The following content is provided under a Creative Commons license. Your support will help MIT OpenCourseWare continue to offer high-quality educational resources for free. To make a donation or view additional materials from hundreds of MIT courses, visit MIT OpenCourseWare at ocw.mit.edu.

JUSTIN CURRY: All right. Hello. Welcome to *Godel, Escher, Bach*-- a Mental Space Odyssey. My name is Justin Curry, and I'm a senior in mathematics here at MIT. I've spent the last year at Cambridge University at UK and, the summer before that, living in Germany. So it's kind of a reverse culture shock coming back, but I'm excited to teach *Godel, Escher, Bach* again.

I taught this course in spring, 2006. It was a 10-week course then. And we attempted the impossible task of trying to get through this thick monster all in one go. And it's impossible.

Most undergrads can't get through it in 13 weeks. I got through it in about seven years. So you're going to be attempting a feat here, not to complete the entire book but to get the essence of *Godel, Escher, Bach* out.

But I want to make sure we introduce everybody, just to get people's names. This will help me take attendance. And it will also-- I also want you to say, what is it when you read the course catalog that interested you most and why, essentially, why you're sitting here today? I'm curious.

So what is the idea behind this book? I interviewed a good many of you this morning, just to make sure that you guys felt comfortable with mathematics. This course isn't directly about mathematics. There's a lot of mathematics being talked about.

Yes, do you have a question?

STUDENT: What's this class about?

JUSTIN CURRY: OK. So that's what I'm going to go through right now. The idea here is that-- Douglas Hofstadter is interested in one primary question. And that question is, how does a self come out of things which have no selves?

How is it that all these carbon atoms and molecules and proteins which make us up in the physical universe, how do they go from being meaningless to developing into an entity which can refer to itself? Like, right now, I'm saying, I think this. I think you like this. I'm meeting all of you, as individuals.

Each one of you claim to have a self. You might remember Descartes' famous quote, "I think, therefore I am." So it seems like the I-- when I say the "I," I mean the things we call ourselves-- is a real, existent thing.

But it's a complex question. How do we get I's out of non-I's? And that's going to be the goal, over here. So I'm just going to call it I.

But how do you get to an I? You get to an I by having a bunch of meaningless primitives, things like atoms, proteins-- molecules, I should say-- et cetera. This is what you're made up of.

But none of these things mean anything. None of these things have I's or selves. But you do. So what's the relationship here?

Douglas Hofstadter wrote this book back in the 70s when he was doing graduate school in physics. And this was after him doing a math undergrad at Stanford. He saw the answer when he was playing around with mathematics in the very formal systems we play with, like when we write down things like 2 plus 2 equals 4. These are just symbols.

And as we go through today, I'll show you completely equivalent ways of doing addition, which will look like this. And these are just logical primitives. And if you've seen any set theory-- and don't feel scared if you haven't seen any of these symbols-- but there exists an x for every-- we give these interpretations. But the idea is that mathematics can be reduced to a bunch of meaningless operations, just symbol shunting.

But what's interesting is that, within mathematics, there exists an equivalent to self-reference. This is a bunch of atoms and proteins referring to itself, calling itself an I. What happens here-- and this is going to be underneath the name of Godel-- is we're going to get to some incompleteness theorems. We're going to get to some statements which, in mathematics, refer to themselves.

And the question of how this happens, we understand this rigorously. Mathematicians have worked out, how do we go from meaningless symbols to something which refers to itself and which has meaning? The claim, then, is that these two systems are equivalent. And this is, really, the profound idea.

I'm going to draw this symbol, and I'm going to use a term called isomorphism. And isomorphism is, basically, an equals to-- and equals in a different sense. But the idea here is, in many ways, we can link atoms and proteins to logical symbolic primitives in mathematics.

And we understand how we get self-reference in mathematics. So maybe we can use this to understand how we get I's, how self comes out of non-self. This is a really tall order, but we're going to try to do it. And that's what this book attempts to do.

And what I've done is isolate the chapters in this book which I think are most pertinent to this stream of thought. Basically, what we're going to do is we're going to learn how it works in mathematics. We're going to go from logical primitives and work up to self-reference and talk about Zen Buddhism consciousness, et cetera.

But that's going to happen as we leap over here. Because we're going to work up, down, and then around. And we'll conclude the course with some interesting questions about artificial intelligence and how intelligent things come out of unintelligent things.

So when I was teaching this course two years ago, or two springs ago, I ran into five things which I viewed as really important tools for thinking. And I've had to condense a little bit into my famous Tools for Thinking lecture.

