MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.901: Astrophysics I **Spring Term 2006**

PROBLEM SET 2

Due: Thursday, March 2 in class

Reading: Chapter 3 in Hansen, Kawaler, & Trimble, especially §3.4. You may also find it useful to read Chapter 8 in Carroll & Ostlie, *Introduction to Modern Astrophysics*. If you are interested in additional reading about the Saha equation, you may also consult Chapter 14 of *Stellar Structure and Evolution* by R. Kippenhahn and A. Weigert (1990, Springer).

1. **Blackbody radiation.** The Planck radiation spectrum is given by

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT)-1} \quad (\text{erg cm}^{-2}\text{ s}^{-1}\text{ Hz}^{-1}\text{ steradian}^{-1}),$$

per unit frequency.

(a) **Wavelength spectrum.** Show by explicit calculation that the equivalent Planck radiation spectrum per unit *wavelength* is given by

$$B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda kT)-1} \quad (\text{erg cm}^{-2}\text{ s}^{-1}\text{ cm}^{-1}\text{ steradian}^{-1}),$$

starting from the expression for B_ν.

(b) **Stefan-Boltzmann law.** Derive the Stefan-Boltzmann law ($F = \sigma T^4$) by integrating the Planck blackbody spectrum over all wavelengths or frequencies. (Note that there is an extra factor of π to convert from brightness per unit solid angle to total brightness, so that $F = \pi \int B_\nu d\nu = \pi \int B_\lambda d\lambda$.) You may use the fact that

$$\int_0^\infty \frac{u^3}{e^u - 1}du = \frac{\pi^4}{15}.$$

Give an expression for the Stefan-Boltzmann constant σ in terms of fundamental physical constants, and check its numerical value and units, $\sigma = 5.67 \times 10^{-5}$ erg cm^{-2} s^{-1} K^{-4}.

(c) **Wavelength of radiation peak.** Derive the Wien displacement law, which relates the wavelength of the radiation at the peak of the Planck function B_λ to the temperature: $T\lambda_{\max} = 0.29$ cm K. [When you differentiate to find the maximum of B_λ, you will obtain a nonlinear equation of the form $5(1-e^{-y}) - y = 0$ which you can solve numerically.]

(d) **Frequency of radiation peak.** Repeat the previous part, but this time find the relation between the *frequency* at the peak of the Planck function B_ν and the temperature: $\nu_{\max}/T = 5.9 \times 10^{10}$ Hz K^{-1}. For a given temperature T, does the photon energy corresponding to $\nu_{\max}$ agree with that for $\lambda_{\max}$ in the previous part? Should they agree? Explain.

2. **Saha equation and pure hydrogen.** Consider a gas of pure hydrogen at fixed density and temperature. The ionization energy of hydrogen is $\chi_0 = 13.6$ eV. You may assume that all the hydrogen atoms (whether neutral or ionized) are in their ground energy state.

(a) Write down the Saha equation relating the number densities of neutral and ionized hydrogen (n_0 and n_1, respectively). Make reasonable approximations to use numerical values for the partition functions.

(b) To find the individual densities, further constraints are required. Reasonable constraints are charge neutrality ($n_e = n_1$) and conservation of nucleon number ($n_1 + n_0 = n$), where the total hydrogen number density n is a constant if the density ρ is fixed. Rewrite the Saha equation in terms of the hydrogen ionization fraction $x = n_1/n$, eliminating n_1, n_0, and n_e. Does this equation have the expected limiting behavior for $T \to 0$ and $T \to \infty$?

(c) Use the relation $n = \rho N_{\rm A}$ (where $N_{\rm A} = 6.023 \times 10^{23}$ is Avogadro's number) to replace n with ρ. Find an expression for the half-ionized ($x = 0.5$) path in the ρ-T plane. Plot this path on a log-log plot for densities in the interesting range from 10^{-10}–10^{-2} g cm^{-3}

3. **Saha equation and pure helium.** (Based on HK&T, Problem 3.1.) Consider a gas of pure helium at fixed density and temperature. The ionization energies for helium are $\chi_0 = 24.6$ eV and $\chi_1 = 54.4$ eV. As above, you may assume that all the helium atoms (whether neutral, singly ionized, or doubly ionized) are in their ground energy state. Let n_e, n_0, n_1, and n_2 be the number densities of, respectively, free electrons, neutral atoms, singly-ionized atoms, and doubly-ionized atoms. The total number density of neutral atoms and ions is denoted by n. Furthermore, define x_e as the ratio n_e/n and, likewise, let x_i be n_i/n where $i = 0, 1, 2$. You should assume that the gas is electrically neutral. You should look up the degeneracy factors you need for the atoms and ions; for example, see pp. 33–36 of *Astrophysical Quantities* by C. W. Allen (3rd edition, 1973, Athlone) or pp. 31–35 of *Allen's Astrophysical Quantities* edited by Arthur Cox (4th edition, 2000, Springer).

(a) As in the hydrogenic case, construct the ratios n_1/n_0 and n_2/n_1 using the Saha equation. In doing so, take care in establishing the zero points of energy for the various constituents.

(b) Apply charge neutrality and nucleon number conservation ($n = n_0 + n_1 + n_2$) and recast the above Saha equations so that only x_1 and x_2 appear as unknowns. The resulting two equations have T and n (or, equivalently, $\rho = 4n/N_{\rm A}$) as parameters.

(c) Simultaneously solve the two Saha equations for x_1 and x_2 for temperatures in the range $4 \times 10^4 \leq T \leq 2 \times 10^5$ K with a fixed value of density from among the choices $\rho = 10^{-4}$, 10^{-6}, or 10^{-8} g cm^{-3}. You may find it more convenient to use the logarithm of your equations. Choose a dense grid in temperature because you will soon plot the results. Once you have found x_1 and x_2, also find x_e and x_0 for the same range of temperature. Note that this is a numerical exercise and you should use a computer.

(d) Plot all your x's as a function of temperature for your chosen value of ρ. (Plot x_0, x_1, and x_2 on the same graph.) Identify the transition temperatures (half-ionization) for the two ionization stages.

4. **Stellar opacity.**

(a) HK&T, Problem 3.2.

(b) (Carroll & Ostlie, Problem 9.7) Calculate how far you could see through the Earth's atmosphere if it had the opacity of the solar photosphere ($\kappa_\odot = 0.264$ cm^2 g^{-1} for a wavelength of 5000 Å and a density of 2.5×10^7 g cm^{-3}). Use 1.2×10^{-3} g cm^{-3} for the density of the Earth's atmosphere.