Practice Final SOLUTIONS 18.01

1 a) $\dfrac{x^2\cdot\frac{1}{x} - \ln x\cdot 2x}{x^4} = \dfrac{1-2\ln x}{x^3}$

b) $\frac{1}{2}(3\sin^2 u+2)^{-1/2}\cdot 6\sin u\cos u$

$= \dfrac{3\sin u\cos u}{\sqrt{3\sin^2 u+2}}$

c) $D^n e^{kx} = k^n e^x$; at 0: $\boxed{k^n}$

2 $D_x(x^2y^2+y^3) = 2xy^2+x^2\cdot 2yy'+3y^2y'$
$= 0.$

At $(1,1)$: $2+2y'+3y'=0$; $y'=-\frac{2}{5}$

Eq'n: $(y-1)=-\frac{2}{5}(x-1)$ or

$y=-\frac{2}{5}x+\frac{2}{5}$

3 $y=\cos^{-1}x$

$x=\cos y$

$1=-\sin y\cdot y'$

$y'=\dfrac{-1}{\sin y}$

$=\dfrac{-1}{\sqrt{1-\cos^2 y}}$

$\therefore\ y'=\dfrac{-1}{\sqrt{1-x^2}}$

(use $+\sqrt{\ }$ since $\sin y>0$)

4 $\begin{cases} x^2+x+a \\ bx+2\end{cases}$ same value at 0 (for continuity)

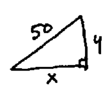
$\Rightarrow \boxed{a=2}$

same deriv. at 0 (for diff'y)

$\Rightarrow 2x+1=b$ at 0

$\therefore \boxed{b=1}$

5



$x^2+y^2=50^2$

$\cancel{2}x\frac{dx}{dt}+\cancel{2}y\frac{dy}{dt}=0$

when $y=30$, $x=40$

$40\cdot\frac{dx}{dt}+30(-2)=0$

$\frac{dx}{dt}=\frac{60}{40}=\boxed{\frac{3}{2}}$ ft/sec

6 $A = 2x\cdot(1-x^2) = 2x-2x^3$

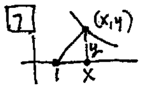
$\frac{dA}{dt} = 2-6x^2 = 0$ if $x^2=\frac{1}{3}$

$x=\frac{1}{\sqrt{3}}$

Area then is:

$2\cdot\frac{1}{\sqrt{3}}\left(1-\frac{1}{3}\right)=\boxed{\frac{4}{3\sqrt{3}}}$

7



$-\dfrac{1}{y'}=\dfrac{y}{x-1}$ (negative recip. slope)

a) $\therefore\ y'=\dfrac{1-x}{y}$

b) $\dfrac{dy}{dx}=\dfrac{1-x}{y}$

$y\,dy=(1-x)\,dx$

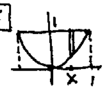
$\frac{1}{2}y^2=x-\frac{1}{2}x^2+c_1$ $\left[\text{or better: } -\frac{1}{2}(1-x)^2+c_2\right]$

$y^2=2x-x^2+c$

c) $y^2+(x-1)^2=c_3$ (completing square)

circles centered at $(1,0)$

8



Volume $= \int_0^1 2\pi x(1-x^2)\,dx$

$= 2\pi\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big]_0^1$

$= 2\pi\cdot\frac{1}{4}=\boxed{\frac{\pi}{2}}$

[or: $\pi\cdot 1^2\cdot 1 - \int_0^1 2\pi x^3\,dx = \pi-\frac{\pi}{2}=\frac{\pi}{2}$]
vol. cylinder, vol. under curve

9

x	0	$\pi/6$	$\pi/3$	$\pi/2$
$\sin x$	0	$1/2$	$\sqrt{3}/2$	1
$\sin^2 x$	0	$1/4$	$3/4$	1

Trap. rule: $\frac{\pi}{6}\left(\frac{1}{2}\cdot 0+\frac{1}{4}+\frac{3}{4}+\frac{1}{2}\cdot 1\right)$

$=\frac{\pi}{6}\cdot\frac{3}{2}=\boxed{\frac{\pi}{4}}$

10 $F(x)=\int_0^x e^{-t^2}dt$

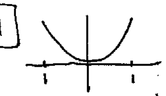
a) $F'(x)=e^{-x^2}$; $F''(x)=-2xe^{-x^2}$