The idea here is that *Godel, Escher, Bach* has an incredible number of conceptual tools for thinking about this complex problem of, how do we go from a non-self to a self? And just to outline these real quick, I'm going to have isomorphisms-- and I'll explain all these terms as we go along; recursion-- I'm going to leave this one mainly up to Curran on the second lecture; paradox; and this is infinity-- and all these concepts are very closely linked. And finally, the main subject for today's lecture is going to be formal systems.

All righty. So first, let me go through definitions of these terms. An isomorphism-- I want you all to be very careful with this. Because when you start talking to mathematicians-- grown-up, professional mathematicians-- they're going to use the term isomorphism to mean something very, very specific.

The way it's used in *Godel, Escher, Bach,* the way it's going to be used in this class, is very loose. We're going to make very intuitive statements, like what's the isomorphism between a car-- I'm not a great artist here-- what's the isomorphism between a skateboard and a car?

And you might say lots of things, like it carries a person, it has four wheels.

So what we do is we construct a map which also has an inverse. And that's the way you think of an isomorphism. You can go either way and preserve information, preserve structure.

If you really feel like following along, I've included, actually, a quote from Douglas Hofstadter on page 7 of your lecture notes. He says-- and this is in the middle of the page-- "The word isomorphism applies when two complex structures can be mapped onto each other in ways that, to each part of one structure, there's a corresponding part in the other structure, where corresponding means that the two parts play similar roles in their respective structures."

This is how we're going to always use the term isomorphism in this class. If you're taking the abstract algebra class, it's going to mean something a lot more specific, and you're going to have a lot more details. You might, actually, think of these as kind of a-- I'll say it, but don't worry about it-- is a homomorphism.

And the idea with the homomorphism is that there are a lot more details here than there are here. And for example, there's no steering wheel. There's a steering wheel in a car, but there's no steering wheel, specifically, in a skateboard. So if you were to create a map from the car to the skateboard, that detail would have to go somewhere else.

But don't worry about those necessities. But when I say the term isomorphism, think of equals. And I'll often use that symbol right there. This is going to be really important because it's going to be how we're going to get meaning out of things. And you'll see it a lot coming up over the book.

But first I want to hop on and talk about recursion. Recursion is, basically-- it's seen everywhere. But it's a list of instructions which you follow but then repeat until you've reached a final case.

So suppose you were cooking. And you could have a recursive algorithm for stirring eggs. And that would be whirl, and then, whirl again. Keep whirling until, essentially, everything looks mixed up. That's a very loose way of understanding it.

But another way, which you all are probably familiar with, and much more rigorous, in term of mathematics, is the Fibonacci sequence. This is where you start with two numbers-- 1 and 1-- and then, you construct the next number by summing the previous two. So you have that, and you have 3, and you have 5, and you have 8, and so on.

And you can create what's called a recursive definition where you define the n-th Fibonacci number-- this is for n greater than or equal to 2. And here, you define the thing in terms of itself. And this is a classic example of recursion. What it is is, really, itself on a smaller level.

I think one of the most exciting applications of recursion are fractals. Because the way we create fractals is through recursion. So I don't know if you all have seen this, but the Sierpinksi triangle, or the Sierpinksi gasket, is a classic fractal.

Here, you divide a triangle up into three. And then, you just repeat the process for an infinite number of times on each remaining triangle. And you create these very beautiful mosaic forms.

But the nice thing about mathematics is that we can be very precise and do things that we can't do in the real world. And that's repeat this infinitely-- and so on. Just for a quick digression, and I really don't want to spend too much time on it because Curran will do more, why is it called a fractal? Does anyone know?

STUDENT: I think it's like a fragment of something.

JUSTIN CURRY: Sure. It was a term coined by Benoit Mandelbrot in 1977, I believe. It, actually, refers to its number of dimensions. So this might be kind of a mind-bending concept for most of you, but we like to think we live in one, two, or three, or four dimensions-- all integers, right?

But my claim is that the Sierpinski gasket actually lives in between one and two dimensions. It lives in, like, 1.63-something dimensions. And I want to help you think about that.

And if you want to hop along to page 9, I've got a recipe for helping you think about dimension. You know what? It's weird because only mathematicians would ever worry about rigorously understanding the concept of what a dimension means.

So here's one way to think about it. If you take a line, and you double it, you have two copies of the line that you started with. This guy's here and there. If you have a square, and you double the sides of the square, you have four copies of the original square.

