PROFESSOR: OK. So what we are talking about is actually matrix elements. If you want to do anything interesting in atomic physics, you have to copy or induce transitions from one state to another state. Well, maybe that should be Hab.

For many phenomenon, which we will cover throughout the rest of this semester, spontaneous emission, coherences, and three-level instances, and super radiance, all we need is a matrix element. And this matrix element will just run through all the equations, and be responsible for a lot of interesting phenomenon.

And for most of the description of those phenomena, we don't have to know where this matrix element comes from. The only thing we have to know, there is a non-seeable matrix element which drives the process. And as you know, the matrix element, with an external field, is called the Rabi frequency. And a lot of physics just depends on the Rabi frequency.

But what is behind? The engine behind the Rabi frequency is a matrix element. So in the unit I started to teach the week before spring break, we talked about matrix elements. And for H, for the Hamiltonian, we used the coupling of an atom to the electromagnetic field. And then we calculated, what is the matrix element induced by the electric field? We made the dipole approximation. And that's your plain, vanilla, generic dipole operator which can connect two states.

But we also consider, what happens when we go beyond the dipole approximation, and we found extra ways of copying two states? For instance, we can copy two states which have the same parity with a quadrupole transition, or we can couple them with a magnetic dipole transition.

So these are other ways to get into matrix elements. For most of the course, you

don't have to understand what is behind the matrix, and you just know, there is a number which drives the process.

So what I want to finish today is to discuss-- and these are called selection rules-- which tell us, when are those numbers, when is this matrix element which couples two states, when is it zero, or when is it non-vanishing. And what is helpful here is, well, as always in physics, use symmetry.

And if you have an operator, let me give you examples immediately. But just think for a moment about the electric dipole. The electric dipole is the position operator R. And you want to know, can the position operator R induce a transition between two states. The way to analyse it is now in terms of symmetry.

And for symmetry, which is always fulfilled for isolated atoms, is angular momentum. Angular momentum is a conserved quantum number, we have rotation symmetry. So therefore, we want to now understand matrix elements in the language of rotation symmetry. And therefore, we don't want to use a precision operator x, y, or z. x, y, z do not have the rotation symmetry. We want to use linear superpositions of x, y, and z-- I'll give you an example in a moment-- in such a way that the operator becomes an element of aesthetic of a spherical tensor.

And spherical tensor, I gave you the definition in the last lecture, the element of spherical tensor, Ln, is defined by-- well, I connect it with something you know, that it transforms on the rotation, like the spherical harmonics, ylm.

So it is pretty much for an operator, what the ylm, what the spherical harmonics are for wave functions. I think I can do it more formal. And Professor Schwann knows much more about it. I think these are elements of the rotational symmetry group, But I don't want to go there.

So what I mean by that is the following, that if you take the position vector r, you can expand it into a basis, which is x and y. But if you use the spherical basis, x plus/minus iy. Then what appears are the spherical harmonics. So in that case, it's rather simple. The position vector has actually in this representation components

which you can even see are the spherical harmonics. And therefore, we transform like the spherical harmonics.

Or just to give you another example, if you have the operator, which is responsible for the quadrupole transition, well, you get the gist, it's a product of two coordinates. So therefore, it's a spherical tensor of rank two. And it so happens, but I'm not deriving that, it is a superposition of two components with Lm quantum number, 2 plus 1, 2 minus 1.

So that's how we should think about it. So, we want to ask, we want to extend the operator, into operators which have rotational symmetry, and these are those, or these are those three. So instead of using the vector Cartesian coordinate, we use its spherical components. And with that, we can take this expression from the last lecture, and rewrite it using the Wigner-Eckart theorem into a way which allows us to immediately formulate selection rules.

So [INAUDIBLE] and primer are the quantum numbers of the state, except for angular momentum, so matricing about principle quantum number of the hydrogen atom. And we want to copy from a total angular momentum J prime to total angular momentum J. Actually, we want to copy from J prime, M prime to a state, JM. And what the Wigner-Eckart theorem tells us that we can factor out the M dependence.

The M dependence just comes from orientation in space. So M is just how you orient wave functions and vectors and space. And you can sort of write this matrix element as a projection. And this is nothing else than the familiar Clebsch-Gordan coefficient. And the Clebsch-Gordan coefficient for coupling the initial state, JM, or to-- let me put it this way-- to start with the initial state J prime, M prime, we have the L and the M of our operator. And that should result in a total angular momentum of J and M.

So we retrieve again the formalism of the addition of two angular momenta. Sometimes, you have two particles you couple into angular momentum and ask, what is the total angular momentum of the composite particles? But what we do here for this selection rule, we have the initial state, we calculate with the angular

momentum of the operator.

