Chapter 2

Tools for Thinking

"Always Be Prepared." – Boy Scout Motto

2.1 Introduction

The premise for this chapter is that before we take off on our journey, we need to read the travel guide, so that we know what kind of creatures to expect to encounter. Also in a weak allusion to the fugal structure of both GEB and this class, I want to lay the theme down for you now (or part of it), so that you might understand it when it is inverted, played backwards, forwards, upside-down, and across the universe.

After intense speculation and discussion, I feel that the following concepts might help aid in your understanding of GEB and the content of this course.

2.2 Formal Systems

From Wikipedia, the free encylopedia:

In the formal sciences of logic and mathematics, together with the allied branches of computer science, information theory, and statistics, a formal system is a formal grammar used for modeling purposes. Formalization is the act of creating a formal system, in an attempt to capture the essential features of a real-world or conceptual system in formal language.

In mathematics, formal proofs are the product of formal systems, consisting of axioms and rules of deduction. Theorems are then recognized as the possible 'last lines' of formal proofs. The point of view that this picture encompasses mathematics has been called formalist. The term has been used pejoratively. On the other hand, David Hilbert founded metamathematics as a discipline designed for discussing formal systems; it is not assumed that the metalanguage in which proofs are studied is itself less informal than the usual habits of mathematicians suggest. To contrast with the metalanguage, the language described by a formal grammar is often called an object language (i.e., the object of discussion - this distinction may have been introduced by Carnap).

It has become common to speak of a formalism, more-or-less synonymously with a formal system within standard mathematics invented for a particular purpose. This may not be much more than a notation, such as Dirac's bra-ket notation.

Mathematical formal systems consist of the following:

1. A finite set of symbols which can be used for constructing formulae.
2. A grammar, i.e. a way of constructing well-formed formulae out of the symbols, such that it is possible to find a decision procedure for deciding whether a formula is a well-formed formula (wff) or not.
3. A set of axioms or axiom schemata: each axiom has to be a wff.
4. A set of inference rules.
5. A set of theorems. This set includes all the axioms, plus all wffs which can be derived from previously-derived theorems by means of rules of inference. Unlike the grammar for wffs, there is no guarantee that there will be a decision procedure for deciding whether a given wff is a theorem or not.

2.3 Isomorphisms

From Wikipedia, the free encyclopedia:

In mathematics, an isomorphism (Greek:isos "equal", and morphe "shape") is a bijective map f such that both f and its inverse f^{-1} are homomorphisms, i.e. structure-preserving mappings.

Informally, an isomorphism is a kind of mapping between objects, which shows a relationship between two properties or operations. If there exists an isomorphism between two structures, we call the two structures isomorphic. In a certain sense, Isomorphic sets are structurally identical, if you choose to ignore finer-grained differences that may arise from how they are defined.

According to Douglas Hofstadter:

> The word "isomorphism" applies when two complex structures can be mapped onto each other, in such a way that to each part of one structure there is a corresponding part in the other structure, where "corresponding" means that the two parts play similar roles in their respective structures. (Gödel, Escher, Bach, p. 49)

2.3.1 Purpose

Isomorphisms are frequently used by mathematicians to save themselves work. If a good isomorphism can be found from a relatively unknown part of mathematics into some well studied division of mathematics, where many theorems are already proved, and many methods are already available to find answers, then the function can be used to map whole problems out of unfamiliar territory over to "solid ground," where the problem is easier to understand and work with.

2.3.2 Physical Analogies

Here are some everyday examples of isomorphic structures:

- A solid cube made of wood and a solid cube made of lead are both solid cubes; although their matter differs, their geometric structures are isomorphic.
- A standard deck of 52 playing cards with green backs and a standard deck of 52 playing cards with brown backs; although the colours on the backs of each deck differ, the decks are

structurally isomorphic – if we wish to play cards, it doesn't matter which deck we choose to use.

- The Clock Tower in London (that contains Big Ben) and a wristwatch; although the clocks vary greatly in size, their mechanisms of reckoning time are isomorphic.
- A six-sided die and a bag from which a number 1 through 6 is chosen; although the method of obtaining a number is different, their random number generating abilities are isomorphic. This is an example of functional isomorphism, without the presumption of geometric isomorphism.

2.4 Recursion

Recursion is probably the most fundamental concept to GEB! In fact recursion has become so tightly associated with GEB that one Andrew Plotkin once defined recursion as:

> If you already know what recursion is, just remember the answer. Otherwise, find someone who is standing closer to Douglas Hofstadter than you are; then ask him or her what recursion is.

But what exactly is recursion? Wikipedia offers us the following definition:

> In mathematics and computer science, recursion specifies (or constructs) a class of objects or methods (or an object from a certain class) by defining a few very simple base cases or methods (often just one), and then defining rules to break down complex cases into simpler cases.

This is an OK definition of recursion, but actually understanding what recursion *is* will take considerable experience and practice recognizing recursive structures everywhere in thought and nature.