Similarly-- and I'm not going to try to draw this because it will get too complicated way too fast-- if you take a cube, and you double each of the sides, you get, if you think about it, eight copies of the original cube. So if you're perceptive enough, you might realize this action of powers going on here.

So here, we had, after our doubling process, two copies. We had 2 to the 1. Here, after our doubling process, we had 2 to the 2. After our doubling process here, we had 2 to the 3-- 8.

So this is weird. Because notice that the cube lives in three dimensions. And the square lives in two dimensions. And the line lives in one dimension. So this might suggest to you the relationship that 2 to the d, where d is the dimension of the space you're living in, equals the number of copies you have after the doubling process.

So let's return to our friend, the Sierpinski gasket. If we start here, and we imagine doubling each of the sides of the Sierpinski gasket, here and here, we're very strangely led to the conclusion that whatever dimension the Sierpinski gasket lives in, it obeys this rule.

So take the logarithms, and d times-- sorry, this is getting crowded. if you take the logarithm of both sides and solve for d, you'll see that the dimension of the Sierpinski gasket is log 3 over log 2, which is approximately 1.585 on to infinity.

So here's an exact example of something which lives somewhere between one and two dimensions. And I think that's a really cool concept.

Moving on for other tools for thinking, we have paradoxes. Paradoxes come in all sorts of different flavors. I don't know if some of you have heard of the birthday paradox, where it's the idea of, OK, what's the probability that someone else in the room has your same birthday?

Everybody thinks it's really small. But if you actually work out the mathematics, it turns out you actually have a good chance. If you're in a room with over 40 people, you have an extremely high chance of finding someone else with your same birthday.

So I've actually list listed out-- this is courtesy of Wikipedia and Mr. Quine-- we have three variants of paradoxes. This is veridical. And these are things which are true but may seem paradoxical at first.

There's falsidical. And I'll give an example of each of these. And then, the classic, the one which we're going to be interested in-- and these are real paradoxes-- are antinomies.

To give you an example of another classic paradox, and one which is visited in *Godel, Escher, Bach* very early on, it's called Zeno's paradox. And the idea is if I want to get from here to my laptop, I first need to walk halfway across the distance.

And then, if I want to walk the remaining distance, I need to walk half of that. And if I want to walk the remaining distance, I need to walk half of that, and then half that, half of that. And eventually, I get stuck in this infinite loop where it seems like I'm not getting to my laptop.

A variant of this paradox is the idea that, if I even want to move at all, if my atoms want to pass in space, first, they have to go halfway. But before it can go halfway, it's got to go halfway by half and halfway of that half and a half of that half. So Zeno, back in Greece, actually used this to prove that motion was impossible and that any motion we saw in the universe was an illusion.

So it's weird. Why? And nobody really could answer Zeno for the longest time. But then it took, essentially, the understanding of limits and calculus to really get an idea of why this wasn't paradoxical.

What, rigorously, did we mean by an infinite number of steps? How could we actually get across the room? It seemed paradoxical, but we knew it had to be true. We knew motion had to be possible.

I'm sure when you all were younger, or even now, you've seen all sorts of falsidical paradoxes where somebody will write out a string of, if you take 1 minus 1 plus 1 minus 1, dot, dot, dot. And the person convinces you, well, look, if you look in groups of this, these are all zeros.

So if you just add a bunch of zeros together, this is necessarily 0. This is an infinite string, right? And we can repeat the pattern. What happens if we add a one?

So suddenly, we get these weird conclusions where 0 equals 1. And they're usually built on doing something illegal involving infinities. And infinity is going to be a very important concept that we'll encounter again and again.

Finally, the antinomy. These are the important paradoxes to think about. I once went out to dinner with a bunch of mathematicians. I don't know how I ended up in that but, let me tell you, it was kind of frightening.

And there was this Korean mathematician who said, well, you know what? Most of these questions don't even matter. We don't understand some of the most fundamental things. And the thing he was most interested in and, I think, which bothers mathematicians the most, is the antimony of the liar and Russell's paradox.

So the Liar's Paradox you probably have heard before. And it's based on, actually, a biblical reference. But it, essentially, says "This sentence is not true." So is it true or is it not true?

Well, if it's true, then it says of itself that it's not true. So true implies not true-- contradiction. So if it's not true, then we know that, if we believe in the law of the excluded middle, which means that things have to either be true or not true, that it's negation is true.

So if it's not true, then the sentence is true. So not true implies true. So we're stuck. The liar paradox still hounds us today. Unlike Zeno's paradox, it hasn't been solved. We still don't know how to deal with it.