You can think the operator is a field which can transfer angular momentum. And then, of course, the final state has to fulfill angular momentum conservation. But one source of the momentum is now the operator, is the external field, is the photon, or the microwave drive, whatever you apply.

And yes, this Wigner-Eckart theorem allows us to write the matrix element as a reduced matrix element. Which really decides whether the transition is non-vanishing or not, times a factor which is just the orientation of the wave function and of the operator in space.

So for the Clebsch-Gordan coefficient, we have a simple selection rule. And this is that for the [INAUDIBLE] number, the M of the final state has to be the M of the initial state, plus a little M of the operator. And for both the Clebsch-Gordan and the reduced matrix element, we have the triangle rule.

Well, if you couple two angular momentum vectors to a final angular momentum, the three vectors have to form a triangle. And the triangle rule says that some-- let me write it down, and then you recognize it-- that the angular momentum construct by the field has to fulfill the triangle rule that J prime and J can be connected. Yes?

AUDIENCE: What is this symbolic meaning of the double bars?

PROFESSOR: It's just how, in many textbooks, the reduced matrix element is written. It's nothing else in a matrix element. But, you know, plus the y that I looked at in the quantum mechanics book. But what happens is these are not states. J and J prime are not states. They have an independence. So we've taken out the independence.

So this is sort of a matrix element between a state which may have been stripped of its independence. So maybe, I don't know if that's 100% correct, but if you have the YLM in certain states, you have an e to the IM and M part. And this has probably been factored out. So these are not really states, and the double line just means it's a reduced matrix element with the meaning I just mentioned. It's a standard way of factorizing matrix elements. And yeah, that means reduced matrix elements.

So in other words, when we talk about selection rules, we want to use the representation of spherical tensors, because the spherical tensor, the rank of the spherical tensor, just tells us how much angular momentum is involved in the photon, is involved in this transition.

So maybe just to give you a question, so if I were to do a multipole expansion, and I have an octupole transition, what is now the angular momentum transferred by a photon?

STUDENT: 3?

PROFESSOR: What?

STUDENT: 3?

STUDENT: 3?

PROFESSOR: 3, yeah. The dipole is L equals 1. Quadrupole is spherical tense of rank 2. L equals 2, so it's L equals 3.

Now, can a photon transfer three units of angular momentum? Can an atom get rid of three units of, let's say, orbital or spin angular momentum. We start in a state which is J prime equals 3. You need one photon, and you go to a state which is J prime equals 0. Is that possible or not?

We don't have [INAUDIBLE], but do you want to volunteer an answer? What's the angular momentum of the photon?

STUDENT: [INAUDIBLE]?

PROFESSOR: Well, be careful. The photon has an intrinsic angular momentum, which is like the spin of the photon. That's plus/minus 1. But just imagine that you have an atom, and the photon is not immediate at the origin. The photon is emitted a little bit further out.

Then, with reference to the origin, the photon has orbital angular momentum. And

that's what we're talking about. In the multipole expansion, we fall in powers of x, and z, and y of the spatial coordinate of the electron. And that actually means we're going away from the origin. And if you emit something which is away from the origin, you have orbital angular momentum.

So, yes. An octupole transition is exactly what I said. It means a photon is emitted, and it changes the angular momentum of the atom left behind by three units. That's what we really mean by that. And that's what we mean by those electrodes.

The question that you should maybe discuss after class is, what happens if you detect this photon? Is that now a supercharged photon, which has three units of angular momentum? Is there something strange in its polarization? Think about it, and if you don't find the answer, we can discuss it in the next class.

OK. So this is the classification. Let's just focus on the simple examples. We have discussed electric dipole and magnetic dipole radiation. These are induced by vectors. Remember, E1 is the dipole vector. For M1, the matrix element was by the angular momentum vector. So

These are vectors. And that means the representation of the spherical tensors, or the quantum numbers of the spherical tensors, are the same of the Y1n. And so for dipole radiation, whether it's electric or magnetic, we have now with the dipole selection rules, which pretty much save you at one unit of angular momentum to state B. Can you reach state A with that?

And these selection rules are that you can change the angular momentum between initial state by 0 and 1. This is the triangle rule. And delta m can be 0 and plus/minus 1, depending on polarization, which we want to discuss in a moment.

So in angular momentum, electric and magnetic dipoles have the same selection rule, where when it comes to the question of parity, we've already discussed that. That an electric dipole connects to a state of opposite parity, whereas, the magnetic dipole connects two states of the same parity. And of course, this comes about because L is an axial vector, and R is a polar vector, which have different symmetry

when you invert the coordinate system.

