Massachusetts Institute of Technology

Physics Department

Physics 8.20 IAP 2005
Introduction to Special Relativity

Problem Set 1

1. **Speeds**

   What fraction of the speed of light does each of the following speeds represent? (If any calculation is required, use Newtonian mechanics; ignore any relativistic effects. In cases where calculation — as opposed to unit conversion — is required, comment on whether your Newtonian results are good approximations to the correct speed.)

   (a) A human being, walking at 4 m/sec.

   (b) The Acela Express (Boston to New York), travelling at 120 miles/hr.

   (c) A satellite orbiting the Earth in low-Earth orbit. The radius of the Earth is 6.4×10^6 m.

   (d) A proton (or a baseball, for that matter) dropped onto the surface of a white dwarf star from rest at a great distance. A white dwarf star is a compact star with a mass of about 1.4 times the mass of the sun and a radius of about 5,000 km. The mass of the sun is 2.0×10^{30} kg. Assume that the proton (or baseball) starts at rest infinitely far from the star. Calculate its kinetic energy when it crashes into the neutron star surface, and then calculate its velocity.

   (e) A spaceship, starting from rest, accelerated at 1m/sec^2 for 100 years. Note this is about 1/10th the acceleration due to gravity at the surface of the earth.

2. **The travels of elementary particles**

   Elementary particles have very short average lifetimes (at least measured on our perceptual scale). In experiments at accelerators the particles are produced and then detected at points away from the point of production. If the rules of Newtonian physics were correct for particles travelling at great speeds, the particles would have to travel much faster than the speed of light in order to go so far before they decay.

   Assuming that the particle in question lives for the average lifetime of that species, compute its average speed using Newtonian mechanics:

   (a) A "π-meson" has a lifetime of 2.6×10^{-8} sec. It is observed 1 kilometers away from its point of production.

   (b) A "Lambda" hyperon has a lifetime of 2.6×10^{-10} sec. It is observed 30 meters away from its point of production.

   (c) A "Λ_B baryon" has lifetime 1.23×10^{-12} sec. It is observed 3 millimeters away from its point of production.

   All of the particles in this problem are built out of quarks. Quarks are the fundamental building blocks of matter – they and the gluons that bind them together account for 99.95% of your mass. However, quarks themselves are never seen. Their interactions are so strong that they are permanently confined in groups of three (particles called "baryons") or as a quark and antiquark (particles called "mesons"). There are six types of quarks, each with a different mass. They have unusual electromagnetic charges: $2/3|e|$ and $-1/3|e|$. The types of quarks are known as "up" (u), "down" (d), "strange" (s), "charm" (c), "bottom" (b) and "top" (t). The lightest quarks are the u and d. All the others decay into these two by the "weak interactions" which are responsible for radioactivity. The π-meson consists of $u\bar{d}$ (where the $\bar{}$ denotes an antiquark). The Λ hyperon consists of uds, and the Λ_B consists of udb. We'll have more to say about these during the course.

3. Galilean transformation on a ship – Case I

A ship is travelling due east at a speed of 30 m/sec.

- A ball is rolled due north on the deck of the ship at a speed of 10 m/sec relative to the ship. What is its velocity relative to the Earth?
- If the ball is rolled 60° east of north at a speed of 5 m/sec relative to the ship, what is its velocity relative to the Earth?

4. Galilean transformation on a ship – Case II

A stone is dropped from rest from the mast of a ship moving with a velocity of 25 m/sec relative to the Earth. Choose the origins of the Earth based ("laboratory") coordinate frame (Σ) and the coordinate frame in which the ship is at rest (Σ') such that they coincide with each other and with the stone at the instant $t = 0$ that the stone is dropped. Choose your coordinate system so that the ship moves along the positive x-axis relative to the Earth and the mast points along the positive y-direction. Find the position of the stone (a) relative to the ship, and (b) relative to the laboratory after 2.5 seconds. [Remember that the acceleration of gravity is 9.8 m/sec^2.] Show that the results can be related using the Galilean transformation.

5. Frame independence of momentum conservation

Taken from Resnick and Halliday, Basic Concepts in Relativity (MacMillan, New York, 1992).

(a) An observer on the ground watches a collision between two particles whose masses are m_1 and m_2 and finds, by measurement, that momentum is conserved. Use the classical velocity addition theorem to show that an observer on a moving train will also find that momentum is conserved in the collision.

(b) Repeat this analysis under the assumption that a transfer of mass from one particle to the other takes place during the collision, the initial masses being m_1 and m_2 and the final masses being m_1' and m_2'. Again, assume that the ground observer finds, by measurement, that momentum is conserved. Show that the train observer will also find that momentum is conserved *only* if mass is also conserved, that is, if

$$m_1 + m_2 = m_1' + m_2'.$$

6. Binomial expansion

We will be using the "binomial expansion" often in 8.20. It reads:

$$\begin{aligned}(1+\epsilon)^a &= 1+\frac{a}{1}\epsilon+\frac{a(a-1)}{2\cdot 1}\epsilon^2+\frac{a(a-1)(a-2)}{3\cdot 2\cdot 1}\epsilon^3 \\ &+ \frac{a(a-1)(a-2)(a-3)}{4\cdot 3\cdot 2\cdot 1}\epsilon^4+\dots \end{aligned} \tag{1}$$

This expansion converges when $|\epsilon| < 1$.

