PROFESSOR: A new way of thinking about this is based on integral equations. So it's a nice method. It's less complicated than it seems, suddenly leads to some expressions for the scattering amplitude. We'll find another formula for this quantity, f of k of theta, when we cannot calculate it with phase shifts. But phase shifts is very powerful if your problem has spherical symmetry. It's probably the first thing you try, because there's nothing to prevent you from finding a solution, in that case.

OK, so integral equations, so integral scattering equation, OK, integral scattering equation. So what are we solving? We're solving minus h squared over 2m Laplacian, plus v of r of potential, that is not necessarily spherical, equals or acts on a psi of r, to give you h squared, k squared over 2m, psi of r.

So it's an equation for eigenstates, the Schrodinger equation. It's good. It's nice. And this is what we've been solving. We've been solving if for some particular cases. The case of physical interest for us was when there was a plane wave coming in from infinity and this spherical wave going out at the infinity.

Now, at this moment, it's probably convenient to clean up the units from this equation a little. So I'll use a notation in which V of r is equal to h squared over 2m, U of r-- U still reminds you of a potential, I think. And it's a good thing. It's a definition of U of r. It's a definition. And in that way, I can get rid of the h bars, in this scattering equation. And what do we get? We get minus Laplacian plus U of r, acting on psi of r, is equal to k squared, acting on psi of r.

OK, that looks nice. It probably is nicer if you put it in a way that the left hand side is kind of a nice, simple operator and the right hand side involves the potential. So I'll move it along, so that I can pass the now left squared to the right, but then make that the left hand side. So this will be Laplacian squared plus k squared on psi of r is equal to U of r, psi of r.

OK, so we've done nothing so far, except put the equation in a way that stimulates our thinking. We kind of think of the right hand side as a source and the left hand side as kind of the equation you want to solve. Sometimes you have an equation like that. Then you can get this zero here. Well that's when the potential is zero. But sometimes there's a bit of potential, so the solution feeds back into this.

Whenever you have an equation of this form, some nice operator-- so this we call nice operator. Acting on psi, giving your something that depends on psi-- maybe it's not that nice or not that simple. We can try to solve this using Green's function. That's what Green's functions are good for. So let's try to use a Green's function.

So what is a Green's function? It is basically a way of understanding what this operator is. So nice operator must have a Green's function. So what is your Green's function here? It's called G. And we'll say it's depends on r minus r prime. r prime is an arbitrary point so far, but here is what it does. It's basically a solution.

You want this Green's function to be almost zero, except they're not quite equal to zero. You want it to be equal to a delta function. So that's the definition of the Green's function, is that thing which is the solution of a similar equation, where you have the nice operator acting on the Green's function being just a delta function. It's almost saying that the Green's function is the thing that solves that equation with zero on the right hand side. Except that it's not really zero, it's a source in the right hand side, it's a delta function of this point.

So you would say maybe, OK, I don't know. Why would I care about this equation? You care for two reasons. The first reason is that this equation is easier to solve than this one, doesn't involve the potential, which is complicated. It just involves a delta function.

So we can have this G and solve it once, because it doesn't involve the potential. And then the great thing about this equation is that it allows you to write the solution for the top equation without having to do any work anymore. Once you have the Green's function, you're done. So let's assume you have the Green's function. How would you write the solution for this equation?

So maybe a superposition, basically, and we'll do it using that. So our aim is to use this Green's function, if we had it, to write a solution for this equation. So here is the claim. You write psi of x. It's going to be given by a beginning one, psi zero of x. That is going to be a funny one. This one solves Laplacian plus k squared on psi zero of x or r is equal to zero. 0.

So whenever you have an equation of this form, you can add any solution. Anything that is killed by this, can be added to whatever solution you have. So let's assume that we have a solution of this form psi zero, of this form. Now, here I'll add one more thing, the important part. It's going to be an integral over r prime, here is the superposition. I'm going to think of this potential, U of r, as kind of existing at every point r prime.

So I'm going go write the solution as a superposition that involves the Green's function. So this will be G of r minus r prime, times U of r prime, times psi of r prime. I claim that this is equivalent to this equation that we have. That this psi provides-- more than equivalent. I think the precise way to say, this provides a solution of this equation, the way I've written. So let's try.

