

PROFESSOR: So, WKB approximation, or semiclassical approximation.

So this is work due to three people-- Wentzel, Kramers, and Brillouin-- in that incredible year, 1926, where so much of quantum mechanics was figured out. As it turns with many of these discoveries, once the discoveries were made, people figured out that somebody did them before. And that person was a mathematician, Jeffreys, who did it three years earlier, in 1923. The work had not become very popular.

So some people write JWKB. But we will not do that. We'll note Jeffreys, but we'll follow this more standard notation.

So these people were dealing with differential equations with slowly-varying spatial coefficients. That was the main thing. So we're continuing our approximation methods. We've done perturbation theory. We will add little pieces to the Hamiltonian. Now we consider things that are slowly varying in space. You might have a very simple Hamiltonian where nothing varies in space-- a constant potential. But as soon as the potential starts varying slowly, you have approximation methods. Those are the methods we're considering now.

We will also consider, after we've finished WKB, time-dependent perturbation theory, where still things start slowly varying in time.

So we'll have many, many things to do still. So this is called, also, the semiclassical approximation. Because classical physics gives you intuition about the quantum wave function. So it is a lesson in which you want to learn something about the quantum wave function, and you learn it by using classical physics.

A quantity that is relevant here is that the Broglie wavelength of a particle. This is the Broglie.

And many people say that semiclassical approximation has to do with the fact that the quantum mechanic effects are not that important. That may happen. The Broglie wavelength is much smaller than the physically relevant sizes of your apparatus. So you have a particle like an electron. And if the Broglie wavelength is very small compared to the aperture in the screen, the letter will go almost like a classical particle. When the Broglie wavelength is comparable with the size of the aperture in the screen, you will get diffraction effects, and the electrons will do quantum mechanical things.

So the semiclassical approximation has to do with λ being smaller than the length scale, L , of your physical problem.

We will refine this. In fact, the whole search of understanding the semiclassical approximation is all about understanding this better. Because it's a little subtle. We will end up deciding that what you need is that the Broglie wavelength, suitably generalized, varies very slowly. We'll have to generalize the concept of the Broglie wavelength. We might get to it today.

Mathematically, you can say, OK, I want this to be sufficiently small. So semiclassical limit was just take \hbar going to 0. You should complain, of course. \hbar is a constant of nature. I cannot take it equal to 0. But on the other hand, they could imagine other universes, maybe, where \hbar has different smaller and smaller values in which quantum mechanical effects don't set in until much smaller scales.

But at the end of the day, I will try to consider \hbar to be small as an idea underlying a semiclassical approximation. And the intuition is that for \hbar small-- we cannot tune it, but we can say it-- λ becomes small. The λ , the Broglie, λ .

You're taking a quantity with units-- \hbar is units. And saying it's small. It goes against lots of things. Things with units are not supposed to be small.

You should compare it with the situation we had before. We had a very nice and clean situation with perturbation theory, where we had a unit-free thing that we consider it to be small. This time, we're going to try to consider \hbar to be small. And it's going to be more delicate because it has units. It's going to be a more complicated story. The physics is interesting, and that's why this approximation is harder in some ways to understand than the ones we've done in perturbation theory.

So how does this begin? It begins by thinking of a particle in a potential, V of x . And the particle has some energy, E . And then, if it's classical, as we're imagining now, it could be a three-dimensional potential. My sketch of course is just for one-dimensional. This is E . And you can solve for p^2 , $2m(E - V)$. This is a notion of local momentum because it depends on x . It's the momentum the particle would have. When it is at some position, x , you will have momentum, p , of x .

And now, nobody forbids you from declaring that you're going to define a position-dependent Broglie wavelength, which is going to be λ of x is your definition, which is equal to $2\pi\hbar/p$

bar over p of x . And it's going to be local the Broglie.

You see that the Broglie wavelength, when you first started in 1924, was considered for a free particle-- always the [INAUDIBLE]. You have a free particle with some momentum it has at the Broglie wavelength.

Why was the Broglie wavelength important? Because when you write the wave function, it's a wave with wavelength equal to the Broglie wavelength. The wave function with the Broglie wavelength solves the Schrodinger equation. That's sort of how it all came about. But it was all defined for a free particle. The Broglie defined it for a free particle with some momentum, p . And it all made sense, because you could write a wave function using the Broglie momentum. It was in fact E to the ipx over \hbar . That was your wave function.

But here, you have a particle moving with varying momenta. And we don't know how to write the solution of the Schrodinger equation, but there's classically this concept, and we could define the local, the Broglie wavelength, and we will have to discover what it means or how it shows up. But it probably shows up in some way.

So there's two cases that probably we should consider. But before that, we looked at the Schrodinger equation-- the time-independent Schrodinger equation.

So it's minus \hbar^2 over $2m$ Laplacian of ψ of x is equal to E minus V of x , ψ of x . All these vectors-- that's why I'm Laplacian, not second derivatives.

But the right-hand side-- well, if I put the $2m$ to the other side, I get minus \hbar^2 Laplacian of ψ of x , is equal to $2m E$ minus Vx . That's p^2 of x , ψ of x . That's the local momentum squared. Maybe it's a curiosity, but it's now nice that the left-hand side is the momentum operator squared-- so it's \hat{p}^2 on the wave function-- is equal to p^2 of x times a wave function.

