Lecture 12 - Topics

- The σ- parameterization
- Equations of motion and Virasoro constraints
- General motion for open strings
- Rotating open string

Reading: Chapter 7

So far:

$$X^0(\tau,\sigma) = ct = c\tau$$

$$v_\perp^2 = \frac{\partial \vec{X}}{\partial t} - \left(\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s}\right)\left(\frac{\partial \vec{x}}{\partial s}\right)$$

$$\sqrt{(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2} = c\frac{ds}{d\sigma}\sqrt{1-\frac{v_\perp^2}{c^2}}$$

$$\dot{x}\cdot x' = \frac{\partial \vec{x}}{\partial t}\cdot\frac{\partial \vec{x}}{\partial s}$$

$$(x')^2 = \left(\frac{\partial \vec{x}}{\partial \sigma}\right)^2 = \left(\frac{ds}{d\sigma}\right)^2$$

$$(\dot{x})^2 = -c^2 + \left(\frac{\partial \vec{x}}{\partial t}\right)^2$$

$$|d\vec{x}| = ds$$

$$\left(\frac{\partial \vec{x}}{\partial t}\right)\cdot\left(\frac{\partial \vec{x}}{\partial \sigma}\right) = 0$$

$$\dot{x}\cdot x' = 0$$

$$v_\perp = \frac{\partial \vec{x}}{\partial t}$$

$$\mathcal{P}^{T\mu} = -\frac{T_0}{c}\frac{-(x')^2\frac{\partial x^\mu}{\partial \tau}}{\sqrt{\cdots}} = \frac{T_0}{c}\frac{ds/d\sigma}{\sqrt{1-v^2/c^2}}\frac{\partial x^\mu}{\partial \tau}$$

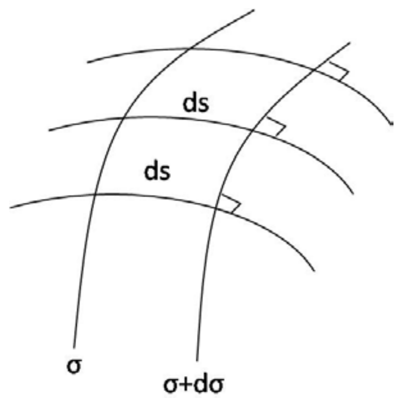
$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c}\frac{-(\dot{x})^2\frac{\partial x^\mu}{\partial \sigma}}{\sqrt{\cdots}} = \frac{T_0\sqrt{1-v^2/c^2}}{ds/d\sigma}\frac{\partial x^\mu}{\partial \sigma} = (0, \vec{\mathcal{P}}^\sigma)$$

$$\frac{\partial \mathcal{P}^{\tau\mu}}{\partial \tau} + \frac{\partial \mathcal{P}^{\sigma\mu}}{\partial \sigma} = 0$$

$\mu = 0$:

$$\frac{\partial \mathcal{P}^{\tau 0}}{\partial t} = 0 \qquad (\mathcal{P}^{\sigma 0} = 0)$$
$$\frac{\partial}{\partial \tau}\left(\frac{T_0}{c}\frac{ds/d\sigma}{\sqrt{1-v^2/c^2}}\right) = 0$$

Consider a constant, fixed $d\sigma$



$$\frac{d}{dt}\left(\frac{T_0 ds}{\sqrt{1-v^2/c^2}}\right) = 0$$
$$\frac{\partial \mathcal{P}^{\tau\mu}}{\partial t} + \frac{\partial \mathcal{P}^{\sigma\mu}}{\partial \sigma} = 0$$
$$\frac{T_0}{c}\frac{ds/d\sigma}{\sqrt{1-v^2/c^2}}\frac{\partial^2 \vec{x}}{\partial t^2} - T_0\frac{\partial}{\partial \sigma}\left(\frac{\sqrt{1-v^2/c^2}}{ds/d\sigma}\frac{\partial \vec{x}}{\partial \sigma}\right) = 0$$

Wouldn't it be nice if the $\sqrt{1-v^2/c^2}$ disappeared? Then we would have a nice wave equation.