(a) For what values of a does the expansion terminate with a finite number of terms?

(b) Use eq. (1) to derive an expansion for $(a+b)^c$ when $|b| < |a|$.

(c) Consider $1/\sqrt{1-v^2/c^2}$. Write this in the form $(1+\epsilon)^a$ and expand it using the binomial expansion and give the first four terms.

(d) If $v/c = 0.4$ how large an error would you make by keeping only the first two terms in part (c)? the first three terms?

7. Numbers for Michelson-Morley

In the Michelson-Morley experiment of 1887, the length, ℓ of each arm of the interferometer was 11 meters, and sodium light of wavelength 5.9×10^{-7} meters was used. Suppose that the experiment would have revealed any shift larger than 0.0025 fringe. What upper limit does this place on the speed of the Earth through the supposed aether? How does this compare with the speed of the Earth around the Sun?

8. Michelson-Morley for a real wind

Taken from Resnick and Halliday, Basic Concepts in Relativity (MacMillan, New York, 1992).

A pilot plans to fly due east from A to B and back again. If u is her airspeed and if ℓ is the distance between A and B, it is clear that her round-trip time t_0 — if there is no wind — will be $2\ell/u$.

(a) Suppose, however, that a steady wind of speed v blows from the west. What will the round trip travel time now be, expressed in terms of t_0, u, and v?

(b) If the wind is from the south, find the expected round-trip travel time, again as a function of t_0, u, and v.

(c) Note that these two travel times are not equal. Should they be? Did you make a mistake?

(d) In the Michelson-Morley experiment, however, the experiment seems to show that (for arms of equal length) the travel times for light *are* equal; otherwise these experimenters would have found a fringe shift when they rotated their interferometer. What is the essential difference between these two situations?

9. Ehrenfest's thought experiment

Taken from Resnick and Halliday, Basic Concepts in Relativity (MacMillan, New York, 1992).

Paul Ehrenfest (1880-1933) proposed the following thought experiment to illustrate the different behavior expected for light under the ether wind hypothesis and under Einstein's second postulate:

Imagine yourself seated at the center of a spherical shell of radius 3×10^8 meters, the inner surface being diffusely reflecting. A source at the center of the sphere emits a sharp pulse of light, which travels outward through the darkness with uniform intensity in all directions. *Explain* what you would expect to see during the three second interval following the pulse under the assumptions that,

(a) there is a steady ether wind blowing through the sphere at 100 km/sec, and

(b) there is no ether and Einstein's second postulate holds.

(c) Discuss the relationship of this thought experiment to the Michelson-Morley Experiment.

10. Aberration, before and after Einstein

Taken from Resnick and Halliday, Basic Concepts in Relativity (MacMillan, New York, 1992).

Show that, according to relativity, the classical aberration equation,

$$\tan \alpha_{\text{classical}} = \frac{v}{c} \quad \text{classical theory}$$

must be replaced by

$$\sin \alpha_{\text{relativity}} = \frac{v}{c} \quad \text{relativity theory.}$$

Thus the ether theory and relativity make different predictions for the aberration of starlight. However the differences are very small. To see this, consider a realistic case. Assume that the Earth's orbital speed is 30 km/sec and take $c = 3.00 \times 10^8$ m/sec. Find the *fractional difference*,

$$f \equiv \frac{\alpha_{\text{classical}} - \alpha_{\text{relativity}}}{\alpha_{\text{relativity}}}$$

Note the differences are so small that your calculator may fail to capture the significant figures. Instead use the series expansions:

$$\begin{aligned} \sin^{-1} x &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 \\ \tan^{-1} x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \end{aligned}$$

11. Aberration due to the Earth's rotation

In class we discussed the stellar aberration generated by the Earth's motion around the Sun. The rotation of the Earth about its axis also causes stellar aberration.

(a) Explain why the amount of stellar aberration generated by the Earth's rotation depends upon the latitude of the observer.

(b) For an observer at a given latitude explain why the amount of aberration depends on the *compass direction* of the star being observed. Compare, for example, the aberration of a star viewed on the eastern or western horizon with one on the northern horizon and with one directly overhead.

(c) What is the largest aberration angle (the tilt of the telescope) due to the Earth's rotation alone for an observer a) at the North Pole, b) at the equator, and c) at latitutde 45° north.

12. Weighing the sun

Taken from Resnick and Halliday, Basic Concepts in Relativity (MacMillan, New York, 1992).

