8.07 E&M Review
on
Waves, Potentials and Radiation

Dec 16, 2012

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Sinusoidal Waves:

Wave equation:

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 \ .$$

$$f(z,t) = A\cos\left[k(z - vt) + \delta\right]$$
$$= A\cos\left[kz - \omega t + \delta\right] \ ,$$

$v = \dfrac{\omega}{k}$ = phase velocity

ω = angular frequency = $2\pi\nu$

ν = frequency

δ = phase (or phase constant)

k = wave number

$\lambda = 2\pi/k$ = wavelength

$T = 2\pi/\omega$ = period

A = amplitude.

General solution to wave equation:

$$f(z,t) = \int_{-\infty}^{\infty} A(k)\, e^{i(kz-\omega t)}\, dk \ ,$$

where $\omega/k = v$, v = wave speed = phase velocity.

Complex amplitude

+, −, ·, ⁄, √….

Problem 9.3

$$
\begin{aligned}
(A_3)^2 &= \left(A_3 e^{i\delta_3}\right)\left(A_3 e^{-i\delta_3}\right) = \left(A_1 e^{i\delta_1} + A_2 e^{i\delta_2}\right)\left(A_1 e^{-i\delta_1} + A_2 e^{-i\delta_2}\right) \\
&= (A_1)^2 + (A_2)^2 + A_1 A_2 \left(e^{i\delta_1} e^{-i\delta_2} + e^{-i\delta_1} e^{i\delta_2}\right) = (A_1)^2 + (A_2)^2 + A_1 A_2 2\cos(\delta_1 - \delta_2); \\
A_3 &= \boxed{\sqrt{(A_1)^2 + (A_2)^2 + 2A_1 A_2 \cos(\delta_1 - \delta_2)}.} \\
A_3 e^{i\delta_3} &= A_3(\cos\delta_3 + i\sin\delta_3) = A_1(\cos\delta_1 + i\sin\delta_1) + A_2(\cos\delta_2 + i\sin\delta_2) \\
&= (A_1\cos\delta_1 + A_2\cos\delta_2) + i(A_1\sin\delta_1 + A_2\sin\delta_2). \quad \tan\delta_3 = \frac{A_3\sin\delta_3}{A_3\cos\delta_3} = \frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}; \\
\delta_3 &= \boxed{\tan^{-1}\left(\frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}\right).}
\end{aligned}
$$

General solution to wave equation:

$$f(z,t) = \int_{-\infty}^{\infty} A(k)\, e^{i(kz-\omega t)}\, \mathrm{d}k \ ,$$

where $\omega/k = v$, $v =$ wave speed $=$ phase velocity.

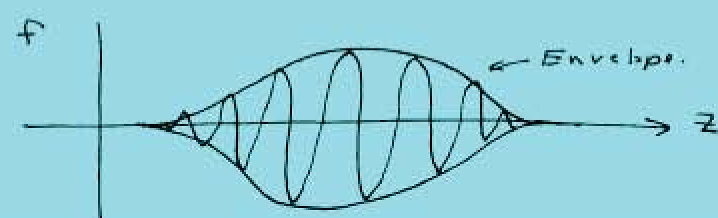
$$\omega(k) = \omega(k_0) + \frac{\mathrm{d}\omega}{\mathrm{d}k}(k_0)(k-k_0) + \ldots$$

$$= \omega(k_0) - k_0\frac{\mathrm{d}\omega}{\mathrm{d}k} + k\frac{\mathrm{d}\omega}{\mathrm{d}k} + \ldots \ .$$

$$f(z,t) = e^{i\left[\omega(k_0) - k_0\frac{\mathrm{d}\omega}{\mathrm{d}k}\right]t} \int_{-\infty}^{\infty} \mathrm{d}k\, A(k) e^{ik\left(z - \frac{\mathrm{d}\omega}{\mathrm{d}k}t\right)} \ .$$

The integral describes a wave which moves with

$$v_{\mathrm{group}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}(k_0) \ .$$



Group velocity:

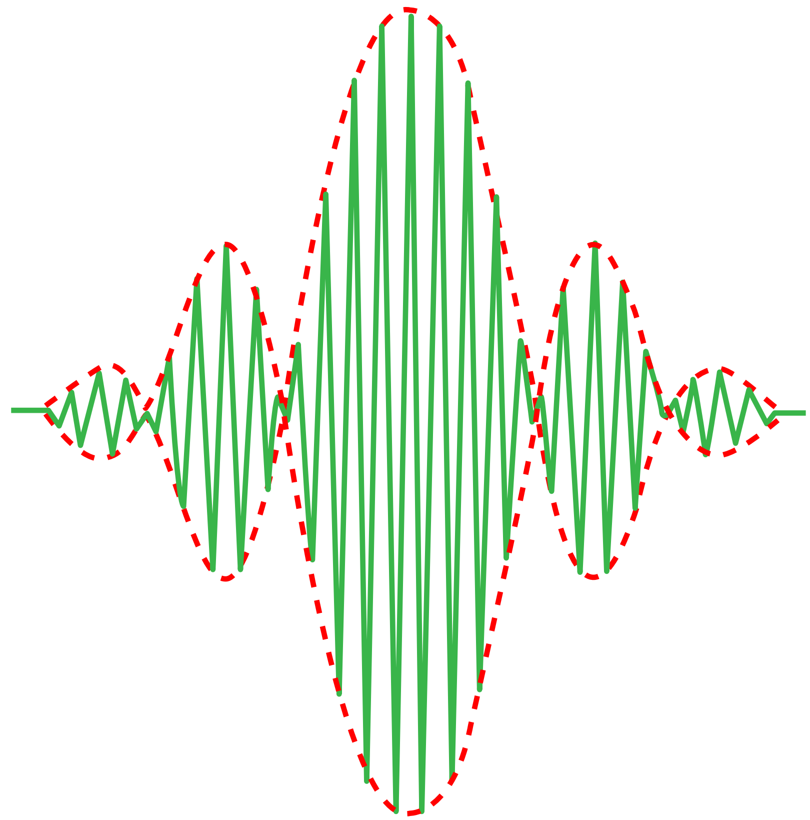
V_g : velocity to convey energy

Anomalous dispersion

V_g can be Negative, zero or > c

V_g & V_p in different directions V_p

Signal velocity still < c



Solid line: A wave packet.
Dashed line: The envelope of the wave packet.
The envelope moves at the group velocity

Image by MIT OpenCourseWare.