The one higher multipole port transition, which we discussed, was the electric quadrupole, E2. And the spherical tensor operators for the quadrupole transition, I gave you already the example of, let's say, xz, products of two coordinates, because we went one order higher than the dipole. They transform as Y2m. And therefore, we have selection rules for quadrupole transitions, which tell us now that we can change the total angular momentum up to 2. And also, delta m can change up to two units.

And again, just to emphasize, because people get confused all the time. When we talk about a quadrupole transition, we mean absolutely positively a transition where one photon is emitted. If you fully quantize the field, there is one creation operator of the photon. It's one photon which is created, and this photon carries away the angular momentum we've just specified.

Questions about that? Let me conclude our discussion of matrix elements by talking about something which is experimentally very relevant. And this is how selection rules depend on the polarization of light. And I only want to discuss it for electric dipole transitions.

So when we wrote down the coupling of the atom to electromagnetic radiation, we had the dipole operator, but we also had, of course, the mode of the electromagnetic field, which was characterized by a polarization epsilon. So until now, when I talked about selection rules, we discussed this part. But now we want to see how it effects polarization.

Well, the epsilon, for instance, for circular polarization-- we'll talk about linear polarization in a moment-- has this representation. So this is the unit vector of the polarization of the electric field when it's circularly polarization.

And now remember, we take this vector r, and expand it in the following way. So if you multiply now the operator r, or the matrix elements created by this vectorate operator, by the polarization, you see that one circular polarization checks out this

component. The other circular polarization projects this out. And later, we'll talk about that linear polarization projects that out.

So when we said that we have matrix elements for dipole transition, which can change angular momentum, or the incremental number, by minus 1, plus 1, and 0, this is now related to the polarization of the light, either the photon which is emitted, or when we use circularly polarized light, we can only drive this transition, that or that, because the scale of product of the polarization vector and the matrix element project out only one component of the spherical tensor.

So if you look at the expansion above, we realize that the left- and right-handed circular light projects now out the spherical tensor operator, T1 plus minus 1. And since it's circularly polarized light, and therefore, we find this election rule that delta m, the Z component, the Z component of the angular momentum, changes by plus/minus 1 when the circular polarized light is sigma plus or sigma minus, right-handed or left-handed.

OK. So this is responsible for circular polarization. These are selection rules for circularly polarized light. Let me conclude by discussing the case of linear polarization. Well, when we ask linear polarization, if we ask for linear polarization along x or y, well, it's linear polarization, but we should regard it as the linear superposition of sigma plus and sigma minus.

So in other words, if you have the quantization axis along z, and you use light which is polarized along x or y, the way how the light talks to the atom with symmetric operators is that the light is a superposition of sigma plus and sigma minus.

So what we have so far is, so we had here the light key, the propagation of the light was along the z-axis. But now, we want to look at the other possibility that z, or the quantization axis, is parallel to the polarization of the electric field, which would mean that the quantization axis is usually defined by an external magnetic field.

If you're talking about the situation that the electric field of the electromagnetic wave is parallel to the magnetic field, then with this polarization, we peak out this spherical

tensor component, which is z, which is r times Y1,0. And that means that this polarization of the light induces a transition for which delta m equals 0. And this is referred to as pi light.

So maybe if that got confusing for you, let me just help out with a drawing. We have our atom here, which is quantized by a magnetic field B. And if you shine light on it, we have the electric field perpendicular to the magnetic field. So this would be x and y. And the natural way to describe it is by using x plus/minus IY. And we have selection rules where delta m is plus/minus 1.

But alternatively, we can also shine light along this direction. And for the electric field, which was perpendicular to B, we retrieve the previous case. We have superpositions of the sigma plus and sigma minus. But the new case now is that the electric field is parallel to B. And then, we drive transitions, which have delta M equals 0. So these are sigma plus and sigma minus transitions. And this here is what is called a pi transition.

Anyway, it's a little bit formal, but I just wanted to present it in this context. Questions?

STUDENT: I have one slightly, maybe, basic question. When we talk about polarization in all these matrix elements-- so for example, photon [INAUDIBLE], right-- these are single photons [INAUDIBLE] elements. And so when we talk about shining a laser, it has a polarization. But we don't talk about polarization for single photons. Or do we?

PROFESSOR: Actually, we talk about-- the question is, what is the polarization? Do we talk about polarization of single photons, or polarization of laser beams? Well, let me back up and say, we talk about polarization of a mode of the electromagnetic field. We will always expand the electromagnetic field into modes. And the mode is the polarization.

It may happen that at some point, a photon is emitting a superposition of modes. But in the most straightforward description, we always do a mode analysis. And often, we simplify the case by saying that the atom interacts only with one mode of

the electromagnetic field. And maybe in the case of spontaneous emission, we then sum over all modes.