And when we talk about Godel's theorem, the way he proves his result is actually going to be intimately linked with a variant on this. So instead of saying, I'm not true, it's going to say, I'm not provable. And that's going to be a very interesting idea. And we'll explore that a little bit later.

The other antinomy I want to look at is Russell's paradox, also known as the barber's paradox. And that's how I'm going to tell it, as the barber's paradox. I think it's a little more friendly.

So you have a town. And there's this male barber. And he abides by the rule that he shaves all people and only people who don't shave themselves. So what does the barber do when his beard is getting as thick as mine? Does he shave himself, or does he not?

Well, let's see. So by definition, the barber only shaves those people who don't shave themselves. So if he shaves himself, then he doesn't. And if he doesn't shave himself, then, by definition, he must shave himself.

A variant of this, which was coined by both Bertrand Russell, Cambridge mathematician and philosopher, and Zermelo, a great German magician, is the idea that you can consider the set-- let's call it omega-- which contains all sets that aren't members of themselves. So remember, a set is just a collection of objects. And mathematicians really believed that set theory was going to be what gave mathematics its ultimate sure and logical foundation.

So let's give an example of a set which contains itself. So let's think of the set of all things which aren't Joan of Arc. Well, sets aren't people. They're people, not sets.

So that set of all things which aren't Joan of Arc includes itself. Because a set can never be a person. So that set is contained in itself. So we have a bunch of things in here which are sets which aren't members of themselves.

And then, we ask the question, is omega an element of itself? And this means "is in." Well, if omega contains itself-- but omega, by definition, only contains things which don't contain themselves. So it can't contain itself.

Well, if it can't contain itself, it doesn't contain itself, and that means it should contain itself-- contradiction. This really, really bothered a lot of mathematicians for a long time. And it's an exact variant on the barber's paradox. So this is kind of interesting things to play around with.

Finally is the concept of infinity. I can't, really, talk too much about it. We're going to look at it more. But I want to introduce you guys to the idea that there are multiple types of infinity.

So you have the integers, and you also have the real numbers. And it is true that you cannot create a direct link. You can't match every real number, like 0.333333-- well, 0.35-something random-- pi. Let's pick pi. You can't put pi directly in connection with a natural number, an integer.

And this is kind of famous-- Cantor's diagonalization argument. So somehow, there are different degrees of infinity. And the real numbers is a higher degree of infinity. So that's an important thing to think about.

Now, we're going to jump ahead to our last tool for thinking. And this is going to be the reason why we ignore the first three chapters of *Godel, Escher Bach.* And it's the idea of a formal system.

The problem is is formal systems are boring. And Douglas Hofstadter takes his sweet, sweet time in introducing you to the concept of a formal system. So I want to try to speed things up because I know you all are smarter than that, and you can get through these concepts very quickly.

We're going to play a game. It's called the mu puzzle, or M-U. And the way you play it is you start with a bag of three letters. And you're going to have a rule you're going to start with

You pull two letters out. And you get M, I. And we're going to have four rules. And these are completely strict typographical rules for deriving new things that we can pull from our bag.

Our first rule is that if we have an I-- so suppose we have MI, or we could have anything and then an I-- we can tack a U on, so IU. So right away, we know that we can create MIU.

Our second rule is, suppose we have M and then a string of letters that are I's and U's, since they're in our bag of alphabet, our alphabet here. Then, you're going to get, for free, Mxx. So just as an example, suppose, somehow, you had MI, which we do. You're going to get MII for free.

Third rule-- suppose somewhere along the way, you end up with a cluster of three I's. They don't have to be at the end. They can be anywhere-- just needs to be three I's all together. And you can replace all three of those I's. They're equal to a U.

And our final rule is that if we have a double pair of U's, we can drop them, and they just go away. So somehow, if we had MUU, we could just have M.

Now, you have these rules. You have these letters. You start with one guy. He's going to be our axiom. An axiom is a starting point for reasoning for applying these rules.

And the game is, can you get MU? Starting from MI, and using only these four rules, can you get MU? I will give $20 to the first person who can derive MU-- that's in this room-- only applying these four rules and starting directly from MI.

Just to give you an idea of where you might be going, where you might be playing, just going off of our rules, we already saw that if we had MI, we can get MIU. We also saw that, using rule two-- that's using rule one-- we can get MII. We saw if we have anything like that, we can repeat it twice. So we can get MIUIU-- that's applying rule two again-- and so on.

