Massachusetts Institute of Technology

8.223, Classical Mechanics II

Pset 4

32. Show that for $F(x_1, ..., x_n)$, the Legendre transformation is

$$G(s_1, ..., s_n) = \sum_{i=1}^{n} x_i s_i - F$$

where

$$s_i = \left(\frac{\partial F}{\partial x_i}\right)_{x_1,...,x_{i-1},x_{i+1},...,x_n}$$

and has the property that

$$x_i = \frac{\partial G}{\partial s_i}$$

33. Starting from the Lagrangian for the simple harmonic oscillator,

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2,$$

find the momentum, the Hamiltonian $H(p, q)$ and Hamilton's equations of motion.

34. Continuing the previous exercise, solve Hamilton's equations and make a plot of typical trajectories in (p, q) space (referred to as "phase space".)

35. Give the Hamiltonian in three dimensions for a system with potential $U(r) = -\alpha/r$ using spherical polar coordinates. Write the equations of motion and identify the conserved momenta. Use this result (and a proper reference frame) to reduce the problem to one dynamical variable and associated conjugated momentum (they will be r and p_r).

36. OPTIONAL Draw phase space diagrams in r and p_r for the previous problem.

37. Show each of the following fundamental Poisson bracket equations:

$$[q_1, q_2] = 0$$
$$[p_1, p_2] = 0$$
$$[q_1, p_2] = 0$$
$$[p_1, q_1] = 1$$
$$[p_1^2, q_1] = 2p_1$$
$$[p_1, q_1^2] = 2q_1$$

38. Canonical transformation:

(a) Show that the transformation on $2n$-dimensional phase space associated with a coordinate transformation on configuration space, namely:

$$q_i \to Q_i(\vec{q})$$

$$p_i \to P_i(\vec{q}, \vec{p}) = \sum_j p_j \frac{\partial q_j}{\partial Q_i}$$

is a canonical transformation.

(b) On a 2-dimensional phase space, show that the transformation

$$q \to Q = \ln(\frac{\sin p}{q})$$

$$p \to P = q \cot p$$

is canonical.

(c) Find the generating function $F_4(p, P)$ of the canonical transformation in (b).

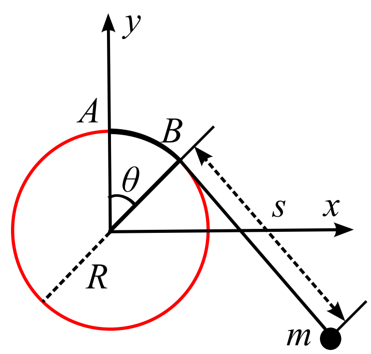

Figure 1:

39. As shown in Fig. 1, consider a mass m attached to a string which in turn is nailed to point A on a circular spool of radius R. The whole system lies on the horizontal plane, and the spool is fixed so it **cannot rotate**. As the mass slides without friction, the string remains taut and either winds or unwinds around the spool. B is the point where the string leaves the spool. Let the total length of the string be l, and let s denote the free length of the string, that is, the length from B to the mass. We align the coordinate axes so that the center O of the spool corresponds to $x = y = 0$ and the radius to OA is in the positive y direction. Let θ denote the angle between OA and OB.

(a) Express the coordinates (x, y) of the mass in terms of s, θ and R. Using the constraint relating s and θ to the total length, find the Lagrangian $L(s, \dot{s})$ of the system in terms of the dynamical coordinates s and its associated velocity $\dot{s}$.

(b) Find the Hamiltonian $H(s,p)$, write Hamilton's equations, and confirm that $\frac{p}{s}$ is a constant of the motion. Use this information to find $s(t)$ in terms of its initial value s_0 and the total energy E.

(c) Consider the following change of coordinates:

$$Q = s^2, \ P = \frac{p}{\lambda s}$$

Find the value of the constant λ so that the transformation is canonical and give $H'(Q,P)$.

40. For a Lagrangian with a velocity dependent potential

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \lambda y\dot{x},$$

a) Show that the solutions have the form

$$x(t) = -R\cos(\frac{\lambda}{m}t + \phi) + x_o$$
$$y(t) = R\sin(\frac{\lambda}{m}t + \phi) + y_o.$$

b) Find the conjugate momenta.