Optimization Methods in Management Science

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Recitation 4

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At the end of this recitation, students should be able to:

1. Interpret the solution and sensitivity report of a problem to take decisions related to modifications in the problem data.

2. Be able to determine the shadow prices, price an activity, determine allowed one-at-a-time changes in the tableau.

Problem 1

(Problem 3 of Applied Mathematical Programming, Chapter 3)

Jean-Pierre Leveque has recently been named the Minister of International Trade for the new nation of New France. In connection with this position, he has decided that the welfare of the country (and his performance) could best be served by maximizing the net dollar value of the country's exports for the coming year. (The net dollar value of exports is defined as exports less the cost of all materials imported by the country.)

The area that now constitutes New France has traditionally made three products for export: steel, heavy machinery, and trucks. For the coming year, Jean-Pierre feels that they could sell all that they could produce of these three items at existing world market prices of $900/unit for steel, $2,500/unit for machinery, and $3,000/unit for trucks.

In order to produce one unit of steel with the country's existing technology, it takes 0.05 units of machinery, 0.08 units of trucks, two units of ore purchased on the world market for $100/unit, and other imported materials costing $100. In addition, it takes .5 man-years of labor to produce each unit of steel. The steel mills of New France have a maximum usable capacity of 300,000 units/year.

To produce one unit of machinery requires .75 units of steel, 0.12 units of trucks, and 5 man-years of labor. In addition, $150 of materials must be imported for each unit of machinery produced. The practical capacity of the country's machinery plants is 50,000 units/year.

In order to produce one unit of trucks, it takes one unit of steel, 0.10 units of machinery, three man-years of labor, and $500 worth of imported materials. Existing truck capacity is 550,000 units/year.

The total manpower available for production of steel, machinery, and trucks is 1,200,000 men/year.

To help Jean-Pierre in his planning, he had one of his staff formulate with Excel the model shown in Figure 1 and solved in Figure 2. A sensitivity report is given in Figure 3. The variables are labeled as follows:

ExportSteel	=	Steel production for export,
ExportMachinery	=	Machinery production for export,
ExportTruck	=	Truck production for export,
ProdSteel	=	Total steel production,
ProdMachinery	=	Total machinery production,
ProdTruck	=	Total truck production.

	ExportSteel	ExportMachinery	ExportTruck	ProdSteel	ProdMachinery	ProdTruck		
Variables	0	0	0	0	0	0		
NetExports	900	2500	3000	-300	-150	-500		
							Rel	Capacity
OutputSteel	-1	0	0	1	-0.75	-1	=	0
OutputMachinery	0	-1	0	-0.05	1	-0.1	=	0
OutputTruck	0	0	-1	-0.08	-0.12	1	=	0
CapacitySteel	0	0	0	1	0	0	<=	300000
CapacityMachinery	0	0	0	0	1	0	<=	50000
CapacityTruck	0	0	0	0	0	1	<=	550000
CapacityManpower	0	0	0	0.5	5	3	<=	1200000

Figure 1: Formulation sheet for Problem 1.

The constraints have labels:

OutputSteel, OutputMachinery, OutputTruck,
CapacitySteel, CapacityMachinery, CapacityTruck, CapacityManpower.

Referring to the three figures, he has asked you to help him with the following questions:

(a) What is the optimal production and export mix for New France, based on Fig. 2? What would be the net dollar value of exports under this plan? In your answer, identify clearly how much is produced of each product, and how much is exported. You should also clearly state what units you are using for each item produced or exported.

(b) What do the first three constraint equations (OutputSteel, OutputMachinery, and OutputTruck) represent? Why are they equality constraints instead of $\leq$ constraints?

(c) The optimal solution suggests that New France produce 50,000 units of machinery. How are those units to be utilized during the year?

(d) What would happen to the value of net exports if the world market price of steel increased to $1250/unit and the country chose to export one unit of steel?

(e) New France wants to identify other products it can profitably produce and export, i.e. include in the optimal production mix. Among the four possible resources (steel, machinery, trucks, manpower), there is one which is the least preferable to be used by new products, and should be used as sparingly as possible. Identify this resource and explain your choice.

(f) There is a chance that Jean-Pierre may have $500,000 to spend on expanding capacity. If this investment will buy 500 units of truck capacity, 1,000 units of machine capacity, or 300 units of steel capacity, what would be the best investment?

(g) If the world market price of the imported materials needed to produce one unit of trucks were to increase by $400, what would be the optimal export mix for New France, and what would be the dollar value of their net exports?

(h) The Minister of Defense has recently come to Jean-Pierre and said that he would like to stockpile (inventory) an additional 10,000 units of steel during the coming year. How will

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$C$16	NetExports	0	490625000

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$C$3	ExportSteel	0	0	Contin
$D$3	ExportMachinery	0	8750	Contin
$E$3	ExportTruck	0	232500	Contin
$F$3	ProdSteel	0	300000	Contin
$G$3	ProdMachinery	0	50000	Contin
$H$3	ProdTruck	0	262500	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$C$18	OutputSteel	0	$C$18=$E$18	Binding	0
$C$19	OutputMachinery	0	$C$19=$E$19	Binding	0
$C$20	OutputTruck	0	$C$20=$E$20	Binding	0
$C$21	CapacitySteel	300000	$C$21<=$E$21	Binding	0
$C$22	CapacityMachinery	50000	$C$22<=$E$22	Binding	0
$C$23	CapacityTruck	262500	$C$23<=$E$23	Not Binding	287500
$C$24	CapacityManpower	1187500	$C$24<=$E$24	Not Binding	12500

Figure 2: Solution sheet for Problem 1.

this change the constraint equation OutputSteel, and what impact will it have on net dollar exports?