Suppose I calculate Laplacian plus k squared on psi of x here? OK, the first term, it's already zero. So Laplacian plus k squared on psi zero, it's already zero. And then I come here and I say, OK, I'm a Laplacian. I care about r, because I'm a Laplacian. I don't care about r prime. So I come in here. And I ignore r prime, ignore r prime. I cannot ignore this thing. So we have plus integral dr prime, Laplacian plus k squared, acting on G of r minus r prime, times U of r prime, psi of r prime.

And now because the Green's function was designed to give you a delta of r minus r prime, this is an integral that can be done, the integral over r prime, and just set the rest of the integrand at the point r, because you integrate over r prime. And this delta function fires when r prime is equal to r. So this gives me U of r, psi of r. And that is the equation I wanted to solve, the equation that we have here.

So we have turned the problem. This cannot be called the solution, because we have not solved it. We have turned the problem into a different kind of problem. It might even seem that we've made the problem worse by turning this into an integral equation. There's no derivatives here. But the function that we're looking for appears outside and inside the integral. So these things are called integral equations.

And the power of an integral equation is the insight it can give you once you have an idea of what the Green's function is. And also it's a good place to do recursive approximations. In that, you can essentially begin and say, OK, I know the wave function is this plus that. But maybe in some sense, I can think of this thing as the leading solution.

I could substitute the leading solution in here and try to make an approximation. That's going to give you a nice approximation, the Born approximation. We'll see that soon. That should be all psi r's. I think I'm using r's, so I please-- so r's and r's everywhere, yeah, no difference at this moment.

OK, so the next step is solving for the Green's function. We need the Green's function. Otherwise, we can't make progress with this equation. So I'm going to do a simple solution of the Green's function, basically by doing a couple of checks and saying that is the answer we're interested in.

And there are several possible Green's functions. And depending on the problem you're solving, you choose the right Green's function. And we'll choose the one that is suitable for us now. This is something that can be done. The general solutions can be obtained by counter integration. And there's all kinds of nice methods to do this.

But in fact, in this case, it's really simple. You don't need any of those complicated things. You can just pretty much write the solution. So that's what I'm going to do.

So what do we need for the Green's function? So we have a Green's function that depends on r minus r prime. And so it depends on a vector. So let me simplify the matter by saying, OK, since it depends on a vector, I'll just first calculate what this G of r, the Green's function of r, when the vector is r. Or you can think it's when r prime is equal to zero. Whatever I find for G of r, this one is obtained by whatever I find here, put in instead r minus r prime. So G of r is enough for what we want to do.

So it should have Laplacian plus k squared. And G of r should be for, not for, but it's delta of r. So we've looked at that in fact at the beginning of this course, not of this course, of the discussion of scattering. We looked at this equation. In fact, we wanted to solve it when the right hand side was zero, to find solutions of this scattering equation. And we found that these G's could be of the form e to the ikr plus, minus ikr over r. Those were the spherical waves that solved this equation with zero potential. Those are our solutions.

Now, it's easy to see that Laplacian plus k squared on this G, that has a plus or a minus, is equal to zero for r different from zero. The formulas for the Laplacian that you can us for r different from zero, you can check that is true. And it's easy. You can use that formula, for example, that the Laplacian is one over r the second-- well, it should be partial. The second dr squared r, so psi, psi, like that. That formula in one 30 seconds, you can see that that works for r different from zero.

But then there's also a formula that you know that Laplacian of one over r from electromagnetism is minus 4pi pi times the delta function of r. It comes from Poisson's equation in electromagnetism. So that's saying Laplacian of the potential is the charge density. This is the potential for a charge. The charge that r equals zero is a singularity. It's something you study.

So here if I put a minus one over 4pi for G plus, minus, a minus one over 4pi, the Laplacian of this whole thing is zero. But when you approach zero, the Laplacian of this function, this function approaches just one over r. That's how it goes to zero. This goes to one. And the Laplacian of one over r gives it this minus 4pi cancels with that. And it will give you a delta function. So the delta function will arise correctly from this quantity.

So I've argued without solving this equation, that this does give you a solution for the Green's function. It's Laplacian plus k squared is zero away from zero. Unless you approach zero, it has a right singularity to give a delta function.