$F'(1)=\frac{1}{e}$ $\quad F''(1)=-\frac{2}{e}$

b) $\int_1^2 e^{-u^2/4}du = \int_{1/2}^1 e^{-t^2}\cdot 2\,dt$

Put $t=u/2$, $dt=\frac{du}{2}$ $\quad = 2\left(F(1)-F(1/2)\right)$

11



$y=x^2/10$

$y'=x/5$

a) $\therefore$ arclength $=\int_{-1}^1\sqrt{1+\frac{x^2}{25}}\,dx$

b) average $=\int_0^1\frac{x^2}{10}dx=\frac{x^3}{30}\Big]_0^1$

$=\frac{1}{30}$ km ≈ 33 m

[or $=\frac{1}{2}\int_{-1}^1\frac{x^2}{10}dx=\dots$]

12 a) $\dfrac{1}{x^2+3x+2}=\dfrac{1}{(x+2)(x+1)}=\dfrac{-1}{x+2}+\dfrac{1}{x+1}$ (by cover up)

$\therefore\int_0^1\left(\frac{-1}{x+2}+\frac{1}{x+1}\right)dx = -\ln(x+2)+\ln(x+1)\Big]_0^1$

$=-\ln 3+\ln 2+\ln 2-0$

$=2\ln 2-\ln 3$ (or $\ln\frac{4}{3}$)

b) $\int x^2\ln x\,dx = \frac{x^3}{3}\ln x-\int\frac{x^3}{3}\cdot\frac{1}{x}dx$

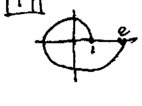
int. by parts $=\frac{x^3}{3}\ln x-\frac{x^3}{9}+c$

13 $\int_0^1\dfrac{dx}{(x^2+1)^2}=\int_0^{\pi/4}\dfrac{\sec^2 u}{\sec^4 u}du=\int_0^{\pi/4}\cos^2 u\,du$

$x=\tan u$ $\quad =\int_0^{\pi/4}\frac{1+\cos 2u}{2}du=\frac{u}{2}+\frac{\sin 2u}{4}\Big]_0^{\pi/4}$

$=\boxed{\frac{\pi}{8}+\frac{1}{4}}$

14



Area $=\frac{1}{2}\int_0^{2\pi}\left(e^{\theta/2\pi}\right)^2d\theta$

$=\frac{1}{2}\int_0^{2\pi}e^{\theta/\pi}d\theta=\frac{\pi}{2}e^{\theta/\pi}\Big]_0^{2\pi}$

$=\frac{\pi}{2}(e^2-1)$

15 a) $\lim_{x\to0}\dfrac{\sin^2x}{1-\cos x}=\lim_{x\to0}\dfrac{2\cancel{\sin x}\cos x}{\cancel{\sin x}}=2$

b) $\lim_{x\to1}\dfrac{(\ln x)^2}{x-1}=\lim_{x\to1}\dfrac{2\ln x\cdot\frac{1}{x}}{1}=0$

c) $\lim_{x\to\infty}\dfrac{x^2}{e^x}=\lim_{x\to\infty}\dfrac{2x}{e^x}=\lim_{x\to\infty}\dfrac{2}{e^x}=0$

16 $\int_1^\infty\dfrac{dx}{x^{3/2}}=\int_1^\infty x^{-3/2}dx=-2x^{-1/2}\Big]_1^\infty=0-(-2)$
$=2$

17 $\dfrac{n}{\sqrt{4+n^p}}\sim\dfrac{n}{n^{p/2}}\sim\dfrac{1}{n^{p/2-1}}$

$\therefore\sum\frac{n}{\sqrt{4+n^p}}$ converges if $\frac{p}{2}-1>1$

or $\boxed{p>4}$

18 $y=(1+x)^{1/2}$ $\quad$ at $x=0$: 1 $\quad$ coeff: 1

$y'=\frac{1}{2}(1+x)^{-1/2}$ $\quad 1/2 \quad 1/2$

$y''=-\frac{1}{2}\cdot\frac{1}{2}(1+x)^{-3/2}$ $\quad -1/4 \quad -1/4\cdot\frac{1}{2!}$

$y'''=-\frac{3}{2}\cdot-\frac{1}{2}\cdot\frac{1}{2}(1+x)^{-5/2}$ $\quad 3/8 \quad 3/8\cdot\frac{1}{3!}$

$\therefore(1+x)^{1/2}=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$

$(1.2)^{1/2}=1+\frac{1}{2}\cdot\frac{2}{10}-\frac{1}{8}\cdot\frac{4}{100}+\frac{1}{16}\cdot\frac{8}{1000}$

$=1+.1000-.005+.0005$

$=\boxed{1.0955}$

19 $y=\tan^{-1}x$

$y'=\dfrac{1}{1+x^2}=1-x^2+x^4-x^6+x^8\dots$

Integrate term by term $\therefore\ y=\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}\dots+c$

$c=0$ since $\tan^{-1}0=0$.