Recitation 3 (Oct. 6, 2017)

3.1 Neutrino Oscillations

This discussion follows Sakurai.

As another example of the dynamics of a two-state quantum mechanical system, we will now discuss neutrino oscillation. Neutrinos are fundamental particles that have no electric charge and very small mass. They interact only through the weak and gravitational forces. In nature, we have observed three flavors of neutrinos, denoted ν_e, ν_μ, and ν_τ, but for simplicity, we will only consider ν_e and ν_μ.

The states $|\nu_e\rangle$ and $|\nu_\mu\rangle$ are referred to as *flavor eigenstates*, because they are eigenstates of the weak force Hamiltonian. In other words, the action of the weak force is diagonal on the flavor eigenstates: the electron neutrino ν_e couples only to the electron via the weak force, and the muon neutrino ν_μ couples only to the muon via the weak force.

However, these flavor eigenstates are not eigenstates of the Hamiltonian H_0 describing a free neutrino. When a neutrino propagates freely (without undergoing interactions), its time evolution is dictated by the free Hamiltonian. We denote the eigenstates of H_0, known as *mass eigenstates*, by $|\nu_1\rangle$ and $|\nu_2\rangle$, with corresponding energy eigenvalues E_1 and E_2, respectively. These are states of well-defined mass.

We can decompose the flavor eigenstates in the mass eigenstate basis as

$$\begin{aligned} |\nu_e\rangle &= \cos\theta|\nu_1\rangle - \sin\theta|\nu_2\rangle\,, \\ |\nu_\mu\rangle &= \sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle\,, \end{aligned} \tag{3.1}$$

for some mixing angle θ that is determined experimentally. For this problem, let us assume that $|\nu_e\rangle$, $|\nu_\mu\rangle$, $|\nu_1\rangle$, and $|\nu_2\rangle$ are all momentum eigenstates of fixed momentum p. Because neutrinos propagate in the mass eigenstate basis, the flavor eigenstates oscillate into one another. That is, time evolution is dictated by the free Hamiltonian H_0, so if an electron neutrino propagates for a time t, its two mass eigenstate components will evolve with different frequencies and thus pick up a relative phase.

We now wish to calculate the probability that an electron neutrino propagates for a time t and is found to still be an electron neutrino. We compute

$$\begin{aligned} \langle\nu_e|U(t,0)|\nu_e\rangle &= (\cos\theta\langle\nu_1| - \cos\theta\langle\nu_2|)e^{-iH_0t/\hbar}(\cos\theta|\nu_1\rangle - \cos\theta|\nu_2\rangle) \\ &= (\cos\theta\langle\nu_1| - \cos\theta\langle\nu_2|)\Big(\cos\theta e^{-iE_1t/\hbar}|\nu_1\rangle - \cos\theta e^{-iE_2t/\hbar}|\nu_2\rangle\Big) \\ &= \cos^2\theta e^{-iE_1t/\hbar} + \sin^2\theta e^{-iE_2t/\hbar}\,. \end{aligned} \tag{3.2}$$

Thus, we have

$$\begin{aligned} P(\nu_e\to\nu_e) &= |\langle\nu_e|U(t,0)|\nu_e\rangle|^2 \\ &= \sin^4\theta + \cos^4\theta + \sin^2\theta\cos^2\theta\Big[e^{i(E_1-E_2)t/\hbar} + e^{-i(E_1-E_2)t/\hbar}\Big] \\ &= \sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta\cos(\Delta Et/\hbar) \\ &= 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta Et}{2\hbar}\right), \end{aligned} \tag{3.3}$$

where $\Delta E = E_1 - E_2$. Because neutrinos have very small mass, they are highly relativistic in most conditions. Thus, to a good approximation, the energy eigenvalue for a neutrino in a momentum eigenstate is

$$E_i = \sqrt{p^2c^2 + m_i^2c^4} \approx pc\left(1 + \frac{m_i^2c^2}{2p^2}\right), \tag{3.4}$$

where m_i is the mass of the mass eigenstate $|\nu_i\rangle$. This gives

$$\Delta E = E_1 - E_2 \approx \frac{\Delta m^2 c^3}{2p}, \tag{3.5}$$

where $\Delta m^2 = m_1^2 - m_2^2$. Therefore,

$$P(\nu_e \to \nu_e) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2 c^3 t}{4p\hbar}\right) = 1 - \sin^2(2\theta)\sin^2\left(\Delta m^2 c^4 \frac{L}{4E\hbar c}\right), \tag{3.6}$$

where $L = ct$ is the distance travelled by the neutrino and $E = pc$ is the neutrino energy, in the relativistic limit. By conservation of probability, this requires that

$$P(\nu_e \to \nu_\mu) = \sin^2(2\theta)\sin^2\left(\Delta m^2 c^4 \frac{L}{4E\hbar c}\right), \tag{3.7}$$

which we can verify by direct computation.

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