'm saying over here is that you recognize a point of inflection by what? It's the situation in which the tangent line to the curve appears-- in fact, it actually does in the manner of speaking, cross the curve.

I think we talked about this in the previous lecture, I'm not sure. But we talked about the idea that a tangent line can cross the curve. And where can it cross the curve and still be a tangent line at that point? At a point of inflection.

By the way, if this is a smooth type of thing in general, what we're saying is that for a point of inflection to occur, the second derivative must be 0. You see because that means what? The curve is neither-- it's going from holding to spilling, so it goes through a transition where it's doing neither. In the same as with first derivatives, the mere fact that the second derivative is 0 does not allow us to conclude that we have a point of inflection. In fact, let me close with this particular illustration.

Let's take the curve 'y' equals 'x to the fourth'. The first derivative is '4 x cubed'. The second derivative is '12 x squared'. The curve is symmetric with respect to the y-axis, et cetera. Using all of the given data you know the second derivative is always positive, what have you. We can sketch this curve. And again, in fact the uninitiated say this curve looks like a parabola. What do you mean it looks like a parabola? Well, he says, the parabola does something like this too. Well, what do we mean by something like this?

I want to mention a few points here. One is, of course, that actually, the parabola 'y' equals 'x squared'. These are going to crisscross very shortly here. It doesn't make any difference. The parabola 'y' equals 'x squared' has the same general shape but with a few different properties, which we'll mention in a little while. But the point that I wanted to mention here first of all is simply this. At the value 'x' equals 0, 'y double prime' is 0. So you notice that the second derivative is 0 over here. Yet even though the second derivative is 0, notice that the curve does not change concavity. The curve here is always holding water.

The concluding remark that I wanted to make is, what is the relationship between 'y' equals 'x squared' and 'y' equals 'x to the fourth'? Or how about 'y' equals 'x to the sixth'? Or 'y' equals 'x to the 12th'? Notice that any curve in that family will look something like this. Only what? As the exponent goes up, the curve becomes broader in the neighborhood of 0. And then once 'x' passes 1, the curve rises more sharply. See what we're saying is, if the magnitude of 'x' is less than 1, the higher a power you raise it to, the smaller 'y' is.

On the other hand, if the absolute value of 'x' is greater than 1, the higher a power you raise it to, the bigger the output becomes. But the idea is this. Notice that the exponent-- in other words, how many derivatives are 0, gives you a way of getting into a problem that will become very, very crucial as this course continues. And it's the idea of, can one curve be more tangent to a line than another curve? You see, all of these curves are tangent to the line what? The x-axis, 'y' equals 0 at 'x' equals 0. How do we distinguish between these curves? Well, it seems that some of these curves fit the x-axis better than others in a neighborhood of the point 'x' equals 0. See the point that I want to bring out as to 'y' curve plotting tells us things that we don't learn in the ordinary physics class is this.

If you study calculus the way it comes up in most physics courses, we essentially don't go past the second derivative. Why? Because in many cases, we're studying distance. And the derivative of distance is velocity. The second derivative of distance, namely the derivative of velocity is acceleration. And we don't usually talk physically beyond acceleration.

But notice that in terms of curve plotting the third, fourth, fifth, sixth, seventh, tenth derivatives all have a meaning that gives you more information than what came before. Don't be deceived by the fact that in other applications that you never go past the second derivative means that there is no value to knowing how higher order derivatives are related to plotting curves. At any rate, I think this is enough of an introduction to the topic of curve plotting and curve sketching. We'll pursue these topics further in our next lectures. So until next time, goodbye.

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