(i) Is it possible with this particular formulation to deal with existing inventories at the start of the year and desired inventories at the end of the year? Suppose we start the year with S_s units of steel in the inventory, M_s units of machinery, and T_s units of trucks. At the end of the year we want to have S_e units of steel in the inventory, M_e units of machinery, and T_e units of trucks. How should we modify the constraints OutputSteel, OutputMachinery and OutputTruck to deal with this situation?

(j) A government R&D group has recently come to Jean-Pierre with a new product, Product X, that can be produced for export with 1.5 man-years of labor and 0.3 units of machinery for each unit produced. What must Product X sell for on the world market to make it attractive for production?

(k) There is the possibility of strikes during the year that will impair Machinery production. In particular, Jean-Pierre wants to know how a reduction of the machinery capacity from 50,000 to 40,000 will affect the total value of net exports. Compute an estimate of the net

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$3	ExportSteel	0	-1350	900	1350	1E+30
$D$3	ExportMachinery	8750	0	2500	10566.66667	281.3953488
$E$3	ExportTruck	232500	0	3000	347.7011494	1350
$F$3	ProdSteel	300000	0	-300	1E+30	1585
$G$3	ProdMachinery	50000	0	-150	1E+30	302.5
$H$3	ProdTruck	262500	0	-500	403.3333333	1350

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$C$18	OutputSteel	0	-2250	0	232500	4166.666667
$C$19	OutputMachinery	1.45519E-11	-2500	0	8750	1E+30
$C$20	OutputTruck	1.16415E-10	-3000	0	232500	1E+30
$C$21	CapacitySteel	300000	1585	300000	3571.428571	252717.3913
$C$22	CapacityMachinery	50000	302.5	50000	4545.454545	8139.534884
$C$23	CapacityTruck	262500	0	550000	1E+30	287500
$C$24	CapacityManpower	1187500	0	1200000	1E+30	12500

Figure 3: Sensitivity report for Problem 1.

dollar exports in this particular scenario, using the information given in Figures 2 and 3. Is the value that you computed the exact value of the net dollar exports in the new scenario, or is it an estimate? If it is an estimate, is it from below or from above (in other words: will the real net dollar exports value be larger or smaller than what you computed)?

Problem 2

Consider the following Linear Program:

$$\left.\begin{array}{rrcl} \max & x_1 + 4x_2 & & \\ \text{subject to:} & & & \\ \text{Resource 1:} & x_1 - x_2 & \leq & 2 \\ \text{Resource 2:} & x_1 + 2x_2 & \leq & 4 \\ \text{Resource 3:} & x_1 & \leq & 3. \end{array}\right\}$$

We introduce slack variables s_1, s_2, s_3 in the three constraints in this order, and we solve the resulting LP. The optimal simplex tableau is given below:

Basic	x_1	x_2	s_1	s_2	s_3	Rhs
$(-z)$	-1			-2		-8
s_1	1.5		1	0.5		4
x_2	0.5	1		0.5		2
s_3	1				1	3

Using the final tableau and the initial formulation, answer the following questions:

(a) What is the optimal solution?

(b) Determine the shadow price for each constraint?

(c) Determine the range on the objective function coefficient of variables x_1 and x_2 for which the current optimal solution stays optimal, assuming that the remaining data stays fixed. (Note: only *one* of the two coefficients should change at a time, while everything else is fixed. Hint: you will have to treat differently the cases where the corresponding variable is basic or nonbasic. When the variable is basic, substitute $c + \delta$ to the old coefficient c, and make sure the final tableau stays in canonical form.)

Problem 3

Assume that we have the following LP in canonical form:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	s_3	Rhs
$(-z)$	3	1	2	-2				0
s_1	1	1	1	1	1			4
s_2	2	-2		-6		1		2
s_2	3		1	-1			1	7

The optimal tableau is:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	s_3	Rhs
$(-z)$	0	0	0	-1	-2	-0.5	0	-9
x_3			1	-4	3	1.5	-2	1
x_1	1			1	-1	-0.5	1	2
x_2		1		4	-1	-1	1	1

(a) What is the optimal solution?

(b) Determine the shadow price for each constraint?

(c) Price a new activity corresponding to the column with coefficients $(0.5, 2, 5)$, using the simplex multipliers (also called shadow prices). (Note: *pricing* an activity means finding out how much it costs to produce one unit of the activity at the current optimal solution, or equivalently, how much revenue for one unit of the activity is required to reach the break-even point.)

(d) By how much can we increase the objective function coefficient of x_4 while leaving the current optimal solution unchanged?