

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Physics

Physics 8.901: Astrophysics I

Spring Term 2006

PROBLEM SET 9

Due: Tuesday, May 9 in class

Reading: Read Chapter 13 on compact X-ray sources in Shapiro & Teukolsky, *Black Holes, White Dwarfs, & Neutron Stars*. (Note that the observational details are very dated, but the theoretical description is good.) You may also find it instructive to skim through Chapters 14 (accretion onto black holes), 15 (accretion onto neutron stars and white dwarfs), and 16 (gravitational radiation).

1. **Supernova explosion in a binary system.** Two stars of mass m_1 and m_2 are in a circular orbit. Star 1 undergoes a supernova explosion in which mass Δm is blown away spherically symmetrically (in the frame of star 1) on a time scale that is very short compared to the orbital period. Show that the condition for the orbit to remain bound is

$$\Delta m < \frac{m_1 + m_2}{2}.$$

Hint: Compute the total energy of the binary after the explosion in the (new) center of mass frame of the binary. Assume that immediately after the explosion, the binary separation and the orbital velocities of both stars are unchanged. A useful relation is that the total kinetic energy is $(1/2)\mu v_{\text{rel}}^2$, where μ is the reduced mass and v_{rel} is the relative velocity.

2. **Changes in orbital separation due to mass transfer.** Two stars of mass m_1 and m_2 are in a circular orbit. Star 2 transfers matter onto star 1 via Roche-lobe overflow. Assume that the mass transfer is *conservative*, i.e. that no mass is lost from the system, and also that the orbit remains circular during mass transfer.

- (a) Using only conservation of mass and orbital angular momentum, find the fractional change in binary separation ($\Delta a/a$) in terms of the mass ratio $q = m_2/m_1$ and the fractional mass change $\Delta m/m$ (where $\Delta m_2 = -\Delta m$ and $\Delta m_1 = +\Delta m$). Show that transfer from the lighter to the heavier star leads to a widening of the orbit ($\Delta a/a > 0$), while transfer from the heavier to the lighter star leads to a shrinking of the orbit ($\Delta a/a < 0$).
- (b) The effective radius of the Roche lobe around star 2 is given approximately by

$$R_{L2} \simeq \frac{a}{2} \left(\frac{M_2}{M_1 + M_2} \right)^{1/3}.$$

Find the change in the Roche lobe radius in response to the mass transfer.

- (c) For simplicity, assume that star 2 is fully convective, and thus well-described by an $n = 3/2$ polytrope. The adiabatic response of the star is obtained from the mass-radius relation for an $n = 3/2$ polytrope, which gives $R \propto M^{-1/3}$. From this relation, find the change in the radius of star 2 in response to the mass loss.
- (d) Argue that the mass transfer will be *dynamically unstable* if $\Delta R_2 > \Delta R_{L2}$. Find the maximum value of the mass ratio q for stable mass transfer.

3. **Magnetospheric radius of an X-ray pulsar.** Define the magnetospheric radius of an accreting neutron star as the distance at which the energy density in the magnetic field ($B^2/8\pi$) is equal to the (Keplerian) kinetic energy density of the orbiting material in the accretion disk ($\rho v_K^2/2$, where ρ is the mass density and v_K is the Keplerian velocity). Assume that the neutron star has a dipolar magnetic field, with dipole moment μ .

Show that the radius of the magnetosphere can be written as

$$r_m \simeq 1.3 \times 10^8 (\eta\Omega)^{2/7} \mu_{30}^{4/7} \dot{M}_{-8}^{-2/7} M_1^{-1/7} \text{ cm},$$

where μ_{30} is the magnetic dipole moment of the neutron star in units of 10^{30} G cm³, \dot{M}_{-8} is the mass accretion rate in units of $10^{-8} M_\odot \text{ yr}^{-1}$, M_1 is the neutron star mass in solar masses, and Ω is the solid angle subtended by the accretion disk as seen from the neutron star.

Note that conservation of mass requires that, in steady state, $\dot{M} \simeq (\rho v_r \Omega)(4\pi r^2)$ (or flux times area), where v_r is the (small) radial velocity component of the disk material. Define the ratio $\eta = v_r/v_K$, where generally $\eta \ll 1$.

4. **Gravitational radiation from the Hulse-Taylor binary pulsar.**

This problem explores material discussed in Ryan Lang's guest lecture on April 27. Consider a binary neutron star system. Assume that both stars have the same mass M and that they move in a circular orbit, separated by a distance R . For convenience, define the period of the orbit to be P and the angular frequency of the orbit to be $\Omega = 2\pi/P$. What are these in terms of M and R ?

- Calculate the (standard Newtonian) energy of the system in terms of M and P .
- Calculate the gravitational quadrupole moment tensor I_{ij} for this system. For simplicity, assume the binary lies in the xy -plane, and take the origin to be the center of mass. Use standard identities to reduce the answer to single trigonometric functions (not products of them). How does the gravitational wave frequency f relate to the orbital frequency Ω ?
- Use the result of part b to calculate the trace-reversed metric perturbation \bar{h}_{ij} generated by the motion of this system. The result you get should already be in transverse-traceless form for a wave propagating in the z direction. Describe the polarization of this wave.
- Apply the projection tensor and remove the trace to find the transverse-traceless representation of a wave propagating in the x direction. What is its polarization? Explain why this makes sense.
- Calculate the energy loss from the system due to gravitational radiation, dE/dt . Then use the result of part a to find the change in period, dP/dt .
- Now plug in some numbers: $M = 1.4M_\odot$ and $P = 7.75$ h. What is the resulting orbital period derivative, dP/dt ? Compare your answer to Hulse and Taylor's answer, -2.42×10^{-12} (dimensionless). What assumptions did we make that might cause these results to disagree?