Leave this as a puzzle. Take your time with it. You'll be working on it for a few hours. But first person that's in this room, derive MU from this, gets $20. Yes?

STUDENT: Fourth rule only applies to U?

JUSTIN CURRY: Yes. Fourth rule only applies to two U's. So yes, if you have two U's, you can remove them. You can subtract them. All right.

And once again, I do urge everyone to buy the book. These rules are listed explicitly in the chapter. And you might gain some insight on how to derive what you want here.

So why is this interesting? We're just playing with letters and strings and things like that. Well, although this seems pretty meaningless and kind of dumb, does anybody feel like when they're just looking at this game, looking at this rules, that they're just, essentially, playing around with algebra that they learned in middle school or high school?

Really, what we're doing here is we've got some statements like 2 plus 2 equals 4 And we all learned that. We have a typographical rule for when we have an equals sign like that, we can add 1 to both sides and preserve equality. So suddenly, we have 2 plus 3 equals 5.

So really, what mathematics reduces to is just playing around with systems of this form and applying these rigorous typographical rules. Except, here, there doesn't seem to be any meaning. It's just meaningless.

One of the important questions we're going to address in this class is, how do things gain meaning? How do we go from meaningless to meaning? This, obviously, seems to have meaning, but I want you to ask yourself why.

Before we proceed, it's necessary-- it's my duty-- to do the boring task of writing down just a few definitions of things, what you can call these, so you have words. So we already saw axiom. That's a definition.

You call any of these guys a string. So a string is just any ordered sequence of, in this case, M, I's, and U's. We already met an axiom. An axiom is a starting point. It's your first thing that you can apply the rules to.

And this, actually, has a lot to do with mathematical logic. Because in math logic, the idea is that we start from really primitive things which seem obvious, like the successor of 0 is 1, and then we work from that concept, and we derive all these truths of number theory and mathematics.

Here, your axiom is MI, and you're trying to prove the theorem-- and that's our next guy here-- we are trying to prove the theorem of MU. So a theorem is, basically, a string which results at the end of a derivation.

And a derivation is like a proof. For those of you who have done geometry, when you're saying, OK, well, this triangle's congruent to this triangle because of side, angle, side and things like that, you're making rigorous justifications for your leaps in logic.

So here, our rigorous justification that MIU was the theorem, well, we applied typographical rule number one. That's a rigorous leap in logic, and we got to this theorem. And you can just call these four rules here, these are rules of inference.

And logic and a lot of things that you'll play around with eventually, on SATs and things like that, if you have the statement that p implies a statement q-- if it's cloudy, then it will rain-- you have that this is equivalent to-- I should use a different arrow here-- to not q implies not p. And these are really nice because they're just typographical rules.

When you see something, like when you have-- well, I've got M followed by any string of letters-- well, then, I can double it. That's a rule of inference, just like this is a rule of inference. If I have p implies q, I can always replace that. It's completely equivalent to not q implies not p.

But for those of you who are scrambling away because you want $20 really fast, I want you to take a break. Because once again, we should focus on what we're saying right now. And we're going to talk a little bit about jumping outside the system. This is the cool renegade stuff that Hofstadter fills his book with. And it's the idea that, as you're playing around with this, right now, you're just playing a game.

And what mathematicians, and what anybody human, does is when they feel like they're caught in loops, just cranking through pages of algebra, and they're not getting anywhere, humans are intelligent enough to stop. They exit the system, and they say, I don't know. I don't think this is going to go anywhere.

Or well, let me think about why I'm not getting, or how might I get, MU? Maybe it has something to do with numbers of I's and U's or things like that. You start doing what I like to call meta-thinking. You're not thinking in the system, applying typographical rules, applying rules of inference to existing strings-- axioms-- and getting theorems.

That's thinking inside the system. That's just thinking. Meta-thinking involves you leaping outside the system and making judgments about it, thoughts which cannot be expressed as any just normal typographical role within the system. You're doing meta-thinking.

One of my favorite parts of this section in *Godel, Escher, Bach* is when Hofstadter says-- and, once again, stop the drive in you-- try to turn to page 24 in your lecture notes. Oops. Somebody's syllabus. Let me get that. No worries. Page 24-- Hofstadter kind of uses this as a life lesson.

He says, look, "Of course, there are cases when only a rare individual will have the vision to perceive a system which governs many people's lives, a system which had never before even been recognized as a system. Then such people often devote their lives to convincing other people that the system really is there and that it ought to be exited from."

It's as if our social customs and our cultures are really just formal games. You know, we say hello. We shake your hand. That's an instance of a formal rule, which we all follow.

