PROFESSOR: It's time for step two. What don't we know here? Well, we don't know anything about the state correction. So n1, what is n1?

So step two is find the piece of n1 k, the piece that is in the space v hat. So I will use this notation, a bar here, saying, of this vector, the piece in that subspace.

Then we, of course, need to find the piece in the degenerate subspace. Remember, the corrections are orthogonal to the original state, but the corrections can have a piece on vn. The vn has more states. So it can have a piece on vn, and that's typically the part that is a little hard to do.

So what do we do here? We take the order one equation, lambda equal 1, and we hit it with p0. So we'll hit with p0 on the order lambda equation. So think of the p0 vector, bra, appearing here. We know the energy of it, so this will become a number. So we get Ep0 minus En0 p0 n1 k.

And indeed, when you see here-- you say, oh, this is my unknown. I don't know the first order correction, and I'm getting the components of this unknown correction along the v hat space. That's why this calculation can only give me the piece of n1 k along v hat.

And then, what do we have here? We have p0 Enk1 minus delta H n0 k. It's just that order one equation calculated here. So what is it? You recognize the challenge here is always sort of trying to remember what the symbols mean and simplify things.

So for example, here. This is a number. So what you must ask is whether this vector is orthogonal to this one because you're going to get an inner product. And this is a vector in v hat, and this is a vector in v capital N. So therefore, they are orthogonal. So this term vanishes by orthogonality of these two.

On the other hand, we are left with this part here. Now, that's a matrix element of delta H between a degenerate state and the rest of the states. So that's something you have to calculate. There's no way around it. So we use our notation. And in our notation, this will be delta Hpnk.

All right. So this gives us-- we're done with this equation. It was simple. No big sweat. We now have these coefficients, because these are numbers, and we can just divide over here. And this factor is nonzero because all the p states that are in v hat have different energy from our degenerate ones.

So there are the components of this. So we have p0 n1 k is equal to minus delta Hpnk divided by this quantity, Ep0 minus En0. And therefore, we can write n1 k. If you have the inner products of a state with p arbitrary, the state is the sum of other p's times that inner product.

So I'll write it as minus because of that minus here. The sum over p of delta Hpnk over Ep0 minus En0 p0 written like that. This is something-- maybe it sounds a little fast, but if you have a basis vector, alpha i with psi being the number beta i, the state psi is equal to the sum of beta i alpha i.

Once you have the components, the vector can be reconstructed as the components times the basis vectors. So that's what we did here. We have the components here. And therefore, the vector is the components times the basis vectors.

You can check that. If that makes you a little uneasy, just put the p0 here, and recalculate that, and you will get that answer. Now, as written it there, it's really not precise. You're making, technically, a mistake. This equation is technically wrong because that's not n1 k. That is only the components of n1 k along v hat. So this is v hat.

If n1 k has something along the degenerate subspace, we haven't found it. But this much we found. We've found part of the correction. So degeneracy is always a complicated thing. So we're going to try to find-- we're going to find, in fact, now the component of the correction around the degenerate space.

Now, in nondegenerate perturbation theory we manage to use this equation to calculate fully the state. But now we've used this equation all the way. We put states in the degenerate subspace, and we found the energies. We put, in the other part of the space, v hat, and we found the part of n1 along v hat. And we ran out of things from that order one equation. It has no more information.

So somehow it must be that the next equation that we usually need to go to second order in energy will tell us something about the missing part of n1. So that's the surprising thing. You have to go to order lambda square to find the first order correction to the degenerate part of the state. That's why degenerate perturbation theory is famous for its complication. You really need to go pretty high to find the things.

So let's do it. So this is step 3. You hit the order lambda squared equation with n0 l. And therefore, the left-hand side is going to be 0 because that operator on the left-hand side always kills those states.

Now, you could have been a little sloppy here. I was sloppy when I first solved those equations. I didn't say, OK, this is v hat. I said, that's the answer. That's the whole state. There's nothing along the degenerate subspace.

But then, if you stop there, you can live happily the rest of your life. But if you go to the next equation and find that it's wrong, there was a piece along the degenerate subspace. So let's see that.

So we'll also write, as we've started to do, n1 k being equal to n1 k along v hat plus the piece of n1 k along the vn. And this one we got. And this one, well, is it there? Do we need it? Does the state receive a correction in that space or not?

Well, let's do the work. Let's hit this with those states. So what happens? We have n0 l Enk 1 minus delta H n1 k. That's our term there. Actually, I will write it twice. So I'll copy it again.

n0 k l Enk 1 minus delta H n1 k. Why do I copy it again? Because I'll just put here on v hat and on vn. So it was one term with an n1 k. But you know, the n1 k vector has components along two subspaces. And therefore, we might as well put each one separately so we can think about them in a clear way.

Then, what else? There's one more term, the En2 k. And that is relatively simple because that's Enk 2. And then we have the overlap of an n0 l with an n0 k which is a delta lk. So this whole thing must be 0. That was the right-hand side of the equation. But the left-hand side of the equation was 0, so that's 0.

OK. So now we have to think about these terms. What do we know? What is 0 to begin with? What is not 0? This term in here, we know n1 k already. We found it there. So that's nice. We know n1 k. We know this corrections. But this is simpler because n0 l is in the degenerate subspace. This is a number, and this is in b hat, so that's 0 again.

OK. So that's a simplification. Here we have a very interesting situation. We have the delta H, and we have this basis state of the degenerate subspace. We know that delta H is diagonal in the degenerate subspace, but is this n0 l an eigenvalue of delta H?