Electromagnetic Plane Waves

Maxwell Equations in Empty Space:

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \; ,$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \; ,$$

where $1/c^2 \equiv \mu_0 \epsilon_0$. Manipulating,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}\underbrace{(\vec{\nabla} \cdot \vec{E})}_{=0} - \nabla^2 \vec{E}$$

$$= \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \; ,$$

so

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \ . \tag{16}$$

This is the wave equation in 3 dimensions. An identical equation holds for $\vec{B}$:

$$\nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0 \ . \tag{17}$$

$$\vec{E}(\vec{r},t) = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\,\hat{n} \ ,$$

where $\tilde{E}_0$ is a complex amplitude, $\hat{n}$ is a unit vector, $\omega/|\vec{k}| = v_{\rm phase} = c$. Then

$$\vec{\nabla}\cdot\vec{E} = i\hat{n}\cdot\vec{k}E_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \ ,$$

so we require

$$\hat{n}\cdot\vec{k} = 0 \qquad \text{(transverse wave)} \ .$$

$$\vec{B} = \frac{1}{c}\hat{k} \times \vec{E} \ .$$

Energy and Momentum:

Energy density:

$$u = \frac{1}{2}\left[\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right] \ .$$

The $\vec{E}$ and $\vec{B}$ contributions are equal.

$$u = \epsilon_0 E_0^2 \underbrace{\cos^2(kz - \omega t + \delta)}_{\text{averages to } 1/2} \ , \quad (\vec{k} = k\,\hat{z})$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = uc\ \hat{z}$$

$$\mathcal{P}_{\text{EM}} = \frac{1}{c^2}\vec{S} = \frac{u}{c}\ \hat{z}$$

$$I \text{ (intensity)} = \left\langle |\vec{S}| \right\rangle = \frac{1}{2}\epsilon_0 E_0^2 \ .$$

REVIEW-MW-EQ-WAVE--SOL-S

Energy and Momentum:

Energy density:

$$u = \frac{1}{2}\left[\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right] .$$

The $\vec{E}$ and $\vec{B}$ contributions are equal.

$$u = \epsilon_0 E_0^2 \underbrace{\cos^2(kz - \omega t + \delta)}_{\text{averages to } 1/2} , \quad (\vec{k} = k\,\hat{z})$$

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$$\mathcal{P}_{\text{EM}} = \frac{1}{c^2}\vec{S} = \frac{u}{c}\ \hat{z}$$

$$I\ (\text{intensity}) = \left\langle |\vec{S}| \right\rangle = \frac{1}{2}\epsilon_0 E_0^2 .$$

In material n

$$u = \frac{1}{2}\left[\epsilon|\vec{E}|^2 + \frac{1}{\mu}|\vec{B}|^2\right]$$

$$\vec{B} = \frac{n}{c}\hat{k} \times \vec{E}$$

$$\vec{S} = \frac{1}{\mu}\vec{E} \times \vec{B} = \frac{uc}{n}\ \hat{z} .$$

Momentum flux =
DxB (c/n)=**S**/(c/n)

Given E
find B, S, intensity and power

$$\vec{E}(r,\theta,\phi,t) = A\frac{\sin\theta}{r}[\cos(kr-\omega t] - (1/kr)\sin(kr-\omega t)]\hat{\phi}\text{ , with }\frac{\omega}{k} = c\ .$$

(This is, incidentally, the simplest possible **spherical wave**. For notational convenience, let $(kr - \omega t) \equiv u$ in your calculations.)

(a) Show that $\vec{E}$ obeys all four of Maxwell's equations, in vacuum, and find the associated magnetic field.

(b) Calculate the Poynting vector. Average $\vec{S}$ over a full cycle to get the intensity vector $\vec{I}$. (Does it point in the expected direction? Does it fall off like r^{-2}, as it should?)

(c) Integrate $\vec{I}\cdot d\vec{a}$ over a spherical surface to determine the total power radiated. [*Answer:* $4\pi A^2/3\mu_0 c$]

Given B?

Potentials

$$\vec{B} = \vec{\nabla} \times \vec{A} \,, \quad \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \,,$$

the source-free Maxwell equations (ii) and (iii),

$$\text{(i)} \quad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \qquad \text{(iii)} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \,,$$

$$\text{(ii)} \quad \vec{\nabla} \cdot \vec{B} = 0 \qquad \text{(iv)} \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \,,$$

gauge transformation:

$$\vec{A}' = \vec{A} + \vec{\nabla} \Lambda \,, \quad V' = V - \frac{\partial \Lambda}{\partial t} \,.$$

Coulomb Gauge: $\boxed{\vec{\nabla} \cdot \vec{A} = 0 \ .}$ (12)

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = -\frac{1}{\epsilon_0}\rho \quad \Longrightarrow \quad \nabla^2 V = -\frac{1}{\epsilon_0}\rho \ . \tag{13}$$

V is easy to find, but $\vec{A}$ is hard. V responds instantaneously to changes in ρ, but V is not measurable. $\vec{E}$ and $\vec{B}$ receive information only at the speed of light.

Lorentz Gauge: $\boxed{\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \ .}$ (14)

$$\Longrightarrow \quad \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho \ . \tag{15}$$

PROBLEM 4: GAUGE CHOICES

gauge transformation:

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda\,, \quad V' = V - \frac{\partial \Lambda}{\partial t}\,.$$

Griffiths Problem 10.7 (p. 422).

> In Chapter 5, I showed that it is always possible to pick a vector potential whose divergence is zero (Coulomb gauge). Show that it is always possible to choose $\vec{\nabla}\cdot\vec{A} = -\mu_0\epsilon_0(\partial V/\partial t)$, as required for the Lorentz gauge, assuming you know how to solve equations of the form 10.16. Is it always possible to pick $V = 0$? How about $\vec{A} = 0$?