But for each mode, there's a specific polarization. And it doesn't matter if this mode is filled with one atom, or with a laser beam, with a classical electromagnetic field, which corresponds to zillions of photons.

STUDENT: [INAUDIBLE] does it always end up being electrical polarization in this case, then? Like because if it's many photons, then there's a lot of [INAUDIBLE] for each of them, or each of them individually-- I don't know.

PROFESSOR: No, it depends. If you have an atom, and it has one unit of angular momentum, and it spontaneously emits a photon, if the photon is emitted along the quantization axis, it can only be sigma plus. If it's emitted in the other direction, it has to be sigma minus.

Now if you go at strange angles, then at this angle, you overlay it with different modes. And you may now find photons in a superposition of polarizations, because we have several modes which are connected with this direction of emission.

I think if you write it down, it's pretty clear. It's just sort of projection operators. And for spontaneous emission, we sum over all modes. But for me, I always think about-- we can always think about what a single photon does by saying, well, if I'm getting confused about a single photon, let me figure out what many, many, many identical photons would be. And that would mean, instead of a single photon in a certain mode, I release a beam in this mode.

And then, suddenly, I can think, classically, I know what the electric field is such. And then you go back to the, what is the electric field of a single photon, and usually make the connection. So I think at least for the discussion of matrix elements, transitions, angular momentum, I don't think you ever have to distinguish between what single photons do and what laser beams do.

But there are important aspects of single photons, non-classical aspects, which we'll discuss in a short while. Other questions? OK. That's all I want to say about

selection rules. So with that now, we can simply take the matrix element and run with it.

So in this lecture and on [INAUDIBLE], I want to talk about basic aspects of atom-light interaction. And what I want to talk today about it is the two important cases when an atom interacts with monochromatic wave, or when it interacts with a broad spectrum.

In one case, when I say monochromatic case, you may just think of the best laser money can buy. Very, very sharp. Very, very monochromatic. When I talk about a broad spectrum, you may just think about black-body radiation, which is an ultra broad spectrum.

And they're two very different cases. And some of it is just related to Nancy's question, that if you have a broad spectrum, we're always talking about many, many modes, and they will be incoherent, and they will be irreversible physics. Whereas for monochromatic light, everything is a pure, plain wave, and everything is coherent.

So we want to sort of talk about that first. And then later this week, I think on Wednesday, we will talk about spontaneous emission. But right now, we focus on the simpler case, where we drive the system with electromagnetic radiation, which is either narrow-band or broadband.

But let's just start with a cartoon. We have an atom. And for that discussion, all we need is two levels. And all we need is that the two levels are connected by some matrix element. And the basic phenomenological situation is that we have one atom, which sits in a vacuum.

So we have volume, V, of vacuum. And what is important now is that the walls of the imaginary boundary of what defines our vacuum is at low temperature. And low temperature means that the atom will irreversibly decay into the ground state with a lifetime tau.

And that means that in some picture, the excited state is the broadening, which is

broadened by the natural lifetime. And in our discussion, we assume-- and this is what I said with the cold walls of the vacuum-- that the energy difference is much, much larger than the relevant temperature.

And this is very well fulfilled for our standard atomic system. The typical excitation energy, even for atoms with loosely bound electrons, as the alkalis, is two electron volt, which corresponds to a temperature of 20,000 Kelvin. And even at the rather hot temperature, definitely hot temperatures, in The Center for Ultracold Atoms, but the KT at room temperature corresponds to 25 milli-electron volt.

So therefore, when we have an atom in isolation, this is what we find. We find an atom which will irreversibly decay to the ground state. And the fact that it irreversibly decays to the ground state is really an inequality between energies. If you will talk about a hyperfine transition or something, there may be a possibility that we have an excited state, which is thermally excited.

But in the following discussion, when we drive the atom, and when we look at spontaneous decay, we always assume that the thermal energies are so small, that we really assume an atom sitting in a cold vacuum. Actually, it's your next homework assignment, where you will consider, what are the effects of black-body radiation. And you will actually find out in your homework that they are non-negligible. So yes, there are corrections. But you will also find out that the corrections are rather small, or it takes a long time before black-body radiation induces any observable transition.

OK. So I'll just try to be a little bit formal here. Give you sort of a sketch of an atom in a cold vacuum. Ground state is stable. Excited state, irreversibly decays. And now, we want to bring life into this situation. Now we add light. And the light-- and this is now our discussion-- has a [INAUDIBLE]. And we want to distinguish the cases of narrow-band and broadband radiation.

So it's clear that if the bandwidth of the light, the only scale-- well, we have the scale of omega. But that's a huge scale. The only smaller scale, which is given by the atom, is the natural linewidth. And depending, in which case we are, we talk about

narrow-band excitation and broadband radiation.

