8.321 Quantum Theory-I Fall 2017

Final Exam

Dec 19, 2017

1. **(25 points)**

A system of 3 qubits is prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \tag{1}$$

This is known as the Greenberger-Horne-Zeilinger (GHZ) state and has some interesting non-classical properties.

(a) Calculate the reduced density matrix of the subsystem made up of the first and second qubit by tracing over the state of the third qubit. (Note: There are four orthogonal basis states in the Hilbert space of the qubits 1 and 2. Your answer should be in the form of an operator on this 2 qubit Hilbert space).

Does this density matrix describe a pure state of the two qubits?

(b) The density matrix you calculated in the previous part has an eigenbasis in this Hilbert space. Describe the general eigenstates that have a non-zero eigenvalue. These are pure states of the two qubit subsystem. Are the two qubits necessarily entangled with each other in these states?

(c) Suppose we measure qubit 3 in the Z basis and find 0. What is the state of the qubits 1 and 2 after the measurement? Are these two qubits entangled in this state?

2. **(25 points)**

Consider two spin-1 magnetic moments coupled together with the Hamiltonian

$$H = J\vec{S}_1 \cdot \vec{S}_2 \tag{2}$$

with $J > 0$.

(a) Show that $\vec{S}_{tot} = \vec{S}_1 + \vec{S}_2$ commutes with the Hamiltonian. Explain why this should have been expected.

(b) The result of the previous part implies that the Hamiltonian can be diagonalized simultaneously with S_{tot}^2, S_{tot}^z. Use this to determine the spectrum and degeneracies of H. Do not attempt to determine the eigenstates. (Hint: Relate H to S_{tot}^2, S_1^2, S_2^2). What are S_{tot}^2, S_{tot}^z for the ground state?

(c) Write the ground state wave function in the $|m_1 m_2\rangle$ basis ($i.e$ the basis of eigenstates of S_1^z, S_2^z) as

$$|\psi\rangle = \sum_{m_1, m_2} a_{m_1, m_2} |m_1, m_2\rangle \tag{3}$$

Use the known action of S_{tot}^z on the ground state to constrain the coefficients a_{m_1, m_2}.

(d) Next use the known action of S_{tot}^+ on the ground state to completely determine the ground state wave function. Make sure that the wave function is normalized.

3. **(25 points)**

If the photon is hypothesized to have a small mass m_γ then the usual Coulomb interaction potential will be modified. In the hydrogen atom the interaction of the electron with the proton will have the form

$$V(r) = -\frac{e^2}{r} e^{-\frac{r m_\gamma}{\hbar c}} \tag{4}$$

(c is the velocity of light). To leading non-vanishing order in m_γ find the change in the energy eigenvalues for the hydrogen atom.

4. **(25 points)**

A spin-1/2 particle is confined to move in two dimensions and has the dispersion

$$H_0 = \frac{(\vec{p})^2}{2m} - \lambda \vec{\sigma} \cdot \vec{p} \tag{5}$$

Here $\vec{p} = (p_x, p_y, 0)$ is the two dimensional momentum and $\vec{\sigma}$ are the usual Pauli matrices.

(a) Find the energy eigenfunctions in the basis $|\vec{p}, \pm\rangle$ and their eigenvalues. Here $\pm$ refer to spin states with $\sigma^z = \pm 1$.

(b) Is H_0 invariant under (a) space translations? (b) two dimensional rotations? If so what are the Hermitian generators of these symmetries?

(c) The particle is subjected to a harmonic potential along the x direction (with no potential in the y direction).

$$V(x) = \frac{1}{2} m\omega^2 x^2 \tag{6}$$

Which of the symmetries you found in the previous part survive in this potential?

(d) At $\lambda = 0$ obtain the eigenvalues of the full Hamiltonian $H_0 + V$.

(e) Consider the ground state when $\lambda = 0$. Calculate the shift of the energy of this state when $\lambda \neq 0$ but is small using perturbation theory upto second order.

Hint: Notice that p_y is a constant of motion and so can be fixed. The motion in the x-direction is described by a perturbed $1d$ harmonic oscillator. You will find it useful to use the representation of momentum in terms of the ladder operators of the oscillator.