

8.05 Quantum Physics II, Fall 2011
FINAL EXAM
Thursday December 22, 9:00 am -12:00
You have 3 hours.

Answer all problems in the white books provided. Write YOUR NAME and YOUR SECTION on your white book(s).

There are seven questions, totalling 100 points.

None of the problems requires extensive algebra.

No books, notes, or calculators allowed.

Formula Sheet

- Conservation of probability

$$\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} J(x, t) = 0$$

$$\rho(x, t) = |\psi(x, t)|^2 ; \quad J(x, t) = \frac{\hbar}{2im} \left[\psi^* \frac{\partial}{\partial x} \psi - \psi \frac{\partial}{\partial x} \psi^* \right]$$

- Variational principle:

$$E_{gs} \leq \int dx \psi^*(x) H \psi(x), \quad \text{for all } \psi(x) \text{ satisfying } \int dx \psi^*(x) \psi(x) = 1$$

- Spin-1/2 particle:

$$\text{Stern-Gerlach : } H = -\vec{\mu} \cdot \vec{B}, \quad \vec{\mu} = g \frac{e\hbar}{2m} \frac{1}{\hbar} \vec{S} = \gamma \vec{S}$$

$$\mu_B = \frac{e\hbar}{2m_e}, \quad \vec{\mu}_e = -2 \mu_B \frac{\vec{S}}{\hbar},$$

$$\text{In the basis } |1\rangle \equiv |z; +\rangle = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle \equiv |z; -\rangle = |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S_i = \frac{\hbar}{2} \sigma_i \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad [S_i, S_j] = i\hbar\epsilon_{ijk}S_k \quad (\epsilon_{123} = +1)$$

$$\sigma_i \sigma_j = \delta_{ij} I + i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad (\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} I + i\vec{\sigma} \cdot (\vec{a} \times \vec{b})$$

$$e^{i\mathbf{M}\theta} = \mathbf{1} \cos \theta + i\mathbf{M} \sin \theta, \quad \text{if } \mathbf{M}^2 = \mathbf{1}$$

$$\exp(i\vec{a} \cdot \vec{\sigma}) = \mathbf{1} \cos a + i\vec{\sigma} \cdot \left(\frac{\vec{a}}{a}\right) \sin a, \quad a = |\vec{a}|$$

$$\exp(i\theta\sigma_3) \sigma_1 \exp(-i\theta\sigma_3) = \sigma_1 \cos(2\theta) - \sigma_2 \sin(2\theta)$$

$$\exp(i\theta\sigma_3) \sigma_2 \exp(-i\theta\sigma_3) = \sigma_2 \cos(2\theta) + \sigma_1 \sin(2\theta).$$

$$S_{\vec{n}} = \vec{n} \cdot \vec{S} = n_x S_x + n_y S_y + n_z S_z = \frac{\hbar}{2} \vec{n} \cdot \vec{\sigma}.$$

$$(n_x, n_y, n_z) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad S_{\vec{n}} |\vec{n}; \pm\rangle = \pm \frac{\hbar}{2} |\vec{n}; \pm\rangle$$

$$|\vec{n}; +\rangle = \cos(\theta/2) |+\rangle + \sin(\theta/2) \exp(i\phi) |-\rangle$$

$$|\vec{n}; -\rangle = -\sin(\theta/2) \exp(-i\phi) |+\rangle + \cos(\theta/2) |-\rangle$$

- Complete orthonormal basis $|i\rangle$

$$\langle i|j\rangle = \delta_{ij}, \quad \mathbf{1} = \sum_i |i\rangle\langle i|$$

$$\mathcal{O}_{ij} = \langle i|\mathcal{O}|j\rangle \leftrightarrow \mathcal{O} = \sum_{i,j} \mathcal{O}_{ij} |i\rangle\langle j|$$

$$\langle i|A^\dagger|j\rangle = \langle j|A|i\rangle^*$$

hermitian operator: $\mathcal{O}^\dagger = \mathcal{O}$, unitary operator: $U^\dagger = U^{-1}$