Let's fix magnitude of σ s.t.

$$ds/d\sigma/\sqrt{1-v^2/c^2} = 1$$

$$\boxed{d\sigma = \frac{ds}{\sqrt{1-\frac{v^2}{c^2}}}}$$

$$d\sigma = \frac{1}{T_0}\frac{T_0 ds}{\sqrt{\cdots}} = \frac{1}{T_0} d\text{Energy}$$

$$\sigma\left[0, \sigma_1 = \frac{E}{T_0}\right]$$

Note σ not equal to the length - more convenient this way, proportional to *energy*.

Our cleverness so far:
Static gauge
Time on world = time on worldsheet
Keep lines orthogonal
Set σ proportional to energy

$$\left(\frac{ds}{d\sigma}\right)^2 = 1 - \frac{v^2}{c^2} \Rightarrow \boxed{\left(\frac{\partial x}{\partial \sigma}\right)^2 + \frac{1}{c^2}\left(\frac{\partial \vec{x}}{\partial t}\right)^2 = 1}$$

Recall:

$$\boxed{\frac{\partial \vec{x}}{\partial t} \cdot \frac{\partial \vec{x}}{\partial \sigma} = 0}$$

These two boxed equations are the parameterization conditions.

Now wave equation is simple:

$$\frac{\partial^2 \vec{x}}{\partial \sigma^2} - \frac{1}{c^2}\frac{\partial^2 \vec{x}}{\partial t^2} = 0$$

For normal non-rel. string, get wave equation.
For new rel. string, get wave equation and 2 parameterization conditions.

Combine the equations:

$$\boxed{\left(\frac{\partial \vec{x}}{\partial \sigma} \pm \frac{1}{c}\frac{\partial \vec{x}}{\partial t}\right)^2 = 1}$$

Now:

$$\boxed{\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\frac{\partial x^\mu}{\partial t}}$$

$$\boxed{\mathcal{P}^{\sigma\mu} = -T_0\frac{\partial x^\mu}{\partial \sigma}}$$

Nice and simple.

Open String Motion Totally Free

$$X(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + \vec{G}(ct-\sigma))$$

This is all the wave equation tell you. x: position of string.

Now BCs:

$$\frac{\partial \vec{x}}{\partial \sigma} = 0$$
$$\sigma = 0, \sigma_1$$

$$\frac{\partial \vec{x}}{\partial \sigma} = \frac{1}{2}(\vec{F}'(ct+\sigma) + \vec{G}'(ct-\sigma))$$

Primes indicate derivative with respect to σ

BC 1:

$$\frac{\partial \vec{x}}{\partial \sigma}_{\sigma=0} = 0$$
$$\vec{F}'(ct) + \vec{G}'(ct) = 0 \Rightarrow \vec{F}'(u) = \vec{G}'(u) \Rightarrow \vec{G}(u) + \vec{a}_0$$

Back to $\vec{X} = \frac{1}{2}(\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma) + \vec{a}_0)$. Absorb $\vec{a}_0$ into $\vec{F}$.

$$\boxed{\vec{X} = \frac{1}{2}(\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma))}$$

$\vec{x}(t,0) = \vec{F}(ct)$. F tells you the motion of one endpoint.