But you know, every once in a while, you get somebody who says, ah, I don't want to shake your hand. I'm going to exit the hand-shaking formal system. But of course, there are much more radical examples of this-- like, I said Karl Marx and communism.

He viewed this idea of, look, you've got these people who are collecting money and property. And they're getting someone else to do all the work, and they're oppressing this whole class of people. Can't people recognize the system?

So then, people like Karl Marx and Fred Engels start writing in pamphlets, encouraging people to overthrow governments, et cetera, because they viewed a system. They said, look, we need to exit the system. For intelligent beings, we can think on a higher level.

Of course, I'm not trying to promote communism here. I'm just showing you an example of historical interest. You know, anarchism, socialism today, working people, the media. Nowadays, I think it's one of the most popular things for people to say is, well, you know, it's just the media trying to do this.

Before, we used to never just refer to this entity as "the media." The media is trying to obscure our understanding of this. The media is trying to scare us. Also, the government. The government's responsible!

Of course, a classic example is also what Karl Marx said, the church. It's the opiate of the masses. That's what he said. And also, school. School is my favorite example of a system which people have encouraged you to exit from.

It's like, well, it's just a daycare that we have. And we don't actually want kids to learn and grow up. And this inspired a lot of new free-thinking educational movements like the Montessoris and things like that.

And I really want you guys to think about, in your daily actions, am I living, perhaps, in a kind of formal system which is acting in a similar way? Try to do some meta-thinking, thinking on a higher level. And is it worth exiting that system?

Hofstadter classifies these three levels of thinking. And he likes to call it a mechanical mode, when you're doing the normal games of the system, an intelligent mode, and, then, just an unmode. Unmode is when you just reject the system. He calls it the zen way of approaching things. And this is something we like to talk about a little more.

I want to quickly introduce you to another-- well, first of all, I want to talk about a concept of what we've previously mentioned. We're eventually going to be talking about artificial intelligence. And it's weird because humans really like to say that their thoughts are logical. We like to say that we do think in this manner.

But a lot of times, we don't. We like to use just inference about collective events. One of our favorite tools of thinking is induction. Well, the sun has rised all these previous days. I'm sure it'll rise tomorrow.

And there's no real formal line of logic that's saying that, well, sun rised yesterday and that, thus, it will rise tomorrow. And I want you to think of whether or not our thoughts are actually just computations in a formal system, much like MIU, p implies q, and things like that.

And that's going to bring me to another formal system which I have to mention just because, in chapter 4, he's going to refer to it. And it's going to lead us to this interesting line of dialogue of when a formal system with meaningless symbols gains meaning. And it's called the pq system.

We're going to have three new letters-- well, three new characters. It's now going to be p, q, and hyphen. And you've, actually, got an infinite number of axioms here. And you've got a definition, and that's that if xp hyphen-- I'm going to make sure I have, just, an underlined p-- qx. And this is going to be an axiom whenever x is just a string of hyphens. So it's just some string of hyphens.

So what's this saying? It's saying that, well, if you have something like this, well, x here was two hyphens, so we know that that's an axiom. All right. It's a little different than MIU. It seems just as meaningless.

And we're going to have different forms for manipulating and playing around with this. And one rule is that if you have x, y, and z, which are just hyphen strings-- xpyqz-- then you can derive, you're given for free, the statement xpy hyphen qz hyphen. Seems meaningless.

But what does it remind you of? We've got this axiom. We, in fact, have a whole infinite list of axioms. And maybe you've noticed that they've got two hyphens here, one hyphen here, got three hyphens here. Now, what does this do?

STUDENT: [INAUDIBLE] .

JUSTIN CURRY: Yeah, exactly. And what it does is it says that, well, if this works, right? So let's apply this rule here. And we'll apply this rule here.

So we can take this and get for free that hyphen hyphen p hyphen-- we can add another hyphen-- q. Now, we had three hyphens here, but this rule says we can tack on another hyphen. What does that say?

Well, this seems to say that 2 plus 2 equals 4. So I want you to realize that the symbolism which mathematicians have been using, and what you've grown up learning, is just shorthand. It's meaningless notation. Yeah?

STUDENT: Did you do those two backwards-- 1 and 2? Should it be 2 and 1?

JUSTIN CURRY: Well, yeah, no. What I meant to say here is that we seem to be inferring this rule that hyphen string 1 plus hyphen string 2 always equals hyphen string 3. And so, just 1 here refers to a whole string of hyphens. And 2 refers to a string of hyphens, like y here. Or better yet, I could say x plus y equals c here.

