Physics 8.321, Fall 2002
Homework #3

Due **Wednesday, October 2** by 4:30 PM in the 8.321 homework box in 4-339B.

Note: for some parts of this problem set, particularly in problem 2, you may wish to use a computer. You can use any software or programming language you like (i.e., Mathematica, Matlab, Maple, C++ etc.), but please include your code and output with your problem set solution.

1. Consider the following two matrices:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix},$$

(a) Show that A and B commute.

(b) Find the eigenvalues and eigenvectors of A and B.

(c) Find the unitarity transformation which simultaneously diagonalizes A and B.

2. N spin-1/2 particles have a total Hilbert space

$$\mathcal{H} = \mathcal{H}_2^{(1)} \otimes \mathcal{H}_2^{(2)} \otimes \cdots \otimes \mathcal{H}_2^{(N)}$$

where $\mathcal{H}_2^{(i)}$ is the (two-dimensional) Hilbert space of the ith particle.

(a) What is the dimension of $\mathcal{H}$?

(b) Define

$$S_z = S_z^{(1)} + S_z^{(2)} + \cdots S_z^{(N)}.$$

What is the spectrum and degeneracy of S_z?

(c) Define an operator I coupling N spins to their nearest neighbors in a ring through

$$I = \mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} + \mathbf{S}^{(2)} \cdot \mathbf{S}^{(3)} + \mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)} + \cdots + \mathbf{S}^{(N-1)} \cdot \mathbf{S}^{(N)} + \mathbf{S}^{(N)} \cdot \mathbf{S}^{(1)}$$

Are S_z and I compatible observables? Prove your answer for any N.

(d) Find the spectrum and degeneracies of I for $N = 2, 3, 4$.

(e) Find the largest positive eigenvalue $\lambda_{\text{max}}^{(N)}$ of I for an arbitrary value of N, and identify an eigenvector with this eigenvalue.

(f) Find the smallest (most negative) eigenvalue $\lambda_{\text{min}}^{(N)}$ for small values of N. What is the largest N for which you can compute this quantity? What can you say about $\lambda_{\text{min}}^{(N)}$ and its associated eigenvector(s) for general N?

(g) Consider N spin-1/2 particles in an external magnetic field, interacting with the external field and one another according to the Hamiltonian

$$H = b\,x S_z - a(1-x)I$$

where a, b are numerical constants with $b = a\hbar$, and $x \in [0,1]$ is a real number. Graph the spectrum of H for $N = 2, 3, 4$ for x in the range $0 \le x \le 1$. Check that your results agree with your answers to the previous parts.

3. Consider 4 spin-1/2 particles, each of which is in an eigenstate $S_x^{(i)} = \hbar/2$. In each part of this problem, a sequence of measurements is performed on these 4 particles. For each part of the problem, give all possible sequences of outcomes of the experiments, and calculate the probability for each sequence of outcomes. In each case, calculate the total probability that the final measurement of the quantity $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ gives each of the possible values. Each part of this problem should be done independently, starting with all spins in the eigenstate $S_x^{(i)} = \hbar/2$ as mentioned above.

(a) $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

(b) $S_z^{(3)}$ is measured, and then $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

(c) $S_z^{(2)}$ is measured, then $\mathbf{S}^{(2)} \cdot \mathbf{S}^{(3)}$ is measured, and then $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

(d) $S_z^{(1)}$ is measured, then $\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$ is measured, then $\mathbf{S}^{(2)} \cdot \mathbf{S}^{(3)}$ is measured, and then $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

(e) $S_z^{(1)}$ is measured, then $\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$ is measured, and then $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.