BC 2:

$$\frac{\partial \vec{x}}{\partial \sigma}_{\sigma=\sigma_1} = 0$$

$$\vec{F}'(ct+\sigma_1) = \vec{F}'(ct-\sigma_1)$$
$$\boxed{\vec{F}'(u+2\sigma_1) = \vec{F}'(u)}$$

F' periodic.

$$\vec{F}(u+2\sigma_1) = \vec{F}(u) + \frac{2\sigma_1}{c}\vec{v}_0$$

$$\frac{\partial \vec{x}}{\partial t}(t,\sigma) = \frac{c}{2}(\vec{F}'(ct+\sigma) + \vec{F}'(ct-\sigma))$$

Let $t \to t + \frac{2\sigma_1}{c}$ then velocity doesn't change! (since $\vec{F}'(u+2\sigma_1) = \vec{F}'(u)$))

$$X(t+\frac{2\sigma_1}{c},\sigma)=\frac{1}{2}(F(ct+2\sigma_1+\sigma)+\vec{F}(ct+2\sigma_1-\sigma))=\vec{x}(t,\sigma)+\frac{2\sigma_1}{c}\vec{v}_0$$

So $\vec{v}_0$ = average velocity of any fixed-σ point on the string.

This explanation is a bit different than the book's.

$$\frac{\partial\vec{x}}{\partial\sigma}+\frac{1}{c}\frac{\partial\vec{x}}{\partial t}=\vec{F}'(ct+\sigma)$$

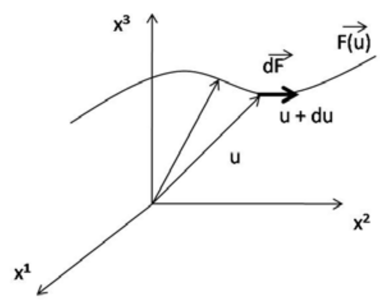
$$\frac{\partial\vec{x}}{\partial\sigma}-\frac{1}{c}\frac{\partial\vec{x}}{\partial t}=-\vec{F}'(ct-\sigma)$$

These yield:

$$\frac{\partial x}{\partial\sigma}\pm\frac{1}{c}\frac{\partial x}{\partial t}=\pm F'(ct\pm\sigma)$$

$$|\vec{F}'(u)|=1$$

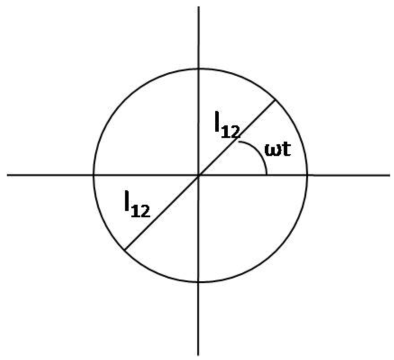
$$|\frac{d\vec{F}(u)}{du}|=1$$



u: length parameter on curve

$$|d\vec{F}|=du$$

Example: Most famous example. Open string doing circular motion.



l: length of string

$$\vec{X}(t, \sigma = 0) = \frac{l}{2}(\cos \omega t, \sin \omega t)$$

Recall:

$$\vec{X}(t, 0) = \vec{F}(ct) \Rightarrow F(u) = \frac{l}{2}(\cos(\omega u/c), \sin(\omega u/c))$$

$$\vec{F}'(u) = \frac{l}{2}\frac{\omega}{c}(-\sin(\omega u/c), \cos(\omega u/c))$$

Unit vector: $\frac{\omega}{c}\frac{l}{2} = 1 \Rightarrow \boxed{\frac{\omega l}{2} = c}$

String endpoints move at speed of light!

Periodicity of $\vec{F}'$: $\frac{\omega(2\sigma_1)}{c} = m(2\pi)$

$m = 1$:

$$x(0, \sigma) = \frac{1}{2}(F(\sigma) + F(-\sigma)) = \frac{l}{2}(\cos(\pi m\sigma/\sigma_1), 0)$$

$$\frac{\omega 2\sigma_1}{c} = 2\pi$$

$$\frac{c}{\omega} = \boxed{\frac{\sigma}{\pi} = \frac{l}{2}}$$

$$\sigma_1 = \frac{E}{T_0} \Rightarrow \boxed{E = \frac{\pi}{2}(lT_0)}$$

String has $\frac{\pi}{2}$ more energy since rotating.