And what makes this system different than MIU? Does anyone have any ideas? Why do you suddenly care a little more about this system than MIU, other than the fact that you have $20 going on the line for deriving MU? Anybody?

What about this fact that I've just showed you this equivalence here? Now, instead of applying these typographical rules, I've showed you that, well, you can also take this as 2 plus 2 equals 4? And then, you're going to say, aha! Well, now, I can do all sorts of things.

Now that I've discovered the meaning of the pq hyphen system, I can go ahead and just create all sorts of new theorems, starting from any of our axioms. And you might even be tempted to say, well, I know it's obvious. I know that 2 plus 2 plus 2 equals 6. And I've discovered this isomorphism between p's and q's and pluses and equal signs.

So I'm tempted to say that hyphen hyphen p hyphen hyphen p hyphen hyphen q hyphen hyphen hyphen hyphen hyphen hyphen-- that's a lot of hyphens. What's wrong with this? Does anyone see a problem? Yes?

STUDENT: [INAUDIBLE] .

JUSTIN CURRY: Exactly, exactly. It doesn't follow the rule. The rules I told you in the axioms, which you start from, you only ever have one p and one q. This is not even what we call-- so this is not what we will refer to as a well-formed formula. So you have to be really careful with what meaning means and when you try to create an isomorphism between what you know about addition and the formal systems you play.

Try to come up with an alternative interpretation. We could have just interpreted these p's, q's and hyphens as, we're going to call p, we're going to say that's horse. And q, that's apple. And one hyphen is happy, and two hyphens is happy happy, and so on.

So suddenly, we have an interpretation for this string. It's not 2 plus 3 equals 4, but it's happy happy horse, happy happy apple, happy happy happy happy happy. Doesn't mean anything, but it's an interpretation.

And there's no reason not to make that interpretation. Perhaps to horses, this is, actually, more sensible than addition. First of all, when we do addition, we're representing these numbers in base 10 because we have 10 fingers.

But horses don't have 10 fingers. And numbers written in base 10 don't mean anything to horses. But perhaps happy, horse, apple really makes much more sense to a horse.

So we're going to throw out-- and I have to be a little rushed about this-- be thinking about where does meaning come from? How do we actually assign meaning to meaningless symbols?

Because that's the goal here. We're going to go from meaningless symbols in mathematics to meaning. And then, we're going to try to create an isomorphism between the universe and our formal systems.

And this leads me perfectly into this idea of is reality a formal system? And if you go to page 29 in your notes, you've got this long quote. It stretches on to 30. I'll go and start reading. It's at the bottom.

It says, "Can all of reality be turned into a formal system? In a very broad sense, the answer might appear to be yes. One could suggest, for instance, that reality is itself nothing but one very complicated formal system.

Its symbols do not move around on paper but, rather, in a three-dimensional vacuum-- space. They are the elementary particles of which everything is composed-- tacit assumption that there is an end to the descending chain of matter, that the expression 'elementary particles' makes sense.

The typographical rules are the laws of physics, which tell how--" we're on page 29, if you just want to catch up-- "The typographical rules are the laws of physics, which tell how, given the positions and velocities of all the particles at a given instant, to modify them, resulting in a new set of positions and velocities belonging to the next instant. So the theorems of this grand formal system are the possible configurations of particles at different times in the universe.

The sole axiom is, or perhaps was, the original configuration of all the particles at the beginning of time. This is so grandiose a conception, however, it has only the most theoretical interest. And besides, quantum mechanics and other parts of physics cast at least some doubt on even the theoretical worth of this idea. Basically, we are asking if the universe operates deterministically, which is an open question."

I think it was Laplace who said, well, look, if you were to give me the position and momentum of every particle in the universe, I could tell you the rest of the future. And this leads to one of the grand philosophical questions which we'll be investigating as part of this class, as well, which is, if the universe operates deterministically, if Newton's laws govern how my arm falls and how all the atoms in my body interact, where does free will creep into? How do I know I have control over these actions, and it's not the fact that, at the Big Bang, there was a denser cluster of atoms over here and a less dense over here, and things evolved according to deterministic laws, much like the formal systems we're playing with here?

So this question, you can really think of on two levels-- one, can the universe be thought of as being modeled by a formal system, having forces, and solving equations for the particles here, and it collides with another particle at this angle, they go off like this, and things like this? but also, I think, likes to ask another question, which is version 2, for those of you who are *Matrix* fans. To what extent is the universe a formal system proper, in a sense?