And once the linewidth is much narrower than gamma, we don't get any new physics when we assume perfectly monochromatic light. So once we are much smaller, we're really discussing the case of, well, we can neglect the spectrum broadening of the light source. Or in the other case, when we have broadband light, we can pretty much make the assumption that the light is infinitely broad, and what matters is only the spectral density of the light.

So in a pictorial representation, if this is the frequency omega, we have the atom with the natural linewidth gamma. Narrow-band means we are much sharper than that. And broadband means really wide distribution.

So if we have broadband light, it doesn't really matter what the total power is. If the light is very broad, there can be infinite power in the wings, but the atoms don't care. What matters when we have broadband radiation is the quantity called the spectral density. And that's what we need in the following. Which is, let me just give you the units. Which is energy per volume and frequency interval.

So we can talk about the spectral density as of omega. Or alternatively, when we have a propagating beam, we don't want to talk about energy, we want to talk about intensity. So it is intensity per unit frequency interval. Which would mean I of omega is the energy density, multiplied with the speed of light. And that becomes energy per area and time. So that's the flow of energy. But because we are talking about board light, it has to be normalized by the frequency interval.

In contrast, monochromatic radiation, it's sort of one monochromatic electric field. And we will specify it by the single frequency, omega, and the electric field amplitude. Which when multiplied by a matrix element becomes the Rabi frequency. Or we can characterize the light by the intensity I. But then it's an intensity which has the units of energy per area time. It's not normalized to any frequency interval, because we have assumed that the frequency interval is 0.

So if you now have a description how these two forms of light interact with the atom,

at this point, and we come to that later this week, we have to make an assumption that we are looking at times which are much smaller than the time for spontaneous emission. So if you now, in a perturbative sense, expose the atom's monochromatic or broadband radiation, unless we have included in the description the many, many modes for spontaneous emission, we are limiting ourselves to a very short time.

This is, you would say, a severe description, because atoms emit photons after a short time. But we already capture, without considering spontaneous emission, a lot of different physics. And we can nicely distinguish between features of monochromatic and features of broadband excitation.

OK. So let's start out with the case of-- give me a second. OK. So if you look at the two cases, in the monochromatic case, we will discuss the idealized situation of an atom interacting only with a single mode. And what we will find out is, we will find out that now, in the optical domain, we will find actually equations for the two-level system which are identical to what we discussed earlier when we discussed spin [INAUDIBLE] in a magnetic field.

So in that sense, a two level system, driven by a laser system, will behave identically to a spin driven by a magnetic field. Shouldn't come as a surprise, but I will show that to you. But I can go over that very quickly. The board-band case will actually follow from the single mode case, because what we assume is broadband means many, many modes. And then we do an averaging over many single modes by assuming random freeze.

But I also want to show it to you because I picked my verbs carefully. You have many, many more things, but we assume that there is a random phase. When we talked about one photon emitted into a angle-- it maybe responds to a question earlier-- this photon may be in a coherent superposition. This is not many modes in a broadband wave. Many modes in broadband wave means that there is no correlation whatsoever between the modes, and all we will be able to talk about is an IMS value of an electric field.

But anyway, the result is sort of predictable, and I wanted to tell you what I'm aiming

for. But it's now really worthwhile to go through those exercises and look at what happens in perturbation theory for short times when we have monochromatic radiation, and when we have broadband radiation.

So the first discussion will show Rabi flopping. I don't know how many times we have looked at Rabi oscillation. But these are now Rabi oscillations between two electronic states covered by a laser beam. And I want to show you how this comes about. And when I said strong driving, well, we have only a limited time window before spontaneous emission happens. We have to discuss the physics we want to discuss in this shot time window.

And if you want to excite an atom, and see Rabi oscillation in a short time, you better have a strong laser beam. So this is why the monochromatic excitation that we discussed will pretty much automatically be in this strong coupling limit.

OK. So what do we have? We have a ground,and we have an excited state. We have a matrix element. We know now where it comes from. And we have a monochromatic time dependence.

In perturbation theory, we build up time-dependent [INAUDIBLE] amplitude in the excited state, because we couple the ground state with the off-diagonal matrix element to the excited state. And we have to integrate from the initial time to the final time. We have the time dependence of the electromagnetic field, and we also need the time dependence of the excited state.

So when I integrate now over t prime, I take out the ground state amplitude, because we're doing perturbation theory, and we assume that for short times, needing order, the ground state amplitude is one, as prepared initially. So this in integral can be solved analytically.

