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Ito process. Ito formula

Content.

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1 Ito process

Observe that trivially $\int_0^t dB_s = B_t$. In the previous lecture we computed

$$I_t(B) = \int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{t}{2},$$

or

$$B^2(t) = 2\int_0^t B(s)dB(s) + t, \tag{1}$$

Observe also that we cannot have $t = \int_0^t X_s dB_s$ for some process $X \in \mathcal{L}_2$ as Ito integral is a martingale, but t is not. Thus we see that applying a functional operation to a process which is an Ito integral we do not necessarily get another Ito integral. But there is a natural generalization of Ito integral to a broader family, which makes taking functional operations closed within the family.

Definition 1. *An* Ito process *or* stochastic integral *is a stochastic process on* $(\Omega, \mathcal{F}, \mathbb{P})$ *adopted to* $\mathcal{F}_t$ *which can be written in the form*

$$X_t = X_0 + \int_0^t U_s ds + \int_0^t V_s dB_s, \tag{2}$$

where $U, V \in \mathcal{L}_2$*. As a shorthand notation, we will write (2) as*

$$dX_t = U_t dt + V_t dB_t.$$

Thus B_t^2 is an Ito process: $B_t^2 = \int_0^t ds + 2\int_0^t B_s dB_s$ or $d(B_t^2) = dt + 2B_t dB_t$. Note the difference from the usual differentiation: $dx^2 = 2xdx$. The additional term dt arises because Brownian motion B is not differentiable and instead has quadratic variation.

Notation Given an Ito process $dX_t = U_t dt + V_t dB_t$, let us introduce the notation $(dX_t)^2$ which stands for $V_t^2 dt$. Equivalently $(dX_t)^2$ is $(dX_t) \cdot (dX_t)$ which is computed using the rules $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, dB_t \cdot dB_t = dt$.

2 Ito formula

We now introduce the most important formula of Ito calculus:

Theorem 1 (Ito formula). *Let X_t be an Ito process $dX_t = U_t dt + V_t dB_t$. Suppose $g(x) \in C^2(\mathbb{R})$ is a twice continuously differentiable function (in particular all second partial derivatives are continuous functions). Suppose $g(X_t) \in \mathcal{L}_2$. Then $Y_t = g(X_t)$ is again an Ito process and*

$$dY_t = \frac{\partial g}{\partial x}(X_t) dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t)(dX_t)^2$$

Using the notational convention for $dX_t = U_t dt + V_t dB_t$ and $(dX_t)^2$, we can rewrite the Ito formula as

$$dY_t = \Big(\frac{\partial g}{\partial x}(X_t) U_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t) V_t^2\Big) dt + \frac{\partial g}{\partial x}(X_t) V_t dB_t.$$

Thus, we see that the space of Ito processes is closed under twice-continuously differentiable transformations.

Proof sketch of Theorem 1. We will do this for a very special case. We assume that the derivatives $\frac{\partial g}{\partial x} \frac{\partial^2 g}{\partial x^2}$ as well as U and V are all bounded simple processes.

The general case is then obtained by approximating U and V by bounded simple processes in a way similar to how we defined the Ito integral.

Let $\Pi_n : 0 = t_0 < t_1 < \cdots < t_n = t$ be a sequence of partitions such that $\Delta(\Pi_n) \to 0$. We use the notation $\Delta B(t_j) = B(t_{j+1}) - B(t_j), \Delta X(t_j) =$

$X(t_{j+1}) - X(t_j)$. Using Taylor expansion of g we obtain

$$\begin{aligned} g(X(t)) &= g(X(0)) + \sum_{j<n} g(X(t_{j+1})) - g(X(t_j)) \\ &= g(X(0)) + \sum_{j<n} \frac{\partial g}{\partial x}(X(t_j))\Delta X(t_j) \\ &+ \frac{1}{2}\sum_{j<n} \frac{\partial^2 g}{\partial x^2}(X(t_j))\Delta^2 X(t_j) + \sum_{j<n} o(\Delta^2 X(t_j)). \end{aligned}$$

Now, we have

$$\Delta X(t_j) = X(t_{j+1}) - X(t_j) = U(t_j)(t_{j+1} - t_j) + V(t_j)(B(t_{j+1}) - B(t_j))).$$