Solution: **A' = A** + delλ = > Δλ = - div **A**

Problem 10.7

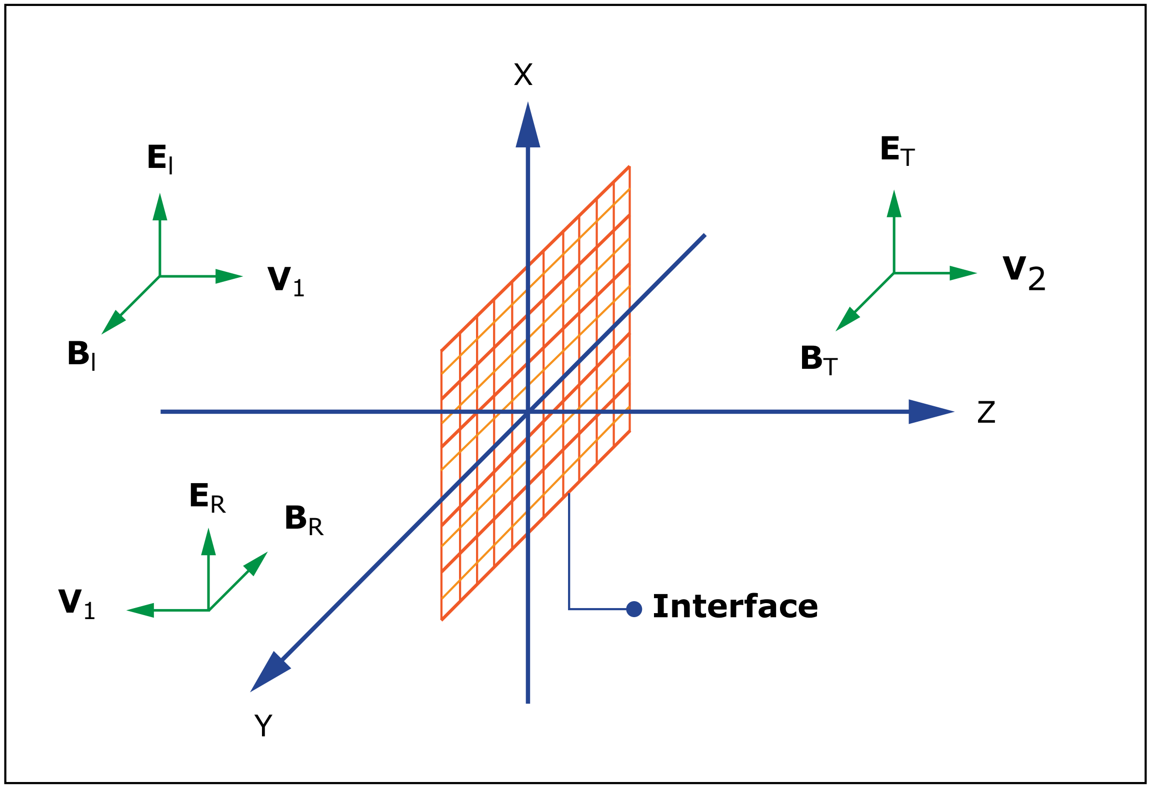
Suppose $\nabla\cdot\mathbf{A} \neq -\mu_0\epsilon_0\frac{\partial V}{\partial t}$. (Let $\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = \Phi$—some known function.) We want to pick λ such that $\mathbf{A}'$ and V' (Eq. 10.7) *do* obey $\nabla\cdot\mathbf{A}' = -\mu_0\epsilon_0\frac{\partial V'}{\partial t}$.

$$\nabla\cdot\mathbf{A}' + \mu_0\epsilon_0\frac{\partial V'}{\partial t} = \nabla\cdot\mathbf{A} + \nabla^2\lambda + \mu_0\epsilon_0\frac{\partial V}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2} = \Phi + \Box^2\lambda.$$

This will be zero provided we pick for λ the solution to $\Box^2\lambda = -\Phi$, which by hypothesis (and in fact) we know how to solve.

We *could* always find a gauge in which $V' = 0$, simply by picking $\lambda = \int_0^t V\,dt'$. We *cannot* in general pick $\mathbf{A} = 0$—this would make $\mathbf{B} = 0$. [Finding such a gauge function would amount to expressing $\mathbf{A}$ as $-\nabla\lambda$, and we know that vector functions *cannot* in general be written as gradients—only if they happen to have curl zero, which $\mathbf{A}$ (ordinarily) does *not*.]

Boundary Conditions



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Boundary Conditions:

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \qquad \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \ ,$$

$$B_1^{\perp} = B_2^{\perp} \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$$

Reflections, transmission, pressure

Incident wave ($z < 0$):

$$\vec{E}_I(z,t) = \tilde{E}_{0,I}\, e^{i(k_1 z - \omega t)}\, \hat{x}$$

$$\vec{B}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0,I}\, e^{i(k_1 z - \omega t)}\, \hat{y}\ .$$

Transmitted wave ($z > 0$):

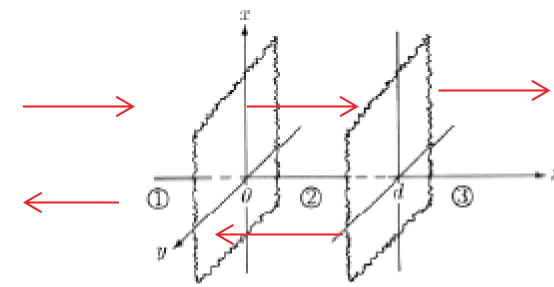
$$\vec{E}_T(z,t) = \tilde{E}_{0,T}\, e^{i(k_2 z - \omega t)}\, \hat{x}$$

$$\vec{B}_T(z,t) = \frac{1}{v_2}\tilde{E}_{0,T}\, e^{i(k_2 z - \omega t)}\, \hat{y}\ .$$

ω must be the same on both sides, so

$$\frac{\omega}{k_1} = v_1 = \frac{c}{n_1}\,, \qquad \frac{\omega}{k_2} = v_2 = \frac{c}{n_2}$$

Problem 9.34



$$z < 0: \quad \begin{cases} \tilde{\mathbf{E}}_I(z,t) = \tilde{E}_I e^{i(k_1 z - \omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_I e^{i(k_1 z - \omega t)}\,\hat{\mathbf{y}} \\ \tilde{\mathbf{E}}_R(z,t) = \tilde{E}_R e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_R(z,t) = -\frac{1}{v_1}\tilde{E}_R e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{y}}. \end{cases}$$

$$0 < z < d: \quad \begin{cases} \tilde{\mathbf{E}}_r(z,t) = \tilde{E}_r e^{i(k_2 z - \omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_r(z,t) = \frac{1}{v_2}\tilde{E}_r e^{i(k_2 z - \omega t)}\,\hat{\mathbf{y}} \\ \tilde{\mathbf{E}}_l(z,t) = \tilde{E}_l e^{i(-k_2 z - \omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_l(z,t) = -\frac{1}{v_2}\tilde{E}_l e^{i(-k_2 z - \omega t)}\,\hat{\mathbf{y}}. \end{cases}$$

$$z > d: \quad \left\{ \tilde{\mathbf{E}}_T(z,t) = \tilde{E}_T e^{i(k_3 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_T(z,t) = \frac{1}{v_3}\tilde{E}_T e^{i(k_3 z - \omega t)}\,\hat{\mathbf{y}}. \right.$$

$$\tilde{E}_{0,R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| \tilde{E}_{0,I} \qquad \tilde{E}_{0,T} = \left(\frac{2n_1}{n_1 + n_2}\right) \tilde{E}_{0,I}\ .$$

What is the pressure on each of the boundary? Use
Momentum flux = **DxB** (c/n)=**S**/(c/n)

Solution to $\Box^2 V = -\dfrac{1}{\epsilon_0}\rho$

Method: Guess a solution and then show that it works.

