Combinatorics: The Fine Art of Counting

Week Eight Problems

1. Diagrams of all the distinct non-isomorphic trees on 6 or fewer vertices are listed in the lecture notes. Extend this list by drawing all the distinct non-isomorphic trees on 7 vertices.

2. Give an example of a 3-regular graph with 8 vertices which is not isomorphic to the graph of a cube (prove that it is not isomorphic by demonstrating that it possesses some feature that the cube does not or vice-versa).

3. For each of the regular polyhedral graphs (tetrahedron, cube, octahedron, dodecahedron, and icosahedron), try to answer the following questions:

- What is its degree and diameter?
- Is it Eulerian? If so, find an Eulerian circuit, otherwise prove it is not.
- Can you find a Hamiltonian circuit?

4. Examples of the graphs of several saturated hydrocarbons are shown in the lecture notes - CH_4 (methane), C_2H_6 (ethane), C_3H_8 (propane), and others. The graph of a saturated hydrocarbon is always a tree where each hydrogen atom is represented by a vertex of degree 1 and each carbon atom is represented by a vertex of degree 4.

Given n carbon atoms, how many hydrogen atoms can be added to make up a saturated hydrocarbon, i.e. what chemical formulas for saturated hydrocarbons are possible?

In the examples listed in the lecture notes, both butane and isobutene have the same chemical formula C_4H_{10}, but different molecular structures. Is there another saturated hydrocarbon besides propane with the molecular formula C_3H_8? How many different saturated hydrocarbons are their with 5 carbon atoms? Can you conjecture a general formula?

5. For each of the graphs K_5 and $K_{3,3}$ count the number of distinct Hamiltonian cycles they contain. For K_5 count the number of distinct Eulerian circuits.

6. Consider the graph of H_n, the n-dimensional hypercube. The vertices of H_n can be labeled with binary strings of length n so that H_n contains 2^n vertices with edges between vertices which differ in 1 bit, e.g. the edges of H_2 are {00,01}, {01,11}, {11,10} and {10,00}. Note that H_n is a regular graph.

- What is the degree and diameter of H_n?
- For which values of n is H_n Eulerian?
- Find a Hamiltonian cycle for H_2, H_3, and H_4.
- Prove that H_n is Hamiltonian for all n (hint: use induction on n).

7. Consider a set of 21 dominos with the ends of each domino labeled with between 1 and 6 dots and each combination uniquely represented. In class we proved that you cannot construct a cycle using all 21 dominos with adjacent edges matching. What is the largest legal domino cycle you can make with this set?

A standard set of dominoes contains 28 dominoes with the ends of each domino labeled with between 0 and 6 dots. Every combination is uniquely represented by one domino. Prove that you *can* construct a legal domino cycle using all the dominos in this set.

8. At an international peace conference two delegates from each of five countries are to be seated for dinner at a round table with 10 places. In order to promote international communication, the delegates should be seated so that the two delegates from the same country are not adjacent, and so that every possible combination of adjacent countries occurs around the table. Find a seating arrangement that satisfies these constraints.

9. With a standard deck of 52 cards, what is the longest sequence of card you can construct so that any two adjacent cards are either the same suit or the same rank, but no three cards in a row have the same suit or rank?