Not quite because being diagonal, these basis vectors make delta H diagonal. So delta H is diagonal in this subspace, but doesn't mean this thing is an eigenvector because it can give you something outside the degenerate subspace. So we cannot quite just say that the eigenvalue of this is the first energy correction.

But actually, we can. Let me explain that. So let's look at this term alone. With this term alone-- now this time I cannot kill this constant because this is in the degenerate subspace, and this is in the degenerate subspace. So we will have to deal with that.

So let's do this. So output here. n0 l delta H n1 k. We're going to try to simplify this. And the way to do it-- this is n1 k, very important in vn-- is to insert the resolution of the identity here. So I'll do it. We'll put n0 l delta H, and now we'll put the whole resolution of the identity. So we'll put the sum over q n0 q n0 q plus a sum over p p0 p0, all acting on n1 k of vn.

I actually want to remark that all what I'm doing in this line-- so let's break this. The equation was up to here, and we decided to try to understand this term. So we're trying to understand this term. Just that term.

Now, delta H here is acting between degenerate eigenstate here and here, some arbitrary state in the degenerate subspace. So we go here and we say, OK, since this is in the degenerate subspace, this inner product with vector in v hat, this is 0. So that part of the resolution of the identity is not relevant.

If that part of the resolution of the identity is not relevant, we are left with n sum over q n0 l delta H n0 q n0 q times n1 k vn. And now you can use that this matrix is, indeed, diagonal in the degenerate subspace. And therefore, this matrix element is the energy Enl 1, the first order correction, times a delta lq.

And then we can do the sum over q because there is a delta function here. And this becomes Enl 1 n0 l n1 k vn. So this is very nice. It took us a little bit of work but look what has happened. This delta H term is going to become the same structure, n0 l n1 k overlap. So it's great progress.

So what does our equation become? Well, from the first term, this is the rest of the end of the aside. From the first term we have minus n0 l delta H n1 k in v hat. That's all that was left from that first term. And this term is known because n1 is known along the rest of the Hilbert space.

The second term over there that was giving us trouble has become something very simple. It has become Enk minus Enl, both 1, multiplied by n0 l n1 k. I could put this vn, or now I may not put it either because I am already projecting to a state in the degenerate subspace. So I am finding the component along the degenerate subspace. So even if n1 k had a piece along v hat it would drop out here. So it's completely legal, and it's simpler to erase the vn.

And then the last term is still there, plus Enk 2 delta lk equals 0. OK. This is our master equation. After thinking of this equation for a while and using our properties, we got this far. It's a very nice equation.

It does give you the second order energies. We were looking for the part of the state along the degenerate subspace. This was our main unknown. But still, we can get the energies. Why? Because we can take l equals to k, in which case the term that we don't know drops out, because nl equals to k, these things cancel.

So when l is equal to k we get that Enk 2, the second order correction to the energy, is n0 k delta H n1 k in v hat. That's it. That's your second correction to the energy, and that's very nice.

Why do we say that's it? Because we actually have the answer for n1. So we can find the complete formula for this. I'll write it here. After a couple of steps of algebra this gives minus the sum over p delta H nkp over Ep0 minus En0 squared.

Look at that formula. It has the same form as the second order correction to nondegenerate states. Same look, except that you only sum over the states that are outside the degenerate space. It was exactly of this form the second order energy correction. So it's kind of nice and simple.

But let's look at this equation again. And here is the thing that would have been shocking if we didn't do this right. If we had set l different from k here, if l is different from k, then this term is 0. If we didn't think-- if we hadn't suspected that the state n1 k had a piece along with the generate subspace, we would have not introduced it, and we would have had not this term. And the equation would have become this equal to 0, which is 0.

You have there n1 of k. And you can look at it, and look at it for hours, and it's just not 0. So unless we have this piece, the equation doesn't make sense. So this proves that the state must develop a component along the degenerate subspace. And let's now finally get it. It's just one line at this point. We don't have to do much.

So when l is different from k, l different from k-- so what do we get for the equation? Look at it there. We can solve directly for the piece n0 l n1 k is equal to-- solve for the second term. The other term goes on the right-hand side. So n0 l delta H n1 k over Enk 1 minus Enl 1.

And here is another thing that ended up working well. If the degeneracy had not been broken, we're doing l different from k. If some of these would have been 0, this would give you 0 in the denominator. This wouldn't work out.

So it was urgent here that all the degeneracy had been broken to first order. Otherwise, you couldn't have computed this state this way. This means that n1 k, to wrap it up here, in the subspace vn, is the sum over l different from k because this inner product only makes sense for l different from k of n0 l times this coefficient, 1 over Enk 1 minus Enl 1 n0 l delta H n1 k.

It's a lot of work to get here. But actually, it's good that we could get there. If you even want to write it more explicitly, you substitute what n1 k is. And it goes there. Now, there is something a little strange at first sight. If you look at it, you say, OK, let's see. This is first order in perturbation theory. We had to go to the second order equation.

And that's why we seem to be counting orders wrong, because this is first order in perturbation. This is another order in perturbation. That's second order in perturbation. That's, indeed, delta H n1 is second order. Delta H n1 appear to second order. So how come we get a first order term that seems to depend on second order stuff? Maybe it's wrong what we did in the-- we thought we were going to get the degenerate first order piece from the second equation, but here this is hitting us back.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yes. The denominator is also first order. So this is second order. This is first order. The ratio is first order. All is good here. This is a good formula. This is a real result.

So this completes our analysis of degenerate perturbation theory when the perturbation, the first order perturbation, splits all the levels. What we're going to do now is degenerate perturbation theory when the first order correction doesn't split any of the levels. Doesn't split them at all. What happens? What can we do?