We know that

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \quad \Longrightarrow \quad V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} \; .$$

We try the guess

$$\Box^2 V = -\frac{1}{\epsilon_0}\rho \quad \Longrightarrow \quad V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|} \; ,$$

where

$$t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c} = \text{retarded time.}$$

Solution to $\Box^2 V = -\frac{1}{\epsilon_0}\rho$

Method: Guess a solution and then show that it works.

We know that

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \quad \Longrightarrow \quad V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} .$$

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where

$$t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c} = \text{retarded time.}$$

Retarded Time Solutions

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}$$

$$\vec{A}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\vec{J}(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|} ,$$

$$t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c} .$$

To differentiate, use change of variables

Gauge invariant?

Coulomb Gauge: $\vec{\nabla} \cdot \vec{A} = 0$. (12)

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = -\frac{1}{\epsilon_0}\rho \quad \Longrightarrow \quad \nabla^2 V = -\frac{1}{\epsilon_0}\rho \ . \tag{13}$$

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Lorentz Gauge: $\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$. (14)

$$\Longrightarrow \quad \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho \ . \tag{15}$$

Problem 10.8 Retarded **A Lorenz** Gauge invariant?

From the product rule:

$$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J})+\mathbf{J}\cdot\left(\nabla\frac{1}{\mathscr{r}}\right),\quad \nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{J})+\mathbf{J}\cdot\left(\nabla'\frac{1}{\mathscr{r}}\right).$$

But $\nabla\frac{1}{\mathscr{r}}=-\nabla'\frac{1}{\mathscr{r}}$, since $\boldsymbol{\mathscr{r}}=\mathbf{r}-\mathbf{r}'$. So

$$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J})-\mathbf{J}\cdot\left(\nabla'\frac{1}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J})+\frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{J})-\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right).$$

But

$$\nabla\cdot\mathbf{J}=\frac{\partial J_x}{\partial x}+\frac{\partial J_y}{\partial y}+\frac{\partial J_z}{\partial z}=\frac{\partial J_x}{\partial t_r}\frac{\partial t_r}{\partial x}+\frac{\partial J_y}{\partial t_r}\frac{\partial t_r}{\partial y}+\frac{\partial J_z}{\partial t_r}\frac{\partial t_r}{\partial z},$$

and

$$\frac{\partial t_r}{\partial x}=-\frac{1}{c}\frac{\partial \mathscr{r}}{\partial x},\quad \frac{\partial t_r}{\partial y}=-\frac{1}{c}\frac{\partial \mathscr{r}}{\partial y},\quad \frac{\partial t_r}{\partial z}=-\frac{1}{c}\frac{\partial \mathscr{r}}{\partial z},$$

tr= t - *r*/c

so

$$\nabla\cdot\mathbf{J}=-\frac{1}{c}\left[\frac{\partial J_x}{\partial t_r}\frac{\partial \mathscr{r}}{\partial x}+\frac{\partial J_y}{\partial t_r}\frac{\partial \mathscr{r}}{\partial y}+\frac{\partial J_z}{\partial t_r}\frac{\partial \mathscr{r}}{\partial z}\right]=-\frac{1}{c}\frac{\partial \mathbf{J}}{\partial t_r}\cdot(\nabla\mathscr{r}).$$

Similarly,

J(r', tr)

$$\nabla'\cdot\mathbf{J}=-\frac{\partial\rho}{\partial t}-\frac{1}{c}\frac{\partial \mathbf{J}}{\partial t_r}\cdot(\nabla'\mathscr{r}).$$

[The first term arises when we differentiate with respect to the *explicit* $\mathbf{r}'$, and use the continuity equation.] thus

$$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}\left[-\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla\mathscr{r})\right]+\frac{1}{\mathscr{r}}\left[-\frac{\partial\rho}{\partial t}-\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\mathscr{r})\right]-\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)=-\frac{1}{\mathscr{r}}\frac{\partial\rho}{\partial t}-\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)$$

(the other two terms cancel, since $\nabla\mathscr{r}=-\nabla'\mathscr{r}$). Therefore:

$$\nabla\cdot\mathbf{A}=\frac{\mu_0}{4\pi}\left[-\frac{\partial}{\partial t}\int\frac{\rho}{\mathscr{r}}\,d\tau-\int\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau\right]=-\mu_0\epsilon_0\frac{\partial}{\partial t}\left[\frac{1}{4\pi\epsilon_0}\int\frac{\rho}{\mathscr{r}}\,d\tau\right]-\frac{\mu_0}{4\pi}\oint\frac{\mathbf{J}}{\mathscr{r}}\cdot d\mathbf{a}.$$

The last term is over the suface at "infinity", where $\mathbf{J}=0$, so it's zero. Therefore $\nabla\cdot\mathbf{A}=-\mu_0\epsilon_0\frac{\partial V}{\partial t}$. ✓

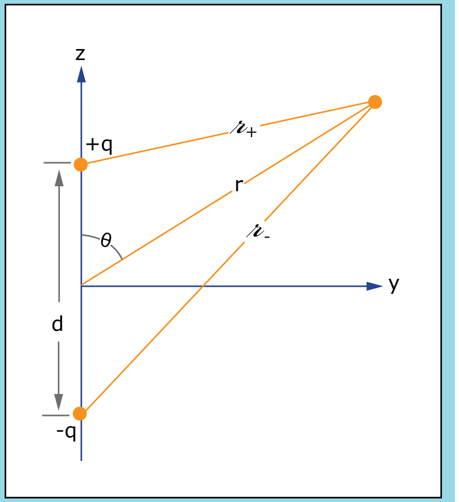
Review-radiation-dipole-V

Electric Dipole Radiation

Simplest dipole: two tiny metal spheres separated by a distance d along the z-axis, connected by a wire, with charges

$$q(t) = q_0 \cos(\omega t) \qquad (5)$$

on the top sphere, and $q(t) = -q_0 \cos(\omega t)$ on the bottom sphere.



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Then

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0 \cos[\omega(t - \mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0 \cos[\omega(t - \mathscr{r}_-/c)]}{\mathscr{r}_-}\right\}$$

Review-radiation-dipole-V

$$V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t-r/c)]$$

Summary of approximations: $d \ll \lambda \ll r$.