Thus, we obtain

$$\sum_{j<n} \frac{\partial g}{\partial x}(X(t_j))\Delta X(t_j) = \sum_{j<n} \frac{\partial g}{\partial x}(X(t_j))\Big(U(t_j)(t_{j+1} - t_j) + V(t_j)(B(t_{j+1}) - B(t_j))\Big)$$

We argue that in $\mathcal{L}_2$ the convergence

$$\sum_{j<n} \frac{\partial g}{\partial x}(X(t_j))\Big(U(t_j)(t_{j+1} - t_j)\Big) \to \int_0^t \frac{\partial g}{\partial x}(X(s))U(s)ds$$

and

$$\sum_{j<n} \frac{\partial g}{\partial x}(X(t_j))\Big(V(t_j)(B(t_{j+1}) - B(t_j))\Big) \to \int_0^t \frac{\partial g}{\partial x}(X(s))V(s)dB(s)$$

takes place. For the first convergence, let us fix any sample ω. Then this convergence follows straight from the definition of Riemann integral since $\Delta(\Pi_n) \to 0$. Thus we have a.s. convergence. Since by our assumptions the integrated variables are bounded then Bounded Convergence Theorem (applied to uniform on $[0, t]$ distribution, just as in Proposition 1 Lecture 12) implies convergence in $\mathcal{L}_2$. To prove the second convergence consider a simple process $\tilde{g}$ which is defined to be $\frac{\partial g}{\partial x}(X(t_j))$ for all $t \in [t_j, t_{j+1})$. Then the left-hand side is Ito integral of $\tilde{g}(s)V(s)$. Then, by definition of Ito integral, the convergence to the right-hand side holds if the following convergence takes place

$$\lim_n \mathbb{E}[\int_0^t (\tilde{g}(s)V(s) - \frac{\partial g}{\partial x}(X(s))V(s))^2 ds = 0.$$

It is a simple exercise in analysis to show this and we skip the details.

Now consider $\sum_{j<n} \frac{\partial^2 g}{\partial x^2}(X(t_j))\Delta^2 X(t_j)$ and identify its $\mathcal{L}_2$ limit. We have

$$\begin{aligned}\sum_{j<n} \frac{\partial^2 g}{\partial x^2}(X(t_j))\Delta^2 X(t_j) &= \sum_{j<n} \frac{\partial^2 g}{\partial x^2}(X(t_j))U^2(t_j)(t_{j+1}-t_j)^2 \\ &+ 2\sum_{j<n} \frac{\partial^2 g}{\partial x^2}(X(t_j))U(t_j)V(t_j)(t_{j+1}-t_j)\Delta B(t_j) \\ &+ \sum_{j<n} \frac{\partial^2 g}{\partial x^2}(X(t_j))V^2(t_j)\Delta^2 B(t_j).\end{aligned}$$

We now analyze these terms when $\Delta(\Pi_n) \to 0$. Recall our assumption that $\frac{\partial^2 g}{\partial x^2}$ and U are bounded. Say it is at most $C > 0$. Therefore the first sum converges to zero provided $\Delta(\Pi_n) \to 0$. To analyze the second sum, we square it and take expected value:

$$\begin{aligned}&\mathbb{E}[2\Big(\sum_{j<n} \frac{\partial^2 g}{\partial x^2}(X(t_j))U(t_j)V(t_j)(t_{j+1}-t_j)\Delta B(t_j)\Big)^2] \\ &\quad = 4\sum_{j<n} \mathbb{E}[\Big(\frac{\partial^2 g}{\partial x^2}(X(t_j))U(t_j)V(t_j)\Big)^2](t_{j+1}-t_j)^3,\end{aligned}$$

where to obtain this equality we first condition on field $\mathcal{F}_{t_j}$, note that the expected value of all cross products vanishes and use $\mathbb{E}[\Delta^2 B(t_j)|\mathcal{F}_{t_j}] = t_{j+1} - t_j$. Again, since the second partial derivative and U, V are bounded, then the entire term converges to zero provided that $\Delta(\Pi_n) \to 0$. In order to analyze the last sum the following result is needed.

Problem 1 (Generalized Quadratic Variation). *Suppose $a(s) \in H^2$ and $\Pi_n : 0 = t_0 < \cdots < t_n = t$ is a sequence of partitions satisfying $\Delta(\Pi_n) \to 0$ as $n \to \infty$. Then the following convergence occurs in $\mathcal{L}_2$:*

$$\lim_n \sum_j a(t_j)(B(t_{j+1}) - B(t_j))^2 = \int_0^t a(s)ds.$$

HINT: use the same approach that we did in establishing the quadratic variation of B.