So this is your Green's function. As I said many ways to derive it. If you want to see griffiths, see other ways in which it can be done, then you can also check this by doing properly the vector calculus. You can think of Laplacian as divergence of the gradient and calculate every step. And it's fun to do it as well. You check that this works. So this works.

And now we have to make choices. And our choices are going to be adjusted to the problem we're trying to solve. We're trying to solve a scattering problem. And psi zero represents a solution. And from the way we thought of our waves, we had psi equal e to the ikz plus f of theta and phi, e to the ikr over r.

So it's reasonable to try to set. And we're going to find solutions by setting psi zero to be e to the ikz. It does solve the equation Laplacian squared plus k squared equals zero. And then we're going to set the Green's function to be G plus. With a plus here, because we want solutions of the type e to the ikr. We've already decided those were the solutions we need. And a Green's function with a plus in there is going to generate that.

So those are fine. You could have chosen another thing here, an arbitrary solution. And you could have chosen a G minus as well. And you would get a solution of the scattering equation. It would not have much to do with the solution we're trying to get.

So let's show that this gives us this kind of solution we want to get and gives us already a formula for the scattering amplitude. So let's do that. For that we just need to write what the Green's function is and approximate. So it will be a first step.

OK, so what do we have? Psi of r is equal to e to the ikz. I'm writing up this stop equation, but we now write it with the right boundary conditions, which correspond to psi zero on a particular choice of the Green's function, v cubed r prime G of r minus r prime, U of r prime, all those are vectors, and psi of r prime. All those are vectors, except G, which is pretty in that it is very spherically symmetric. This G plus, minus of r just depends on the magnitude of r, which is very nice.

OK, so a couple of things that we can do here and most everybody would do. Let's G plus of r minus r prime is equal to minus one over 4pi, e to the ik, length of r minus r prime-- that's what this is supposed to be. Over the length of r minus r prime. Remember I said to you that whenever we have G of r, then to get the one we need, we'll just have to replace the r by r minus r prime. And here this little r or r without the vector is the magnitude of this vector. So we've put the magnitude of those vectors there.

Now, people look for approximations in here. So here is what we need to do. Imagine you are the origin. Here is the potential. And you have your system of coordinates. You are integrating over places where the potential exists. So the r prime that you're integrating remains inside the range of the potential. Here is zero. And then there is a r, which is where you are looking at.

And there is an angle between these two. Maybe I should make the angle a little bigger, smaller, I mean, r. And in general, we look always far enough. This is an exact solution, but we might as well consider looking at psi's that are located far from the potential. So we have the idea of what should we do with this terms.

One thing you can do is to say that in the denominator here, r minus r prime is good enough to set it to r in the denominator. And for the numerator, however, it's a lot more sensitive, because you have a k here. And depending on whether k is large or small-- a small difference in r minus r prime, if you're estimating this and you make an error comparable with the wavelength of your particle, you make a big mistake. If you're far way, you're 100 times farther than the range of the potential, here you're making a 1% error, no problem. But if you're making a 1% error in the estimate of this, that might be still comparable to the wavelength of this k. So you have to be a lot more careful in the phase, than you have to be on the other one.

So for the phase, phase, we will take r minus r prime equal to r minus n, r prime. So what is n? n is a unit vector in this direction. And indeed, approximately, this distance here is approximately equal to the distance r minus the projection of r prime into this one. So that is your approximate distance.

So here is the final formula we're going to write today. I'm running out of time. But that's what we wanted to end up with. So here is what we get.

So G plus of r minus r prime has become minus one over 4pi r. The denominator is r. There's an e to the ikr and then e to the minus ikn dot r prime. So that's-- and the arrows are crucial here. If I don't put an arrow somewhere, it probably means something.

So psi of r is equal to e to the ikz plus minus one over 4pi r. The integral over r prime doesn't care about this, the cubed r prime, e to the minus ikn dot r prime, times U of r prime, times psi of r prime, all that multiplied by-- let me take the r out. e to the ikr over r.

So look at what we obtain. I just rewrote, finally, this expression for the Green's function over there, with the integral over r prime and the Green's function here. You have this quantity in brackets. This quantity in brackets here is nothing but the f of theta phi of the scattering amplitude. You have this equal to that, plus e to the ikr over r. So we'll continue that next time and finish up with scattering probably in the first half of the next lecture.