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{-q_0\omega\sin[\omega(t-\mathscr{r}/c)]}{\mathscr{r}}\,dz$$

$$= -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t-r/c)]\,\hat{z}\,.$$

Review-radiation-dipole-E-B

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\theta} .$$

Max at θ=π/2

$$\vec{B}(\vec{r},t) = \frac{1}{c}\,\hat{r} \times \vec{E}(\vec{r},t) .$$

Total Power:

Integrate over a sphere at large r.

$$\langle P \rangle = \int \left\langle \vec{S} \right\rangle \cdot d\vec{a} = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right) \int \frac{\sin^2\theta}{r^2}\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{\mu_0 p_0^2 \omega^4}{12\pi c} .$$

= power radiated by of an antenna

Poynting Vector:

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{r}\,. \tag{16}$$

Intensity:

Average the Poynting vector over a complete cycle: $\langle\cos^2\rangle = 1/2$.

$$\left\langle\vec{S}\right\rangle = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{r}\,. \tag{17}$$

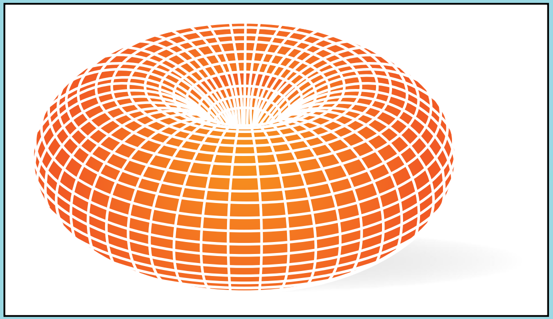


Image by MIT OpenCourseWare.

$$\langle P\rangle = \int\left\langle\vec{S}\right\rangle\cdot d\vec{a} = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\int\frac{\sin^2\theta}{r^2}\,r^2\sin\theta\,d\theta\,d\phi$$

$$= \frac{\mu_0 p_0^2\omega^4}{12\pi c}\,.$$

= power radiated by of an antenna

PROBLEM 7: RADIATION RESISTANCE of an antenna

Griffths Problem 11.3 (p. 450).

> Find the **radiation resistance** of the wire joining the two ends of the dipole. (This is the resistance that would give the same average power loss—to heat—as the oscillating dipole in fact puts out in the form of radiation.) Show that $R = 790(d/\lambda)^2\,\Omega$, where λ is the wavelength of the radiation. For the wires in an ordinary radio (say, $d = 5$ cm), should you worry about the radiative contribution to the total resistance?

Solution:

$I = dq/dt = q_0\,\omega \sin \omega t$

Problem 11.3

$P = I^2 R = q_0^2\omega^2 \sin^2(\omega t) R$ (Eq. 11.15) $\Rightarrow \langle P\rangle = \frac{1}{2}q_0^2\omega^2 R$. Equate this to Eq. 11.22:

$$\frac{1}{2}q_0^2\omega^2 R = \frac{\mu_0 q_0^2 d^2\omega^4}{12\pi c} \Rightarrow \boxed{R = \frac{\mu_0 d^2\omega^2}{6\pi c};} \text{ or, since } \omega = \frac{2\pi c}{\lambda},$$

$$R = \frac{\mu_0 d^2}{6\pi c}\frac{4\pi^2 c^2}{\lambda^2} = \frac{2}{3}\pi\mu_0 c\left(\frac{d}{\lambda}\right)^2 = \frac{2}{3}\pi(4\pi\times 10^{-7})(3\times 10^8)\left(\frac{d}{\lambda}\right)^2 = 80\pi^2\left(\frac{d}{\lambda}\right)^2\,\Omega = \boxed{789.6(d/\lambda)^2\,\Omega.}$$

For the wires in an ordinary radio, with $d = 5\times 10^{-2}$ m and (say) $\lambda = 10^3$ m, $R = 790(5\times 10^{-5})^2 = 2\times 10^{-6}\,\Omega$, which is negligible compared to the Ohmic resistance.

PROBLEM 8: A ROTATING ELECTRIC DIPOLE

Griffiths Problem 11.4 (p. 450).

A *rotating* electric dipole can be thought of as the superposition of two *oscillating* dipoles, one along the x axis, and the other along the y axis (Fig. 11.7), with the latter out of phase by 90°:

$$\vec{p} = p_0[\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}] \ .$$

Using the principle of superposition and Eqs. (11.18) and (11.19) (perhaps in the form suggested by Prob. 11.2), find the fields of a rotating dipole. Also find the Poynting vector and the intensity of the radiation. Sketch the intensity profile as a function of the polar angle θ, and calculate the total power radiated. Does the answer seem reasonable? (Note that power, being *quadratic* in the fields, does not satisfy the superposition principle. In this instance, however, it *seems* to. Can you account for this?)

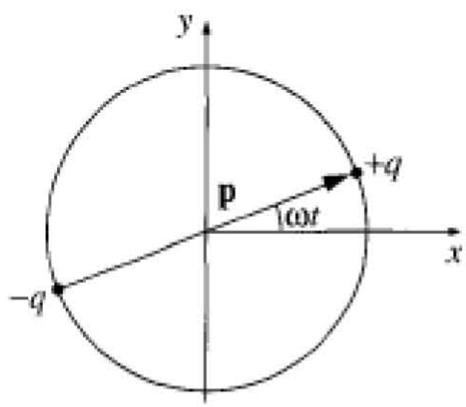


Figure 11.7

Magnetic Dipole Radiation

Consider a wire loop of radius b, with alternating current

$$I(t) = I_0 \cos(\omega t) \;. \qquad (19)$$

with magnetic dipole moment

$$\vec{m}(t) = \pi b^2 I(t)\,\hat{z} = m_0 \cos(\omega t)\,\hat{z} \;, \qquad (20)$$

where

$$m_0 \equiv \pi b^2 I_0 \;. \qquad (21)$$

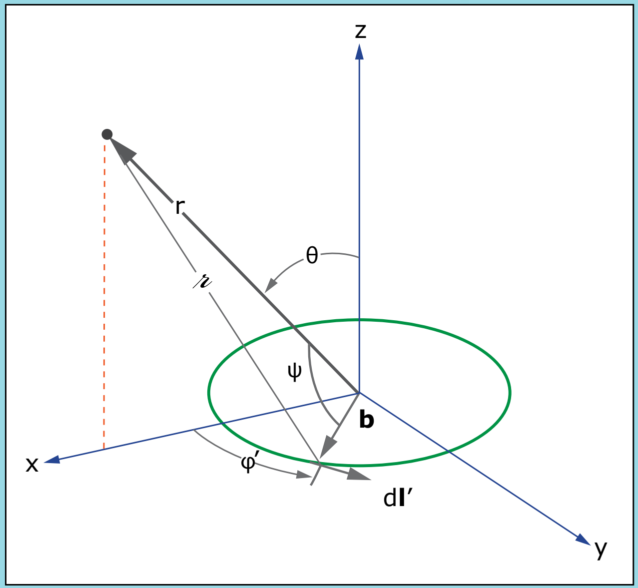


Image by MIT OpenCourseWare.