Some of you may remember that the minus 1 has something to do with the lower bound of the integral. And when we discuss the easy polarizability, we said, this is a transient, and we neglected it for good reasons. But now, we're really interested in the time evolution of the system, so now we have to keep it.

OK. We are interested in the probability in the excited state. So we take the above expression and square it. And we find the well-known result, with sine squared, divided by omega minus omega eg.

OK. So this is pretty much just straightforward, writing down an analytic expression. But now, let's discuss it. For very short times, and this is an important limiting case, the probability in the excited state is proportionate to times squared. And this is important. We're not getting a rate which is proportional to time. We're obtaining something which is time square. And the proportionality to t square means it's a fully coherent process.

So whenever somebody asks you, you switch on a strong coupling from a ground to the excited state, what is the probability in the excited state? It starts out quadratically. The linear dependence-- famous golden rule, [INAUDIBLE] or such-- only come later. This is a very universal feature. And even if you use broadened light, for a time window, delta T, which is shorter than the inverse bandwidth of the light, talking about Fourier's theory, you don't have time to even figure out that your light is broad and not monochromatic.

For very short times, the Fourier limit does not allow you to distinguish whether the light is broad or monochromatic. So what I just derived for you, an initial quadratic dependence, is the universal behavior of a quantum system at very short times. Because it simply says the amplitude in the excited state goes linearly in time, and the probability, quadratic.

OK. So this is for a very short times. But if you look at it now for longer times, we have actually-- we'll see the atomic behavior, and these are Rabi oscillations. But there is one caveat.

So we have derived. However, we have derived them only perturbatively by assuming that the ground state has always a population close to 100%, which means we have assumed that the probability in the excited state is much smaller than 1. Otherwise, we wouldn't keep the ground state.

And this is only fulfilled if you inspect the solution. The solution is only self-consistent if you have an off-resonant case, where the Rabi oscillation only comes from a small fraction of the ground state population of the excited state. Of course, you all know that Rabi oscillations, this formula, is also varied on-resonance. And you can have full Rabi flopping.

But I want to make a case here, distinguish carefully between monochromatic radiation and broadband radiation. For that, I need for perturbation theory. And therefore, I'm telling you what perturbation theory gives us at short times, and in terms of Rabi oscillations.

STUDENT: So you're saying we assume strong coupling with respect to the atomic linewidth, but weak coupling with respect to the resonance, for instance, in [INAUDIBLE]?

PROFESSOR: It's simple, but subtle. Yes. So what we have is, we assume we switch on a monochromatic laser. Since we do not include spontaneous emission, which will actually damp out Rabi oscillation-- we'll talk about that later-- we are only limited, we are limited here to short times, which are shorter than the spontaneous decay.

And now, I gave you one universal thing. At very, very short times, it's always quadratic. It's a coherent process. So that's one simple, limiting, exact case you should keep in your mind. But now the question is, if you let the time go longer, something will happen. And there are several options. One is, if times go longer, spontaneous emission happens. OK. We are invalid.

The other possibility is, when time gets longer, and we are on-resonance, we deplete the ground state, or [INAUDIBLE] perturbation theory doesn't deal with that. But if we are off-resonance, we can allow time to go over many Rabi periods and observe perturbative Rabi oscillations.

So this is how we have formulated it. We do perturbation theory of the system without spontaneous emission. And eventually, we violate our assumptions, either because spontaneous emission kicks in, or because we deplete the ground state when we drive it too hard, or if we go too close to resonance. But the later

assumption, of course, that we can't drive it hard, as you know, is artificial. We can actually discuss the monochromatic case. Not just in perturbation theory, but we can do it exactly.

STUDENT: I want to go back again to--

PROFESSOR: And this is what I want to do now. But first, we can go back.

STUDENT: So when we are talking about non-B resonance and B-resonance, so if we decrease the detuning, then we are getting close to resonance. So again, this gets invalid.

But if we increase the detuning, we could exceed the spontaneous emission rate. So then, we won't see any Rabi oscillations again, because, at those time periods, this oscillation would [INAUDIBLE] detuning. So to observe Rabi oscillations, we have to be at times more than the detuning, or more than [INAUDIBLE] detuning.

PROFESSOR: Oh, yeah. Of course.

STUDENT: So the detuning has to be less than [INAUDIBLE], but more [INAUDIBLE] that we are still [INAUDIBLE] resonant.

PROFESSOR: No. The detuning has to be larger than the natural linewidth, because then the Rabi oscillations are fast, and we have Rabi oscillations which are faster than any damping due to spontaneous decay. That's an image we are talking about. So in the limit of our detuning, you can detune very, very far, and you never reach the limit of our perturbative abode.

STUDENT: Yes. OK.