Using this result we establish that the last sum converges in $\mathcal{L}_2$ to

$$\int_0^t \frac{\partial^2 g}{\partial x^2}(X(s))V^2(s)ds.$$

It remains to analyze $\sum_j o(\Delta^2 X(t_j))$ and using similar techniques, it can be shown that this term vanishes in $\mathcal{L}_2$ norm as $\Delta(\Pi_n) \to 0$. Putting all of this together, we conclude that $g(X(t))$ is approximated in $\mathcal{L}_2$ sense by

$$g(X(0)) + \int_0^t \frac{\partial g}{\partial x}(X(s))dX(s) + \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x^2}(X(s))V^2(s)ds.$$

But recall that $V^2(s)ds = (dX(s))^2$. Making this substitution, we complete the derivation of the Ito formula. $\square$

Let us apply Theorem 1 to several examples.

Exercise 1. Verify that in all of the examples below the underlying processes are in $\mathcal{L}_2$.

Example 1. Let us re-derive our formula (1) using Ito formula. Since $B_t = \int_0^t dB_s$ is an Ito process and $g(x) = \frac{1}{2}x^2$ is twice continuously differentiable, then by the Ito formula we have

$$\begin{aligned} d(\frac{1}{2}B_t^2) = dg(B_t) &= \frac{\partial g}{\partial x}dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dB_t)^2 \\ &= B_t dB_t + \frac{1}{2}(dB_t)^2 \\ &= B_t dB_t + \frac{dt}{2}, \end{aligned}$$

which matches (1).

Example 2. Let us apply Ito formula to B_t^4. We obtain

$$d(B_t^4) = 4B_t^3 dB_t + \frac{1}{2}12B_t^2 dt = 4B_t^3 dB_t + 6B_t^2 dt,$$

namely, written in an integral form

$$B_t^4 = 4\int_0^t B_s^3 dB_s + 6\int_0^t B_s^2 ds.$$

Taking expectations of both sides and recalling that Ito integral is a martingale, we obtain

$$\mathbb{E}[\int_0^t B_s^2 ds] = \frac{1}{6}\mathbb{E}[B_t^4]$$

which we find to be $(1/6)3t^2$ as the 4-th moment of a normal zero mean distribution with std σ is $3\sigma^4$. Recall an earlier exercise where you were asked to compute $\mathbb{E}[\int_0^t B_s^2 ds]$ directly. We see that Ito calculus is useful even in computing conventional integrals.

3 Multidimensional Ito formula

There is a very useful analogue of Ito formula in many dimensions. We state this result without proof. Before turning to the formula we need to extend our discussion to the case of Ito processes with respect to many dimensions, as so far we have we have considered Ito integrals and Ito processes with respect to just one Brownian motion. Thus suppose we have a vector of d independent Brownian motions $B_t = (B_{i,t}, 1 \le i \le d, t \in \mathbb{R}_+)$. A stochastic process X_t is defined to be an Ito process with respect to B_t if there exists $U_t \in \mathcal{L}_2$ and $V_{i,t} \in \mathcal{L}_2, 1 \le i \le d$ such that $X_t = U_t dt + \sum_i V_{i,t} dB_{i,t}$, in the sense explained above. The definition naturally extends to the case when X_t is a vector of processes.

Theorem 2. *Suppose $dX_t = U_t dt + V_t dB_t$, where vector $U = (U_1, \dots, U_d)$ and matrix $V = (V_{11}, \dots, V_{dd})$ have $\mathcal{L}_2$ components and B is the vector of d independent Brownian motions. Let $g(x)$ be twice continuously differentiable function from $\mathbb{R}^d$ into $\mathbb{R}$. Then $Y_t = g(X_t)$ is also an Ito process and*

$$dY_t = \sum_{i=1}^{d} \frac{\partial g}{\partial x_i}(X_t) dX_{i,t} + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 g}{\partial x_i x_j}(X_t) dX_{i,t} \cdot dX_{j,t},$$

where $dX_{i,t} \cdot dX_{j,t}$ is computed using the rules $dtdt = dtdB_i = dB_i dt = 0, dB_i dB_j = 0$ for all $i \ne j$ and $(dB_i)^2 = dt$.