$$\vec{E} = -\frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\phi} \;.$$

Max at θ=π/2

Compared to the electric dipole radiation,

$$p_0 \longrightarrow \frac{m_0}{c} \;, \qquad \hat{\theta} \longrightarrow -\hat{\phi} \;.$$

M1/E1 ~ c p_o/m_0 ~v/c << 1

As always for radiation,

L-C radiator

$$\vec{B}(\vec{r}, t) = \frac{1}{c}\,\hat{r} \times \vec{E}(\vec{r}, t) \;.$$

Consider an arbitrary time-dependent charge distribution $\rho(\vec{r}, t)$. Then

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} d\tau' . \qquad (25)$$

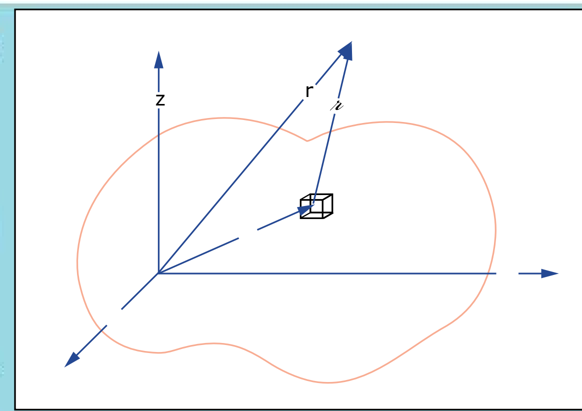


Image by MIT OpenCourseWare.

Expand $1/|\vec{r} - \vec{r}'|$ and t_r in powers of $\vec{r}'$, using similar approximations as before. Minor difference: here we have no ω. Previously we assumed that $d \ll \lambda$ or equivalently $\omega \ll c/d$. Here we need to assume that $|\ddot{p}/\dot{p}| \ll c/d$, with similar bounds on higher time derivatives.

$$V(\vec{r}, t) \simeq \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r} + \frac{\hat{r} \cdot \vec{p}(t_0)}{r^2} + \frac{\hat{r} \cdot \dot{\vec{p}}(t_0)}{rc} \right] . \qquad (26)$$

where t_0 is the retarded time at the origin. Final result:

$$\begin{aligned} \vec{E}(\vec{r}, t) &\simeq \frac{\mu_0}{4\pi r} [(\hat{r} \cdot \ddot{\vec{p}})\hat{r} - \ddot{\vec{p}}] \\ \vec{B}(\vec{r}, t) &\simeq -\frac{\mu_0}{4\pi rc} [\hat{r} \times \ddot{\vec{p}}] . \end{aligned} \qquad (27)$$

Although this looks different, it is really the same as what we had for the simple electric dipole, changing to vector notation and replacing $-\omega^2 \vec{p}_0$ by $\ddot{\vec{p}}$.

The Liénard-Wiechert Potentials

Finally,

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}-\vec{r}_p|\left(1-\frac{\vec{v}_p}{c}\cdot\frac{\vec{r}-\vec{r}_p}{|\vec{r}-\vec{r}_p|}\right)} \,,$$

where $\vec{r}_p$ and $\vec{v}_p$ are the position and velocity of the particle
Similarly, starting with

$$\vec{J}(\vec{r},t) = q\vec{v}\delta^3\left(\vec{r}-\vec{r}_p(t)\right)$$

for a point particle, we find

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{q\vec{v}_p}{|\vec{r}-\vec{r}_p|\left(1-\frac{\vec{v}_p}{c}\cdot\frac{\vec{r}-\vec{r}_p}{|\vec{r}-\vec{r}_p|}\right)} = \frac{\vec{v}_p}{c^2}V(\vec{r},t) \,.$$

Review-radiation-point-Q-E

Radiation by Point Charges

Recall the fields (found by differentiating the Liénard-Wiechert potentials):

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{|\vec{r}-\vec{r}_p|}{(\vec{u}\cdot(\vec{r}-\vec{r}_p))^3}\left[(c^2-v_p^2)\vec{u} + (\vec{r}-\vec{r}_p)\times(\vec{u}\times\vec{a}_p)\right] \ , \qquad (28)$$

where

$$\vec{u} = c\hat{\boldsymbol{\mathscr{r}}} - \vec{v}_p \ , \qquad \vec{\boldsymbol{\mathscr{r}}} = \vec{r} - \vec{r}_p \ . \qquad (29)$$

If $\vec{v}_p = 0$ (at the retarded time), then

Vp << c

$$\vec{E}_{\rm rad} = \frac{q}{4\pi\epsilon_0 c^2|\vec{r}-\vec{r}\,'|}\left[\hat{\boldsymbol{\mathscr{r}}}\times(\hat{\boldsymbol{\mathscr{r}}}\times\vec{a}_p)\right] \ . \qquad (30)$$

The Fields of a Point Charge

Differentiating the Liénard-Wiechert potentials, after several pages, one finds

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{|\vec{r}-\vec{r}_p|}{(\vec{u}\cdot(\vec{r}-\vec{r}_p))^3}\left[(c^2-v_p^2)\vec{u}+(\vec{r}-\vec{r}_p)\times(\vec{u}\times\vec{a}_p)\right] \;, \qquad (39)$$

where

$$\vec{u} = c\frac{\vec{r}-\vec{r}_p}{|\vec{r}-\vec{r}_p|}-\vec{v}_p \;. \qquad (40)$$

=> c-v for linear

And

$$\vec{B}(\vec{r},t) = \frac{1}{c}\frac{\vec{r}-\vec{r}_p}{|\vec{r}-\vec{r}_p|}\times\vec{E}(\vec{r},t) \;. \qquad (41)$$

For Vp << c

Poynting Vector (particle at rest):

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0 c}|\vec{E}_{\text{rad}}|^2\,\hat{\boldsymbol{\imath}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\boldsymbol{\imath}^2}\right)\hat{\boldsymbol{\imath}}\,.$$

Total Power (Larmor formula):

g-2 experiment to
measure e dipole moment

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}\,.$$

Total Power (Larmor formula):

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \,. \tag{32}$$

Liénard's Generalization if $\vec{v}_p \neq 0$:

For Vp → c

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v} \times \vec{a}}{c}\right|^2\right) = \underbrace{\frac{\mu_0 q^2}{6\pi m_0^2 c}\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}}_{\text{For relativists only}} \,. \tag{33}$$

P = rate at which energy that is destined to become radiation is leaving the particle.

PROBLEM 9: RADIATION AND THE BOHR ATOM

Griffiths Problem 11.14 (pp. 464–465).

> In Bohr's theory of hydrogen, the electron in its ground state was supposed to travel in a circle of radius 5×10^{-11} m, held in orbit by the Coulomb attraction of the proton. According to classical electrodynamics, this electron should radiate, and hence spiral in to the nucleus. Show that $v \ll c$ for most of the trip (so you can use the Larmor formula), and calculate the lifespan of Bohr's atom. (Assume each revolution is essentially circular.)

