8.321 Quantum Theory-I Fall 2017

Prob Set 2

1. **Sakurai Prob 1.14**

2. **Adapted from Sakurai Prob 1.17** Two observables A_1 and A_2, which do not involve time explicitly, are known not to commute,

$$[A_1, A_2] \neq 0 \tag{2}$$

yet we also know that A_1 and A_2 both commute with a third Hermitian operator H (the 'Hamiltonian'):

$$[A_1, H] = 0, \quad [A_2, H] = 0. \tag{3}$$

Prove that the eigenstates of H are, in general, degenerate.

3. (a) Show that the non-Hermitian matrix $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only real eigenvalues, but its eigenvectors do not form a complete set.

   (b) Being non-Hermitian, this matrix must violate the real-valuedness condition that we have found for the expectation values of physical observes. Find a vector $|v\rangle$ such that $\langle v|M|v\rangle$ is complex. (This example illustrates the need to represent real observables by Hermitian operators, and not merely by operators that have real eigenvalues. Since $\langle v|M|v\rangle$ can be complex, it clearly cannot be interpreted as an average of the eigenvalues of M.)

4. Consider the following two matrices

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix} \tag{4}$$

   (a) Show that A and B commute.
   (b) Find the eigenvalues of A and B.
   (c) Find the unitary transformation which simultaneously diagonalizes A and B.

5. (a) For a spin-1/2 atom show that the eigenvalues of the spin operator along a general axis $\hat{n}$

$$S_{\hat{n}} = \vec{S} \cdot \hat{n} \tag{5}$$

   are $\pm\frac{\hbar}{2}$. Find the corresponding eigenvectors. You may take the unit vector $\hat{n}$ to be

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{6}$$

   (b) Represent the two eigenstates as points on the Bloch sphere.
   (c) If the atom is prepared to initially be in the eigenstate of $S_{\hat{n}}$ with eigenvalue $+\frac{\hbar}{2}$, what is the probablity that a measurement of S_z will return the value $+\frac{\hbar}{2}$? If instead of S_z, S_x is measured what is the probability of obtaining $+\frac{\hbar}{2}$?

6. (a) Show that it is impossible for an electron to be in a state such that $\langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0$
   (b) Can a spin-1/2 particle be in a state with $(\langle S_x \rangle)^2 + (\langle S_y \rangle)^2 + (\langle S_z \rangle)^2 < \left(\frac{\hbar}{2}\right)^2$?