

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Physics

Physics 8.901: Astrophysics I

Spring Term 2006

PROBLEM SET 4

Due: Tuesday, March 21 in class

REMINDER: The mid-term exam will be in class on Thursday, March 23.

Reading: Chapter 6, Chapter 7 (§7.1–7.2) in Hansen, Kawaler, & Trimble. (I encourage you to at least skim through the rest of Chapter 7 as well.) We will also start covering the material in Chapter 2 (§2.1–2.5), although homework problems on this material will be on Problem Set 5. (You should still do this reading prior to the midterm exam, however.)

1. **Uniqueness of solutions to the stellar structure equations.** HK&T, Problem 7.1.

2. **Properties of polytropic stars.** Recall that in polytropic stars, pressure and density are simply related as $P = K\rho^{(1+1/n)}$, where K and n are constants and n is called the polytropic index. These stars satisfy the Lane-Emden equation,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n,$$

where the dimensionless variables θ and ξ are defined in terms of radial coordinate r and density ρ as $r = a\xi$ and $\rho = \rho_c \theta^n(r)$ and ρ_c is the central density. The relevant boundary conditions are $\theta(0) = 1$ and $\theta'(0) = 0$ at the center ($\xi = 0$) and $\theta(\xi_1) = 0$ at the surface (the first zero-crossing of θ , which occurs at $\xi = \xi_1$).

Show the following:

- (a) For an ideal gas equation of state, the variable θ is a dimensionless temperature, such that $T(r) = T_c \theta(r)$.
- (b) The total mass of a polytropic star is $M = -4\pi a^3 \rho_c \xi_1^2 (\theta')_{\xi_1}$.
- (c) The ratio of the mean density to the central density is $\frac{\langle \rho \rangle}{\rho_c} = -(3/\xi_1) (\theta')_{\xi_1}$.
- (d) The central pressure is $P_c = [4\pi(n+1)(\theta')_{\xi_1}^2]^{-1} (GM^2/R^4)$.

3. **Solutions for polytropic models.**

- (a) Show that solutions to the Lane-Emden equation for polytropes of index n can be expanded as

$$\theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} + \dots,$$

for ξ small (i.e. near the center of the star). To do this, first show that polynomial expansions of $\theta(\xi)$ contain only even terms of ξ , and then substitute such a polynomial into the Lane-Emden equation and find the first three coefficients.

- (b) Numerically integrate the Lane-Emden equation for polytropic indices of $n = 1.0, 1.5, 2.0, 2.5, 3.0,$ and 3.5 . Do this by breaking up the second-order differential equation into a pair of first-order coupled equations in $u = d\theta/d\xi$ and $du/d\xi$, and use a numerical integration scheme (e.g.,

4th-order Runge-Kutta or some other equivalent scheme) to find $\theta(\xi)$, applying the appropriate boundary conditions. To help start the integration near the center, use the first two terms of the analytic expansion you derived above for $\theta(\xi)$.

Plot the dimensionless temperature $\theta(\xi)$ and the dimensionless density $\theta^n(\xi)$ for all six values of n . (Put all the temperature curves on one plot and all the density curves on another.)

4. Classical Coulomb collisions.

- What temperature would be required for two protons to collide if quantum mechanical effects are neglected? Assume that nuclei having ten times the rms value for the Maxwell-Boltzmann distribution can overcome the Coulomb barrier. Compare your answer with the estimated central temperature of the Sun.
- Calculate the ratio of the number of protons having velocities ten times the rms value to those moving at the rms velocity for a Maxwell-Boltzmann distribution.
- Assuming (incorrectly) that the Sun is pure hydrogen, estimate the number of hydrogen nuclei in the Sun. Are there enough protons moving with a speed ten times the rms value to account for the solar luminosity?

5. **Coulomb barrier penetration.** Consider a head-on collision between two atomic nuclei whose charges are Z_1e and Z_2e and whose masses are A_1 and A_2 (in atomic mass units). Using the WKB approximation, show that the quantum mechanical tunneling probability for this encounter is given by

$$T \simeq \exp\left(\frac{-2\pi Z_1 Z_2 e^2}{v\hbar}\right),$$

where v is the relative velocity of the two nuclei and e is the proton charge. Show also that this can be written in the form

$$T \simeq \exp(-31.28 Z_1 Z_2 A^{1/2} E^{-1/2}),$$

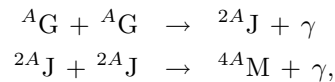
where A is the reduced mass of the two nuclei (in atomic mass units) and E is the center-of-mass kinetic energy (in keV) of the two nuclei at a large separation before the collision.

6. **Nuclear binding energies.** The Q value of a nuclear reaction is the amount of energy released (or absorbed) in the reaction, defined such that energy released is positive. Compute the Q values (in MeV) for each of the following nuclear reactions. Indicate whether the reaction is exothermic or endothermic.

- $^{12}\text{C} + ^{12}\text{C} \rightarrow ^{24}\text{Mg} + \gamma$
- $^{12}\text{C} + ^{12}\text{C} \rightarrow ^{16}\text{O} + 2^4\text{He}$
- $^{19}\text{F} + ^1\text{H} \rightarrow ^{16}\text{O} + ^4\text{He}$
- $^1\text{H} + ^1\text{H} \rightarrow ^2\text{H} + e^+ + \nu$
- $^{15}\text{N} + ^1\text{H} \rightarrow ^{12}\text{C} + ^4\text{He}$

You may compute the atomic mass excesses for each isotope yourself, but it is easier to consult a table (see, e.g., Table 4-1 in Clayton, *Principles of Stellar Evolution and Nucleosynthesis* or Table 38 in Lang, *Astrophysical Formulae*, 2nd ed. 1980).

7. **Nuclear burning.** Consider the hypothetical nuclear reactions



where G, J, and M are hypothetical elements, γ is a photon, A is the atomic mass number of element G (an integer with units of amu), and the atomic numbers Z for each element are taken to be exactly half their atomic mass number.

Suppose that we have two stars, one in which the first reaction is occurring and another in which the second reaction is occurring, and that the reactions proceed at the same rate at the respective stellar centers. Estimate the ratio of the central temperatures of the two stars. Comment on whether your answer seems reasonable or not, and why.

Note: In the spirit of ruthless approximation, you may ignore factors which multiply exponentials, e.g., the densities of the reactants and the astrophysical cross-section factor $S(E)$.

8. **Relative velocity distribution of two Maxwell-Boltzmann populations.** In class we calculated the weighted average $\langle \sigma v \rangle$ for nuclear reaction rates by assuming that the relative velocity of the two fusing nuclei has a Maxwell-Boltzmann distribution. Here, you will prove that this was justified.

If the velocities of a set of identical, distinguishable particles with mass m has a Maxwell-Boltzmann distribution, then the fraction of particles that have velocity in the range $[v, v + dv]$ is given by

$$f(v) dv_x dv_y dv_z = \left(\frac{m}{2\pi kT} \right)^{3/2} \exp \left(-\frac{mv^2}{2kT} \right) dv_x dv_y dv_z.$$

Suppose there are *two* different sets of particles, with masses m_1 and m_2 , with Maxwellian velocity distributions v_1 and v_2 , respectively. Show that the distribution of relative velocities $v = v_1 - v_2$ is also Maxwellian. (You will find it useful to rewrite v_1 and v_2 in terms of the center of mass and relative velocities.)