2.4.1 Recursion in Math

Recursion probably got its start in mathematics, especially in defining interesting sequences of numbers such as the Fibonacci sequence

$$\{1, 1, 2, 3, 5, 8, 13, 21, ...\}$$

, which can be defined by

$$f(n) = f(n-1) + f(n-2) \qquad n \geq 2 \tag{2.1}$$

$$f(0) = f(1) = 1 \tag{2.2}$$

However, in my humble opinion some of the most interesting applications of recursion (besides Gödel's Incompleteness Theorem) are fractals. Fractals appear everywhere in nature and are self-similar. They exist in a fractional number of dimensions (thus *Fractal*) and look really cool.

You might be confused by the notion of fractional (i.e. 1.7, 2.5, etc.) number of dimensions, and there many possible ways of rewiring your brain to think differently about dimensions, but here is one simple way.

- Imagine a line. When you double its length, you now have **two** copies of the original.
- Imagine a square. When you double its sides, you now have **four** copies of the original.
- Imagine a cube. When you double its sides, you now have **eight** copies of the original.

If you are perceptive enough, maybe you'll notice that the number of dimensions, d of each object, and the number of copies, N followed the following relationship:

$$2^d = N \qquad d \geq 0 \tag{2.3}$$

However consider the sierpinski triangle (or gasket) – a very famous fractal of dimension 1.585. It has the very strange property that if you double its sides (or scale them by $\frac{1}{2}$) you have only 3 copies. Thus we need a dimension that satisfies

$$2^d = 3 \Rightarrow d = \frac{log(3)}{log(2)} \simeq 1.585... \tag{2.4}$$

Although this argument is very imprecise and informal, it will give you a flavor for fractals. If you want to study a more rigorous method for calculating dimension, look up the **Box Counting Dimension** or the **Minkowski-Bouligand Dimension**.

2.4.2 Recursion in Other Fields

Recursion is everywhere! Whether it is developing an algorithm in computer science, or understanding how language works, or how evolution works, *recursion appears to have a deep importance in the universe.*

2.5 Paradox

We will also find tightly knit with our exploration of recursion, are encounters with paradox. Paradoxes are a genuinely difficult thing to study, and we will explore several famous paradoxes, and try to understand its role in GEB. One tool which Hofstadter finds useful for considering paradoxes is the Zen Koan – an apparently contradictory statement or story to help quiet the mind – and I have attached an appendix of cool Zen Koans.

2.5.1 Definition of Paradox

From Wikipedia

A paradox is an apparently true statement or group of statements that seems to lead to a contradiction or to a situation that defies intuition. Typically, either the statements in question do not really imply the contradiction, the puzzling result is not really a contradiction, or the premises themselves are not all really true or cannot all be true together. The recognition of ambiguities, equivocations, and unstated assumptions underlying known paradoxes has led to significant advances in science, philosophy and mathematics.

The word paradox is often used interchangeably and wrongly with contradiction; but where a contradiction by definition cannot be true, many paradoxes do allow for resolution, though many remain unresolved or only contentiously resolved, such as Curry's paradox. Still more casually, the

Image of Mandelpart2.jpg removed due to copyright restrictions.
Available at: http://en.wikipedia.org/wiki/Image:Mandelpart2.jpg

Figure 2.1: Part of the Mandelbrot Set – A Picture of God? – From Wikipedia

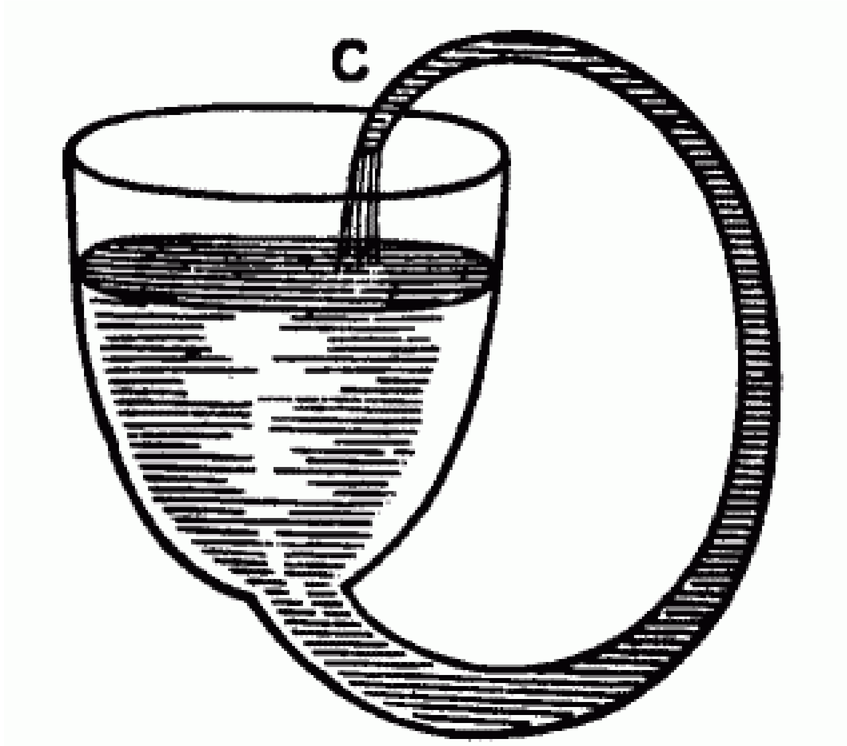

Figure 2.2: Boyle's perpetual motion scheme – From Wikipedia

term is sometimes used for situations that are merely surprising, albeit in a distinctly "logical" manner, such as the Birthday Paradox. This is also the usage in economics, where a paradox is an unintuitive outcome of economic theory.