(a) Show that M, the mass of the sun, is related to the aberration constant, α (see problem 10) by

$$M = \frac{\alpha^2 c^2 R}{G}$$

in which R is the radius of the Earth's orbit (which we assume to be circular) and G is Newton's constant ($G = 6.67 \times 10^{-11}$N m/kg^2. Hint: apply Newton's second law to the Earth's motion around the sun.

(b) Calcuate M given that $\alpha = 20.5''$ and $R = 1.50 \times 10^{11}$m.

13. "Emission theories"

One of the early responses to the realization that electromagnetism predicts that the speed of light is a "constant" took the form of "emission theories". These theories make the hypothesis that the speed of light is a constant, c, with respect to the source that emits it. They then have to deal with what happens when the emitted light reflects off a mirror — what velocity counts then? Here are three versions of emission theories that differ in their predictions of what the speed of light will be upon reflection from a mirror:

- The "original source" theory: the speed remains c relative to the original source.
- The "ballistic" theory: the speed is originally c relative to the original source, but upon reflection from the mirror it becomes c relative to the mirror.
- The "new source" theory: the speed is originally c relative to the original source, but becomes c relative to *the mirror image of the source* upon reflection.

Now suppose that a source of light, S, and a mirror M are moving away from one another. To be explicit, assume that the source is moving to the left with speed u in the laboratory, and a mirror, M is originally moving to the right with speed v. What is the speed of a light beam (as measured in the laboratory) originally emitted by the source after it has been reflected from the mirror according to each of the three "theories"?

According to Einstein's second postulate what would be the the measured speed of a light pulse (either before or after reflection from the mirror) as viewed from i) the lab frame; ii) the rest frame of the mirror; iii) the rest frame of the source.

Note: Einstein's theory is not only right, it's simpler!

14. Invariance of the wave equation

Starting from Maxwell's equations it is possible to derive a *wave equation* whose solutions represent electromagnetic waves. The equation, for the electric field, E, is

$$\frac{\partial^2}{\partial z^2}E(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(z,t) = 0 \tag{2}$$

where c is the speed of light.

(a) Show that $E(z,t) = f(z \pm ct)$ is a solution to this equation *for any function, f*.

(b) Explain why this solution represents a wave travelling to the right or left (toward increasing or decreasing z) with speed c. Which sign corresponds to which direction?

(c) Show that eq. (2) *is not invariant* under the Galilean transformation $z' = z - vt$, $t' = t$.

(d) Show however that eq. (2) *is invariant* under the Lorentz transformation $z' = a(z - vt)$, $t' = a(t - vz/c^2)$, where $a = 1/\sqrt{1 - v^2/c^2}$.

Hint: you may find the "chain rule for partial derivatives" useful:

$$\left.\frac{\partial f(x,t)}{\partial x}\right|_t = \left.\frac{\partial f(x',t')}{\partial x'}\right|_{t'} \left.\frac{\partial x'(x,t)}{\partial x}\right|_t + \left.\frac{\partial f(x',t')}{\partial t'}\right|_{x'} \left.\frac{\partial t'(x,t)}{\partial x}\right|_t$$

15. Relativity of simultaneity

A plane flies overhead an observer on the Earth. Treat both the Earth and the plane as inertial frames for this problem. The speed of the plane is v. When the plane is overhead a light signal is emitted from the center of the plane. Subsequently it is detected by observer A in the front of the plane and observer B in the rear of the plane. Both observers measure their distance from the center of the plane to be d.

(a) Assume the speed of light is c as measured by the observers in the plane. Explain why observers A and B agree that the light signal reaches them simultaneously. How much time does the light take to reach them?

(b) Assuming that the speed of light is also c as measured by an observer on the Earth, explain why the Earth-bound observer would say that the arrival of the light signal at A and at B *were not simultaneous events.*

16. A feeling for the Lorentz factor

The "Lorentz factor", $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ determines the magnitude of many of the most unusual consequences of relativity (time dilation and length contraction, for example). What must an object's velocity be relative to you, the observer, for it's Lorentz factor to be:

(a) 1.0001

(b) 1.1

(c) 2

(d) 100

(e) 10^6

17. Inverse Lorentz transformation

Suppose two inertial frames are related by a Lorentz transformation:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - vx/c^2) \end{aligned}$$

Solve for x, y, z, t in terms of x', y', z', t' and show that the transformation is identical except for $v \to -v$.

18. Lorentz transformation in an arbitrary direction

Suppose two inertial frames, Σ and Σ' move such that their coordinate axes are parallel, their origins coincide at $t = t' = 0$, and the origin of Σ' is observed to move with velocity $\vec{v}$ in Σ. Starting from the form of the Lorentz transformation when the relative motion is along a coordinate axis, derive the Lorentz transformation relating x', y', z', t' to x, y, z, t. [Hint: it will be useful to decompose $\vec{x}$ into $\vec{x}_{\parallel} = \hat{v}(\hat{v} \cdot \vec{x})$ and $\vec{x}_{\perp} = \vec{x} - \vec{x}_{\parallel}$.]