Solution:

Charges in a given quantum state re-absorbs its own fields

Problem 11.14

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} = ma = m\frac{v^2}{r} \Rightarrow v = \sqrt{\frac{1}{4\pi\epsilon_0}\frac{q^2}{mr}}.$$ At the beginning ($r_0 = 0.5$ Å),

$$\frac{v}{c} = \left[\frac{(1.6\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.11\times10^{-31})(5\times10^{-11})}\right]^{-1/2}\frac{1}{3\times10^8} = 0.0075,$$

and when the radius is one hundredth of this v/c is only 10 times greater (0.075), so for *most* of the trip the velocity is safely nonrelativistic.

From the Larmor formula, $P = \frac{\mu_0 q^2}{6\pi c}\left(\frac{v^2}{r}\right)^2 = \frac{\mu_0 q^2}{6\pi c}\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{mr^2}\right)^2$ (since $a = v^2/r$), and $P = -dU/dt$, where U is the (total) energy of the electron:

$$U = U_{\text{kin}} + U_{\text{pot}} = \frac{1}{2}mv^2 - \frac{1}{4\pi\epsilon_0}\frac{q^2}{r} = \frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{r}\right) - \frac{1}{4\pi\epsilon_0}\frac{q^2}{r} = -\frac{1}{8\pi\epsilon_0}\frac{q^2}{r}.$$

So $-\frac{dU}{dt} = -\frac{1}{8\pi\epsilon_0}\frac{q^2}{r^2}\frac{dr}{dt} = P = \frac{q^2}{6\pi\epsilon_0 c^3}\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{mr^2}\right)^2$, and hence $\frac{dr}{dt} = -\frac{1}{3c}\left(\frac{q^2}{2\pi\epsilon_0 mc}\right)^2\frac{1}{r^2}$, or

$$dt = -3c\left(\frac{2\pi\epsilon_0 mc}{q^2}\right)^2 r^2\,dr \Rightarrow t = -3c\left(\frac{2\pi\epsilon_0 mc}{q^2}\right)^2\int_{r_0}^{0} r^2\,dr = \boxed{c\left(\frac{2\pi\epsilon_0 mc}{q^2}\right)^2 r_0^3}$$

$$= (3\times10^8)\left[\frac{2\pi(8.85\times10^{-12})(9.11\times10^{-31})(3\times10^8)}{(1.6\times10^{-19})^2}\right]^2(5\times10^{-11})^3 = \boxed{1.3\times10^{-11}\text{ s.}}$$ (Not very long!)

Radiation Reaction Force

For a nonrelativistic particle, radiation power is given by the Larmor formula:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \, .$$

The run-away solution
In the textbook
misused the formula

Maybe then,

$$\vec{F}_{\rm rad} \cdot \vec{v} \overset{?}{=} -\frac{\mu_0 q^2 a^2}{6\pi c} < 0 \, . \tag{34}$$

But not necessarily, since $P_{\rm Larmor}$ represents the power destined to become radiation. There can be other energy exchanges with the near fields. Since the RHS of Eq. (34) does not depend on $\vec{v}$, there is no way to match the two sides.

Radiation reaction Power =? Radiated power

$$\frac{\mu_0 q^2}{6\pi c}\dot{\vec{a}} \cdot \vec{v} \stackrel{?}{=} -\frac{\mu_0 q^2 a^2}{6\pi c}$$

V * da/dt = a^2?

PROBLEM 10: THREE EXAMPLES OF RADIATION REACTION

Griffiths Problem 11.17 (p. 468).

V * da/dt = a²?

Problem 11.17

(a) To counteract the radiation reaction (Eq. 11.80), you must exert a force $\mathbf{F}_e = -\frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}$. *For circular motion*, $\mathbf{r}(t) = R[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}]$, $\mathbf{v}(t) = \dot{\mathbf{r}} = R\omega[-\sin(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t)\,\hat{\mathbf{y}}]$;

$\mathbf{a}(t) = \dot{\mathbf{v}} = -R\omega^2[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}] = -\omega^2\mathbf{r}$; $\dot{\mathbf{a}} = -\omega^2\dot{\mathbf{r}} = -\omega^2\mathbf{v}$. So $\boxed{\mathbf{F}_e = \frac{\mu_0 q^2}{6\pi c}\omega^2\mathbf{v}.}$

$\boxed{P_e = \mathbf{F}_e\cdot\mathbf{v} = \frac{\mu_0 q^2}{6\pi c}\omega^2 v^2.}$ This is the power *you* must supply.

Meanwhile, the power *radiated* is (Eq. 11.70) $P_{\text{rad}} = \frac{\mu_0 q^2 a^2}{6\pi c}$, and $a^2 = \omega^4 r^2 = \omega^4 R^2 = \omega^2 v^2$, so $P_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\omega^2 v^2$, and the two expressions agree.

(b) *For simple harmonic motion*, $\mathbf{r}(t) = A\cos(\omega t)\,\hat{\mathbf{z}}$; $\mathbf{v} = \dot{\mathbf{r}} = -A\omega\sin(\omega t)\,\hat{\mathbf{z}}$; $\mathbf{a} = \dot{\mathbf{v}} = -A\omega^2\cos(\omega t)\,\hat{\mathbf{z}} = -\omega^2\mathbf{r}$; $\dot{\mathbf{a}} = -\omega^2\dot{\mathbf{r}} = -\omega^2\mathbf{v}$. So $\boxed{\mathbf{F}_e = \frac{\mu_0 q^2}{6\pi c}\omega^2\mathbf{v};\ P_e = \frac{\mu_0 q^2}{6\pi c}\omega^2 v^2.}$ But this time $a^2 = \omega^4 r^2 = \omega^4 A^2\cos^2(\omega t)$,

whereas $\omega^2 v^2 = \omega^4 A^2 \sin^2(\omega t)$, so

$$P_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\omega^4 A^2\cos^2(\omega t) \neq P_e = \frac{\mu_0 q^2}{6\pi c}\omega^4 A^2\sin^2(\omega t);$$

the power you deliver is *not* equal to the power radiated. However, since the time *averages* of $\sin^2(\omega t)$ and $\cos^2(\omega t)$ are equal (to wit: 1/2), *over a full cycle* the energy radiated is the same as the energy input. (In the mean time energy is evidently being stored temporarily in the nearby fields.)

Paradox of radiation reaction force?

$$\vec{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{\vec{a}} \tag{37}$$

is the Abraham-Lorentz formula for the radiation reaction force.

(c) Consider the case of a particle in free fall (constant acceleration g). What is the radiation reaction force? What is the power radiated? Comment on these results.