2.5.2 Examples

Sometimes supernatural or science fiction themes are held to be impossible on the grounds that result in paradoxes. The theme of time travel has generated a whole family of popular paradoxes, supposed to arise from a person's interference with the past. Suppose Jones, who was born in 1950, travels back in time to 1900 and kills his own grandfather. It follows that neither his father nor he himself will be born; but then he would not have existed to travel back in time and kill his own grandfather; but then his grandfather would not have died and Jones himself would have lived; etc. This is known as the Grandfather paradox.

Paradoxes that arise from apparently intelligible uses of language are often of interest to logicians and philosophers. This sentence is false is an example of the famous liar paradox: it is a sentence which cannot be consistently interpreted as true or false, because if it is false it must be true, and if it is true it must be false. Russell's paradox, which shows that the notion of the set of all those sets that do not contain themselves, was instrumental in the development of modern logic and set theory.

2.5.3 Etymology

The etymology of paradox can be traced back the use of the word paradoxo, used in Plato's Parmenides by the Greek philosopher Zeno of Elea, who lived at 490-430 BC. The word was used to describe seminal philosophic ideas posited by Zeno, known as Zeno's paradoxes, which exerted a poignant effect on Greek thinkers that has survived to modern day. Zeno sought to illustrate that equal absurdities followed logically from the denial of Parmenides' views. There were apparently 40 paradoxes of plurality and other paradoxes that Zeno used to attack the Greek understanding of the physical world. In fact, Zeno's paradoxes of multiplicity and motion revealed some problems in space and time that cannot be resolved without the mathematical methods discovered in the 19th century and perhaps beyond. Although it is unknown if Zeno coined the word, he can certainly be attributed as popularizing it. It is unknown if incarnations of paradox were used before Zeno of Elea. Later and more frequent usage of the word has been traced to the early Renaissance. Early forms of the word appeared in the late Latin paradoxum and the related Greek paradoxos meaning 'contrary to expectation', 'incredible'. The word is composed of the preposition para which means "against" conjoined to the noun stem doxa, meaning "belief". Compare orthodox (literally, "straight teaching") and heterodox (literally, "different teaching"). The liar paradox and other paradoxes were studied in medieval times under the heading insolubilia.

2.5.4 Common Themes

Common themes in paradoxes include direct and indirect self-reference, infinity, circular definitions, and confusion of levels of reasoning. Paradoxes which are not based on a hidden error generally happen at the fringes of context or language, and require extending the context or language to lose their paradox quality.

In moral philosophy, paradox plays a central role in ethics debates. For instance, an ethical admonition to "love thy neighbour" is not just in contrast with, but in contradiction to an armed neighbour actively trying to kill you: if he or she succeeds, you will not be able to love him or her. But to preemptively attack them or restrain them is not usually understood as loving. This might be termed an ethical dilemma. Another example is the conflict between an injunction not to steal and one to care for a family that you cannot afford to feed without stolen money.

2.5.5 Types of Paradoxes

W. V. Quine (1962) distinguished between three classes of paradoxes.

- A veridical paradox produces a result that appears absurd but is demonstrated to be true nevertheless. Thus, the paradox of Frederic's birthday in The Pirates of Penzance establishes the surprising fact that a person may be more than Nine years old on his Ninth birthday. Likewise, Arrow's impossibility theorem involves behaviour of voting systems that is surprising but all too true.

- A falsidical paradox establishes a result that not only appears false but actually is false; there is a fallacy in the supposed demonstration. The various invalid proofs (e.g. that $1 = 2$) are classic examples, generally relying on a hidden division by zero. Another example would be the Horse paradox.

- A paradox which is in neither class may be an antinomy, which reaches a self-contradictory result by properly applying accepted ways of reasoning. For example, the Grelling-Nelson paradox points out genuine problems in our understanding of the ideas of truth and description.

2.6 Infinity

We will discuss infinity at great lengths in this course and will find that there exists a lovely trinity between recursion, paradox, and infinity. Meditate on the following quote from Douglas Adams "Hitchhiker's Guide to the Galaxy" as we read GEB.

> "Bigger than the biggest thing ever and then some, much bigger than that, in fact really amazingly immense, a totally stunning size, real 'Wow, that's big!' time. Infinity is just so big that by comparison, bigness itself looks really titchy. Gigantic multiplied by colossal multiplied by staggeringly huge is the sort of concept we are trying to get across here."