Is it a program running in the background of some hyperdimensional alien who's playing WoW, and he's just running our universe as a simulation on his supercomputer cluster that he's got in his basement? Who knows?

I mean, if the universe is deterministic, or he's just coded up-- hacking away in Python-- all of our rules of our universe, and he said, all right, let's let this simulation go. And here we are, in his computer, having all these dramatic interactions with people, et cetera, et cetera, and he's just, oh, well, a bug came up, et cetera. It's kind of interesting to think about.

So we've, now, really hit home these five tools for thinking. And we're going to be revisiting all of these ideas throughout the entire book. And one of the things that Douglas Hofstadter does is he structures his book in its own kind of recursive fashion.

And I only gave you a few specific instances of where recursion shows up. And this represents my bias. For me, I'm very much an art person and a math person. But I'm not so much of a music person. And I really encourage you guys to bring in different elements.

Because *GEB* has such high-dimensional structure to it, everybody contributes their own slice to it. And one thing which I would hate to deny you guys from is the music aspect of this book.

Each one of Douglas Hofstadter's dialogues is, actually, structured and based upon a piece of Bach's music. And if you listen to Bach's music, and you read the dialogue, he might, actually, hint at some of the connections, some of the isomorphism that Hofstadter's alluding to.

But first of all, you should know why he chose Bach, how recursion acts in music. And that's why I have this whole speaker setup, here. So allow me to play. So this is Bach's Little Fugue in G minor.

[MUSIC - BACH, "LITTLE FUGUE IN G MINOR"]

Just as a nice anecdote, who here has seen *A Beautiful Mind,* the movie? All right. So John Nash, the mathematician who went crazy, Princeton, et cetera, the story goes that he used to actually stalk around the halls of the math department, smoking cigarettes and whistling this song constantly.

And what were some of the things which you noticed about this piece? For those of you with good auditory abilities, what did you notice?

STUDENT: They're, sort of, patterns.

JUSTIN CURRY: OK. Elaborate a little bit on these patterns.

STUDENT: I don't know. I'm not a music person, either. I don't know. Maybe, just, after a certain number of notes, it repeats.

JUSTIN CURRY: Exactly. So you heard it come in at a different tone, at a different volume. And you noticed it was the same theme. It's the same theme that he played-- stretched, inverted, backwards, on higher levels, on lower levels. So *GEB* is, actually, very much structured like a fugue.

Hofstadter lays out for us-- and what I did in this first lecture was I'm laying out the entire book for you, all in one go, so that way, you understand it when I play it stretched out, inverted, backwards, and at different volumes. So this is nice. You have a musical illustration. You have artistic illustrations of the ideas we're talking about. But we need to, actually, settle into the book itself.

So Curran Kelleher and I, or anyone else who's really excited about reading-- anybody really excited about volunteering for reading the dialogue? Anybody have the book with them right now? Oh, good job. Would you like to read? You don't have to.

STUDENT: [INAUDIBLE] Yeah, sure.

JUSTIN CURRY: You want to? OK. So we're going to spend the last 15 minutes going through a dialogue. I, actually, have another copy. Good.

And so I need two characters-- one to be Achilles and one to be Tortoise. These are two characters we're going to meet in this dialogue. They're going to play a prominent role throughout the entire book.

So does anyone else want to be-- well, see, I like the tortoise, so I'd like to be the tortoise. But someone else can be the tortoise if they want to be. OK. So we only have one soul that's brave enough to do it. All right. All righty. So page 79. Yeah, sorry.

So I'm going to give you some quick background on this dialogue. So Hofstadter, like me, believes that it's important to introduce the idea of a topic conceptually first, before you start really diving into it. So he prefaces every chapter with a dialogue. And the dialogue is kind of a conceptual introduction to the ideas we're talking about.

To go ahead and give you an idea of what this dialogue's based on, it's going to be the conflict of two mathematicians, Kurt Godel and David Hilbert. David Hilbert believed that mathematics could be put into a formal system very rigorously, and it could also be proved to be consistent and complete. Those are two words which I'm going to have to define at the end of this dialogue.

But let's go and start it off and try to work quickly through this. I'm going to ask that when you have the italics, you go ahead and read it as part of your section, so people have an idea of what's going on in the book.

All right, excellent. So we don't have, really, any time left. But I want to say one thing. It's a challenge. Pay attention to Tortoise's quote on page 81 when she talks about acrostics. if you can find two acrostics in this dialogue, I'll [AUDIO OUT].