(c) *In free fall,* $\mathbf{v}(t) = \frac{1}{2}gt^2\,\hat{\mathbf{y}}$; $\mathbf{v} = gt\,\hat{\mathbf{y}}$; $\mathbf{a} = g\,\hat{\mathbf{y}}$; $\dot{\mathbf{a}} = 0$. So $\boxed{\mathbf{F}_e = 0;}$ the radiation reaction is zero, and hence $\boxed{P_e = 0.}$ But there *is* radiation: $\boxed{P_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}g^2.}$ Evidently energy is being continuously extracted from the nearby fields. This paradox persists even in the *exact* solution (where we do *not* assume $v \ll c$, as in the Larmor formula and the Abraham-Lorentz formula)—see Prob. 11.31.

???

Two errors cause this paradox

1. a =? g = constant?
2. Wrong application of the formula

(c) *In free fall*, $\mathbf{v}(t) = \frac{1}{2}gt^2\,\hat{\mathbf{y}}$; $\mathbf{v} = gt\,\hat{\mathbf{y}}$; $\mathbf{a} = g\,\hat{\mathbf{y}}$; $\dot{\mathbf{a}} = 0$. So $\boxed{\mathbf{F}_e = 0;}$ the radiation reaction is zero, and hence $\boxed{P_e = 0.}$ But there *is* radiation: $\boxed{P_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} g^2.}$ Evidently energy is being continuously extracted from the nearby fields. This paradox persists even in the *exact* solution (where we do *not* assume $v \ll c$, as in the Larmor formula and the Abraham-Lorentz formula)—see Prob. 11.31.

- Rest m = gravitational mass? ma = mg?
- $ma = mg - F_{rad}$ →
- $a = g\,[1 - e^{-t/\alpha}]$ $a = g - \alpha\, da/dt$
- a = 0 at t = 0!!!! da/dt not = 0!

Radiation reaction Power =? Radiated power

$$\frac{\mu_0 q^2}{6\pi c}\dot{\vec{a}} \cdot \vec{v} \stackrel{?}{=} -\frac{\mu_0 q^2 a^2}{6\pi c}$$

Derivation of Radiation Reaction Force

For cyclic motion, the total energy loss over one cycle should match the energy loss described by the Larmor formula. If a cycle extends from t_1 to t_2 then

$$\int_{t_1}^{t_2} a^2\,\mathrm{d}t = \int \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\,\mathrm{d}t = \underbrace{\left(\vec{v}\cdot\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right)\Big|_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2}\frac{\mathrm{d}^2\vec{v}}{\mathrm{d}t^2}\cdot\vec{v}\,\mathrm{d}t\ . \quad (35)$$

So, energy conservation holds if

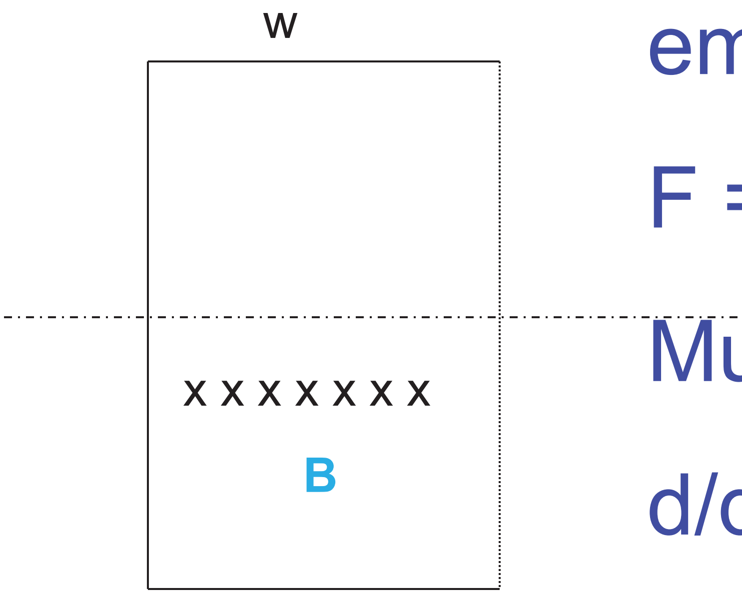
$$\int_{t_1}^{t_2}\left(\vec{F}_{\mathrm{rad}} - \frac{\mu_0 q^2}{6\pi c}\,\dot{\vec{a}}\right)\cdot\vec{v}\,\mathrm{d}t = 0\ , \quad (36)$$

which will hold if

$$\vec{F}_{\mathrm{rad}} = \frac{\mu_0 q^2}{6\pi c}\,\dot{\vec{a}}\ . \quad (37)$$

Free fall is NOT cyclic! The v * a terms at t1 and t2 must be kept to make Eq. 35 hold!

Last example: against free fall
A vertical bouncing loop in B-field



emf = -B w dy/dt –L dI/dt = IR

F = ma = I w B – mg + F_{rad}

Multiply by v

d/dt[1/2 mv^2 + mgy +1/2 LI^2]

=-I^2R - power radiated

Pset10 2c

Since the fringing fields can be ignored, this implies, like most capacitors, the linear dimension L >> d, which is the distance between the plates, so Ampere loop integral yields

$\sim\ 2\int_0^L E_{\text{ind},\,y}(z = d/2)\, dy + 2\int_{-d/2}^{d/2} E_{\text{ind},\,z}(y = L)\, dz$

$\sim 2\int_0^L E_{\text{ind},\,y}\left(z = \frac{d}{2}\right) dy$

$\sim\ 2L * E_{\text{ind},y}(z=d/2)$

which equals $d\Phi_B/dt$ and Problem #2-C can be solved accordingly just like in the original version of the solution. As long as L>>d, $E_{\text{ind},y}(z=d/2)$ can be shown to be constant, independent of y by taking many smaller loops, similar to the case of an infinite solenoid.

Pset10 2c

Faraday's law is independent of the charge distributions on the plates. Even if the electrons in the conductor will rearrange to their equilibrium configuration under the influence of the induced electric field to stop the current, Using Faraday's law we still have $E_{ind,y} \sim d\Phi_B/dt\,/(2L+2d) \sim d\Phi_B/dt\,/(2L)$. E_{ind} does not vanish inside the conducting plates, and $E_{ind,y}$ does not vanish just outside the plates on the horizontal surfaces. Surely E_y must be continuous.

To further elaborate, let me define $E_{total\,y} = E_{ind,y} + E_{static\text{-}y}$

Where $E_{static\text{-}y}$ is the static field produced by the (re-distributed) charges along one plate and it will produce an internal force between the charges on the plate and thus will not produce any net horizontal force. $E_{ind,y}$ will produce a net horizontal force