

Problem 1. (15 points) Evaluate $\int \frac{dx}{x(x+1)^2}$

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

"Cover-up" method yields $\boxed{A=1}$ $\boxed{C=-1}$
 $(x=0)$ $(x=-1)$

Let $x=1$:

$$\frac{1}{4} = \frac{1}{1} + \frac{B}{2} + \frac{-1}{4} \Rightarrow \boxed{B=-1}$$

$$\int \frac{dx}{x(x+1)^2} = \int \left[\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} \right] dx$$

$$= \boxed{\ln|x| - \ln|x+1| + \frac{1}{x+1} + \text{const.}}$$

(Also ok: $\ln x - \ln(x+1) + \frac{1}{x+1} + \text{const.}$)

Problem 4. a. (10 points) Find an integral formula for the arclength of the curve $y = 2\sqrt{x+1}$ for $0 \leq x \leq 1$. Do not evaluate.

$$y' = \frac{1}{\sqrt{x+1}}$$

$$\text{Length} = \int_0^1 \sqrt{1+y'^2} dx = \int_0^1 \sqrt{1 + \frac{1}{x+1}} dx$$

b. (10 points) Find an integral formula for the surface area of the curve in part (a) rotated around the x-axis. Simplify the integrand, and evaluate the integral.

$$\text{Area} = \int_0^1 2\pi y ds = \int_0^1 2\pi (2\sqrt{x+1}) \sqrt{1 + \frac{1}{x+1}} dx$$

$$= 4\pi \int_0^1 \sqrt{(x+1) + 1} dx$$

$$= 4\pi \int_0^1 (x+2)^{1/2} dx$$

$$= 4\pi \left[\frac{2}{3} (x+2)^{3/2} \right]_0^1$$

$$= \boxed{\frac{8\pi}{3} \left(3^{3/2} - 2^{3/2} \right)}$$

Problem 2. (15 points) Evaluate $\int (\ln x) x^2 dx$

$$\int \underbrace{\ln x}_u \underbrace{x^2}_{v'} dx = (\ln x) \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$$

$$\left[u' = \frac{1}{x}, v = \frac{x^3}{3} \right] = \boxed{\frac{x^3}{3} \ln x - \frac{x^3}{9} + C}$$

Problem 3. (20 points) Use a trigonometric substitution to evaluate $\int_0^1 \frac{dx}{(4+x^2)^{3/2}}$. (Be careful evaluating the limits.)

$$x = 2 \tan u, dx = 2 \sec^2 u du$$

$$\int_0^1 \frac{dx}{(4+x^2)^{3/2}} = \int_{u_0}^{u_1} \frac{2 \sec^2 u du}{(4+4 \tan^2 u)^{3/2}} \quad \left(\begin{array}{l} 2 \tan u_1 = 1 \\ 2 \tan u_0 = 0 \\ \Rightarrow u_0 = 0 \end{array} \right)$$

$$= \int_0^{u_1} \frac{2 \sec^2 u du}{8 \sec^3 u} = \frac{1}{4} \int_0^{u_1} \cos u du$$

$$= \frac{1}{4} \sin u_1 = \frac{1}{4} \cdot \frac{1}{\sqrt{5}}$$

$$2 \tan u_1 = 1 \Rightarrow \tan u_1 = 1/2$$

$$\sin u_1 = 1/\sqrt{5}$$

Problem 5. a. (7 points) Sketch the spiral $r = \theta^2, 0 \leq \theta \leq 3\pi$. Say how many times the curve meets the x-axis counting $\theta = 0$ as the first time, and mark those points with X's. (Your sketch need not be accurate to scale.)



b. (8 points) On your picture, shade in the region $0 \leq r \leq \theta^2, 0 \leq \theta \leq 2\pi$, and find its area.

$$\text{Area} = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} \theta^4 d\theta = \frac{1}{10} \theta^5 \Big|_0^{2\pi}$$

$$= \boxed{\frac{(2\pi)^5}{10}}$$

or $\frac{16\pi^5}{5}$

Problem 6. a. (10 points) Find the equation in polar coordinates for the line $y = x - 1$ in the form $r = f(\theta)$.

$$r \sin \theta = r \cos \theta - 1$$

$$\Leftrightarrow 1 = r \cos \theta - r \sin \theta$$

$$\Leftrightarrow \boxed{r = \frac{1}{\cos \theta - \sin \theta}}$$

b. (5 points) Find the range of θ for the portion of the line $y = x - 1$ in the range $0 \leq x < \infty$. (It helps to draw a picture.)