Let us now do a quick example illustrating the use of the Ito formula. Consider $g(t, B_t) = e^{tB_t}$. We will use Ito formula to find its derivative. Since both t and B_t are Ito processes and $g(t,x) = e^{tx}$ is twice continuously differentiable function $g : \mathbb{R}^2 \to \mathbb{R}$, the formula applies. $\frac{\partial}{\partial t} g = xe^{tx}$, $\frac{\partial^2}{\partial t^2} g = x^2 e^{tx}$, $\frac{\partial}{\partial x} g = te^{tx}$, $\frac{\partial^2}{\partial x^2} g = t^2 e^{tx}$. Then we can find its Ito representation using the Ito formula as

$$\begin{aligned} d(e^{tB_t}) &= e^{tB_t} B_t dt + \frac{1}{2} e^{tB_t} B_t^2 (dt)^2 + te^{tB_t} dB_t + \frac{1}{2} t^2 e^{tB_t} (dB_t)^2 \\ &= e^{tB_t}(B_t + \frac{1}{2} t^2) dt + te^{tB_t} dB_t. \end{aligned}$$

Problem 2. *Use the Ito formula to find $\int e^{B_t} dB_t$. In other words, you need to represent this integral in terms of expressions not involving dB_t (as we did for $\int B_t dB_t$).*

Suppose f is a continuously differentiable function. Let us use Ito formula to find $\int_0^t f_s dB_s$ and derive the integration by parts formula. In other words we look at a special simple case when X is a deterministic process i.e., $X_s = f_s$ a.s. First we observe that $f \in \mathcal{L}_2$. Indeed, it is differentiable and therefore continuous. This implies that f is bounded on any finite interval and therefore $\mathbb{E}[\int_0^t f_s^2 ds] = \int_0^t f_s^2 ds < \infty$.

Introduce $g(t, x) = f_t x$. We find that $\frac{\partial g}{\partial t} = x \frac{df}{dt}$, $\frac{\partial g}{\partial x} = f_t$, $\frac{\partial^2 g}{\partial t^2} = x \frac{d^2 f}{dt^2}$ and second order partial derivatives with respect to x disappear. Therefore, using Ito formula, we obtain

$$d(g(t, B_t) = \frac{df}{dt} B_t dt + \frac{d^2 f}{dt^2} B_t (dt)^2 + f_t dB_t + 0 = \frac{df}{dt} B_t dt + f_t dB_t$$

This implies (since $g(0, B(0)) = 0$)

$$\int_0^t f_s dB_s = f_t B_t - \int_0^t B_s \frac{df}{ds} ds.$$

This does look like integration by parts.

It turns out that a more general version of the integration by parts formula holds in Ito calculus. We start by recalling the definition of Stieltjes integral. We are given a function g which has bounded variation and another function f, which we assume for simplicity is continuous. We define the Stieltjes integral $\int_0^t f_s dg_s$ as an appropriate limit of the sums $\sum_j f_{t_j}(g_{t_{j+1}} - g_{t_j})$ where $\Pi_n : 0 = t_0 < \cdots < t_n = t$ is a sequence of partitions with $\Delta(\Pi_n) = \max_j (t_{j+1} - t_j) \to 0$. We skip the formalities of the construction of such a limit. They are similar to (and simpler than) those of the Ito integral.

Now let us state without proving the integration by parts theorem.

Theorem 3. *Suppose f_s is a continuous function on $[0, t]$ with bounded variation. Then*

$$\int_0^t f_s dB_s = f_t B_t - \int_0^t B_s df_s,$$

where the second integral is the $Stieltjes$ integral.

Is there a generalization of integration by parts when f is not necessarily deterministic? The answer is positive, but, unfortunately, it does not reduce an Ito process to a process not involving Brownian component dB.

Theorem 4. *Let X_t, Y_t be Ito processes. Then $X_t Y_t$ is also an Ito process:*

$$X_t Y_t = X(0)Y(0) + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \int_0^t dX_s dY_s$$

Proof. The proof is direct application of multidimensional Ito formula to the function $g(X_t, Y_t) = X_t Y_t$. $\square$

4 Additional reading materials

- Øksendal [1], Chapter IV.

References

[1] B. Øksendal, *Stochastic differential equations*, Springer, 1991.

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