Kind of a nice result. Nice-looking Schrodinger equation. I don't know if you've seen it like that before. It's an operator on the left. And on the right, almost like an eigenvalue. It's an illegal eigenvalue. If it were a real eigenvalue problem, there should be a number, not a function here. But it's a function that acts a little like an eigenvalue. It's a nice way of thinking of the Schrodinger equation in the semiclassical approximation.

Well, no. No approximation here so far in the semiclassical language, in which you call this the

local momentum. And thus defined, it certainly is an exact statement-- no approximation whatsoever.

OK, now I want to know these two circumstances of course. If you have E greater than v , you're in an allowed region. And p squared is really $2m(E - v)$. And it's convenient to define k squared of x , the wave number. You remember that $p = \hbar k$ was good notation. We use the wave number sometimes. We'll have here a local wave number as well. And if you're in the forbidden region, which energy is less than v -- forbidden region-- then p squared is positive, is $2m(v - E)$. That is positive this time, because v is greater than E . And that, we always used to call the penetration constant, the κ , for wave functions that decay exponentially.

So we call this κ squared of x . A local decaying factor wave number. It can be thought as an imaginary wave number. But that's notation.

So, so far, so good. We will need one more piece of intuition as we work with this semiclassical approximation. We have not done any approximation yet. The approximation will come soon, as we will begin solving the Schrodinger equation under those circumstances. And we will take \hbar as an expansion parameter-- will be fine and correct, but a little subtle.

So the thing that will help us many times to understand these things is to write the wave function as a complex number. So suppose you have a wave function. We will write it as a complex number in the polar form-- radius times $e^{i\phi}$. So the radius is going to be ρ of x . The probability density, I claim, is this. And there is a phase that I will write as S of x and t .

It looks a little like an action. And it's, to some degree, the beginning of the path into rule formulation. It has lots of connections with the action principle.

So here, ρ and S are going to be real. So this is truly scale factor in front that determines the value. This is a pure phase, because it's an imaginary number times a real thing. S has units of \hbar , or angular momentum. And indeed, ψ squared is equal to ρ of x and t . If you loop ψ squared.

The reason we focus on this wave function is that our solutions of WKB are going to have exactly that form. So we need to have an intuition as to what the observables of this wave function are.

So the other observable is the current density. If you remember, it's \hbar over m , imaginary part of $\psi^* \text{grad} \psi$. So this must be calculated.

So what is this? Gradient of ψ -- we must take the gradient of this. Gradient's a derivative, so it acts on one, acts on the other. When it acts on the first, it first acts as the relative $1/2$ squared of ρ of x and t , times the gradient of ρ , times the phase factor, plus now I have to take the gradient on this quantity, and this will bring down an i , an \hbar , the gradient of S , and then multiplied by the whole wave function, because this factor remains and the exponential remains.

Now we can multiply by ψ^* to form what we need to get for the current. So $\psi^* \text{grad} \psi$ [INAUDIBLE] ψ . If I multiply by that, I'm multiplying by the top line. For the first factor, I get $1/2$ gradient of ρ . The exponentials cancel. And for the second part, we get plus i over \hbar , gradient of S , times ψ^2 , which is ρ . The imaginary part of this is equal 1 over \hbar , ρ gradient of S .

So finally, the current, which is \hbar over m , times that imaginary part, is ρ gradient of S , over m .

A very nice formula. Basically, it says that the phase factor in the wave function determines the probability current. And it also says that if you want to think of this, here are the surfaces of constant phase. Here is our space. And S constant. So here is one valley of the phase, another valley of the phase. Those are surfaces in space of constant phase. The current is orthogonal to that. So the current is proportional to the gradient. The gradient of a function is always proportional. It's a normal vector to the surfaces of constant values. So the current is orthogonal to the surfaces of constant phase.

If you have a fluid mechanics interpretation, J is ρv in fluids. So, so far, everything I've said could have been said in 804. These are properties of a general wave function. This is how you compute the current. The useful thing is that our WKB wave functions are going to be presented in that language. So if you think of the analogy with fluid mechanics. The current is the charge density times the velocity, and therefore the velocity would be identified with gradient of S over m . Or the momentum would be identified with gradient of S .

That's not a quantum mechanical rigorous identification. Because gradient of S is a function. And therefore, p there would be a function. And a function of momentum-- momentum in

quantum mechanics is an operator, and it has eigenvalues, which are numbers. They're not functions. But we already have seen the beginning of some momentum function. So that analogy is actually quite nice.

Let me give you an example, and conclude with that. If you have a free particle, you have a wave function ψ of x and t , which is E , to the $i p x$ over \hbar , times minus $i E t$ over \hbar .

So in this case, ρ is equal to 1, an S -- remember, S is read by having an i over an \hbar out. And that gives you $p \cdot x$ minus $E t$.

So for a free particle, the gradient of S is indeed just the momentum. You take this gradient, and it's that. And therefore, that's a rigorous interpretation when you have a free particle, that the gradient of S is going to be the momentum. Interestingly, the derivative of S with respect to time is minus the energy.

What will happen in the semiclassical approximation is that this S over there will depend on x , and this p will depend on x , and it will be this p that depends on x here. And we will see how to solve this equation in an approximation scheme where the changes are a little slow. And the notation S here is also motivated because actions in classical mechanics actually have this property.

The gradient of the action-- if you think of the action as a function of coordinates, which is something you don't usually do, but if you do, in somewhat advanced classical mechanics, you see that the derivatives of the action-- spatial derivatives are the momentum, and the time derivatives are the energy. It's a nice relation between classical mechanics, and justifies once more the name of semiclassical approximation, which we will continue to develop next time. See you then.