PROFESSOR: Anyway, I want to do perturbation theory of the broadband case. And the broadband case will be an incoherent sum over the single mode case. So this is why I had to bore you with, what do we get out of perturbation theory for the monochromatic case?

Of course, you know already that in a two-level system, we can do it exactly. And I just want to outline it, mainly to introduce some notation. So our Hamiltonian here,

which couples the ground in the excited state is given by the dipole matrix element, the electric field vector, and we call this the Rabi frequency.

And then we have a sinusoidal or co-sinusoidal frequency dependence. And all I want to do is to show you that a two-level system driven by an electromagnetic field is identical to spin 1/2, which we discussed earlier, and then we are done.

There is one technical or little trick we have to do, which is trivial, but I want to mention it. So if you want to compare directly with spin 1/2, we are now shifting the ground state to half the excitation frequency. In other words, just to make the key analogy with the spin, usually we say for an electronic transition, we start at 0, and we go up. But now we shift things that the zero of energy is in the middle between the ground and the excited state. And then, it looks like the excited state, we spin up, the ground state, we spin down.

So with that, our Hamiltonian is now excited, excited, minus-- so all I've done is I've shifted the origin. And the coupling, using our definition of the Rabi frequency is couples ground and excited state. And excited ground state. These are the two off-diagonal matrix element. And the time dependence is cosine omega t.

So we are now very close to exploit the correspondence with spin 1/2. Because after shifting the ground state energy, this is the z component of the spin operator, the [INAUDIBLE] matrix. And this here is the x component. So therefore, for driving an electronic transition with a laser beam, we have actually spin Hamiltonian, which has the standard form.

So let me just write it down, because it's an important result. The Hamiltonian for driving and dipole transition with a linearly polarized laser beam corresponds, or is identical, to the Hamiltonian for spin 1/2, in a static magnetic field along the Z direction, which causes a splitting between spin up and spin down. And the splitting is now omega eg plus a linearly polarized oscillating field along the x direction.

And you probably remember that when we discussed the spin problem, what we liked actually most was that we had a rotating magnetic field, because it made

everything simpler. And we are doing that now by formally writing the [INAUDIBLE] polarized field as a superposition of left-handed and right-handed, or counter-rotating and co-rotating magnetic field.

So let me just do that. So we have the Z part. And now, instead of having just sigma x, cosine omega t, I add sigma y, sine omega t, and I subtract sigma y, sine omega t.

So now we have shown that there is something in addition to the spin problem. We discover when we had a rotating magnetic field, that we have two components here which rotate. And these are the co and counter rotating magnetic fields in the spin problem.

And the counter rotating, you remember in the spin problem, we solved the problem exactly by going into a frame which rotated at the Larmor frequency, which becomes now omega eg. And the co-rotating term became stationary on resonance in this rotating frame, whereas the counter-rotating term rotates at a very high frequency in this frame at the Larmor frequency.

So if this frequency, if you fulfill the inequalities that the co-rotating term is close to resonance, or in other words, we are close to resonance, and we are not using an infinite intensity of the laser beam, that we broaden everything in co and counter rotating terms of boson resonance. So if you fulfill those two conditions, then we can neglect the last term. And this is the rotating wave approximation.

So in other words, in the spin problem, we can always assume we haven't circularly polarized the rotating magnetic field, and we have an exact solution. I say a little bit more about it later. But in many situations, when you excite an atom with a laser beam, you get both terms. And usually, you proceed by neglecting one term, and by making the rotating wave approximation.

will, in one or two lectures, discuss whether there are situations where the counter-rotating term is exactly 0 due to angular momentum selection rules, but that's a separate discussion. In many situations, it cannot be avoided, and it's always there.

It's actually always there to the point that when I talk to some colleagues and say, I can create a situation, an atom, where the counter-rotating term is exactly 0, some colleagues reacted with disbelief, and then eventually felt that the situation I created for angular momentum conservation was somewhat artificial. But we'll get there. It's an interesting discussion.

But anyway, just remember that for magnetic drive, if you use a rotating magnetic field, you don't need a rotating wave approximation. Everything rotates at one frequency. But usually, when you drive a two-level system with lasers, we usually have an extra term which needs to be neglected.

OK. But if you do the rotating wave vapor approximation, we have now exactly the situation we discussed for spin 1/2 in a rotating magnetic field. And then, the same equation has the same results. And then, our results for spin 1/2 are now as expected. Rabi oscillations without making any assumptions about perturbation theory. So this is an exact result for the initial conditions that we start in the ground state, and the initial population of the excited state is 0. And as usual, I have used here the generalized Rabi frequency, which is the quadrature sum of these matrix elements squared and the detuning.

OK. A lot of it was to get ready for the broadband case. So that's-- yes, we have a little bit more than five minutes. So, so far, we have discussed the monochromatic case. What I really needed as a new result, because I carried over for the board-band case, was a perturbative result. But I also wanted to show you that the perturbative result is one limiting case of the exact solution, which I just derived by analogy to spin 1/2.

OK. So we just had the result that in perturbation theory, for sufficiently short times, we discussed all that, that the excited state amplitude has the following dependence. So this is nothing else than-- I want to make sure you recognize it-- Rabi oscillations at the generalized Rabi frequency.

The generalized Rabi frequency is simply the detuning, because it's a perturbative result. In perturbation theory, you don't get power broadening, because you assume

that your drive field is perturbatively weak. So therefore, the Rabi oscillation, our now Rabi oscillation where the Rabi frequency, the generalized Rabi frequency, is delta the detuning.

And this is just rewriting. Let me just scroll up. This is this result here. I wasn't commenting on it. But this is nothing else than the detuning. Look. I'm just reminding you what you get from perturbation theory. Power broadening is not part of perturbation theory.

OK. So this is our perturbative result. And now, we want to integrate over that because we have a broadband distribution of the light. So what we have to use now is the energy density, W of omega. The electric field is related, the energy of the electromagnetic field, is 1/2 epsilon the energy density of the electromagnetic field, is 1/2 epsilon naught times the electric field squared.

Well, if you have many modes, we add the different modes in quadrature. And we still have the same reaction between the electric field squared and the total energy. But the total energy is now an integral over d omega. We integrate over frequency over the spectral distribution of the light.

So this is how we go from energy density to electric feels. But now, we want to evaluate this expression. And what appears in this expression is the Rabi frequency. Well, what we have to do now is we have to go back from the Rabi frequency. We assume linearly polarized light in the x direction to the electric field.

And that means, now, that when we-- OK. We want to now take this expression, and sum it up over all modes, which means we integrate over, we write the Rabi frequency squared as an electric field squared. And the electric field squared is obtained as an integral over the spectral distribution of the light.

So this means we will replace the Rabi frequency in this formula by an integral over the energy density of the radiation. We have the matrix element squared as a prefactor. I just try to re-derive it, but I think the prefactor is 2 over epsilon naught.

So, yes. With that, in perturbation theory, the probability to be in the excited state is-

- let's just take all of the prefactors. Now, I change the integration variable from omega to detuning, we just go from resonance-- we integrate relative to the resonance. So our energy density is now at the resonance, omega naught plus the detuning. And we have this Rabi oscillation term.

OK. So this is nothing else than taking our perturbative Rabi oscillation formula, which is coherent physics, and indicate over many moles. . I'm one step away from the final result. If the energy density is flat, is broadband-- so for the extreme broadband case, we can pull that out of the integral. And then, we are left only with this function, F of t.

And you can discuss this function, F of t, is a standard result. And we have seen many discussions in perturbation theory. If I plot this function, versus delta, we have something which has wiggles. Then, there is a maximum, and it has wiggles. The width here is t to the minus 1. And the amplitude is t squared. And this is the excited state amplitude squared.

So if we integrate that over delta, we get something which is linear in t. Something which goes as t-square, and has a width 1/t. Yes, time is over. So the function F of t, which is under the integrand, starts out at short times, proportion to t squared, as we discussed. Maybe my drawing should reflect that.

But then it becomes linear. So for long times, the function F of t becomes linear in t and the delta function in the detuning delta. This is what you have seen many times in the derivation of Fermi's golden rule.

I'm running out of time now. I'll pick up the ball on Wednesday, and we'll discuss that result and put it into context. But the take-home message-- and what I really wanted to show you is that we do have coherent Rabi oscillations. And by just performing the integral over this broad spectrum of the light, we lose the Rabi oscillations, and we find rate equations, Fermi's golden rule, and excitation probability proportional to t.

And we have done the transition from coherent physics to irreversible physics. This

is all hidden in this one formula, but I want to fully explain it when we start on Wednesday. Any last second question about that? Cody?

STUDENT: It looks like we're integrating right over the point to where perturbation theory becomes an exact, because we're integrating over delta equals 0. And that's the most important part.

PROFESSOR: We are integrating over it, but we are integrating over it with the [INAUDIBLE]. So therefore, since we have-- perturbation theory remains valid, actually. Perturbation theory remains valid, as long as the excitation probability is less than one. So I have not put a scale on it, but we can go from a quadratic dependence to linear dependence. As long as the probability of being in the excited state is smaller than 1, perturbation theory is exactly valid.

So I think what confuses you here is, we can do resonant excitation. The broadband includes resonant excitation. But for sufficiently short times, we reach the rate equation before we run out of [INAUDIBLE] perturbation theory.