- Position and momentum representations: $\psi(x) = \langle x|\psi\rangle$; $\tilde{\psi}(p) = \langle p|\psi\rangle$;

$$\hat{x}|x\rangle = x|x\rangle, \quad \langle x|y\rangle = \delta(x-y), \quad \mathbf{1} = \int dx |x\rangle\langle x|, \quad \hat{x}^\dagger = \hat{x}$$

$$\hat{p}|p\rangle = p|p\rangle, \quad \langle q|p\rangle = \delta(q-p), \quad \mathbf{1} = \int dp |p\rangle\langle p|, \quad \hat{p}^\dagger = \hat{p}$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right); \quad \tilde{\psi}(p) = \int dx \langle p|x\rangle \langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \exp\left(-\frac{ipx}{\hbar}\right) \psi(x)$$

$$\langle x|\hat{p}^n|\psi\rangle = \left(\frac{\hbar}{i} \frac{d}{dx}\right)^n \psi(x); \quad \langle p|\hat{x}^n|\psi\rangle = \left(i\hbar \frac{d}{dp}\right)^n \tilde{\psi}(p); \quad [\hat{p}, f(\hat{x})] = \frac{\hbar}{i} f'(\hat{x})$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx) dx = \delta(k)$$

- Generalized uncertainty principle

$$(\Delta A)^2 \equiv \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$$

$$(\Delta A)^2 (\Delta B)^2 \geq \left(\langle \Psi | \frac{1}{2i} [A, B] | \Psi \rangle \right)^2$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\Delta x = \frac{\Delta}{\sqrt{2}} \quad \text{and} \quad \Delta p = \frac{\hbar}{\sqrt{2}\Delta} \quad \text{for a gaussian wavefunction } \psi \sim \exp\left(-\frac{1}{2} \frac{x^2}{\Delta^2}\right)$$

$$\int_{-\infty}^{+\infty} dx \exp(-ax^2) = \sqrt{\frac{\pi}{a}}$$

$$\Delta H \Delta t \geq \frac{\hbar}{2}, \quad \Delta t \equiv \frac{\Delta Q}{\left| \frac{d\langle Q \rangle}{dt} \right|}$$

- Commutator identities

$$[A, BC] = [A, B]C + B[A, C],$$

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots,$$

$$e^A B e^{-A} = B + [A, B], \quad \text{if } [[A, B], A] = 0,$$

$$[B, e^A] = [B, A]e^A, \quad \text{if } [[A, B], A] = 0$$

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A, B]} = e^B e^A e^{\frac{1}{2}[A, B]}, \quad \text{if } [A, B] \text{ commutes with } A \text{ and with } B$$

- Time evolution

$$|\Psi, t\rangle = U(t, 0)|\Psi, 0\rangle, \quad U \text{ unitary}$$

$$U(t, t) = \mathbf{1}, \quad U(t_2, t_1)U(t_1, t_0) = U(t_2, t_0), \quad U(t_1, t_2) = U^\dagger(t_2, t_1)$$

$$i\hbar \frac{d}{dt} |\Psi, t\rangle = \hat{H}(t) |\Psi, t\rangle \quad \leftrightarrow \quad i\hbar \frac{d}{dt} U(t, t_0) = \hat{H}(t) U(t, t_0)$$

Time independent \hat{H} : $U(t, t_0) = \exp\left[-\frac{i}{\hbar} \hat{H}(t - t_0)\right] = \sum_n e^{-\frac{i}{\hbar} E_n(t-t_0)} |n\rangle \langle n|$

$$[\hat{H}(t_1), \hat{H}(t_2)] = 0, \quad \forall t_1, t_2, \quad U(t, t_0) = \exp\left[-\frac{i}{\hbar} \int_{t_0}^t dt' \hat{H}(t')\right]$$

$$\langle A \rangle = \langle \Psi, t | A_S | \Psi, t \rangle = \langle \Psi, 0 | A_H(t) | \Psi, 0 \rangle \quad \rightarrow \quad A_H(t) = U^\dagger(t, 0) A_S U(t, 0)$$

$$[A_S, B_S] = C_S \quad \rightarrow \quad [A_H(t), B_H(t)] = C_H(t)$$

$$i\hbar \frac{d}{dt} \hat{A}_H(t) = [\hat{A}_H(t), \hat{H}_H(t)], \quad \text{for } A_S \text{ time-independent}$$

- Two state systems

$$H = -\gamma \vec{S} \cdot \vec{B} \quad \rightarrow \quad \text{spin vector } \vec{n} \text{ precesses with Larmor frequency } \vec{\omega} = -\gamma \vec{B}$$

$$\text{NMR magnetic field } \vec{B} = B_0 \vec{e}_z + B_1 (\cos \omega t \vec{e}_x - \sin \omega t \vec{e}_y)$$

$$|\psi(t)\rangle = \exp\left(\frac{i}{\hbar} \omega t S_z\right) \exp\left(\frac{i}{\hbar} \gamma \vec{B}_{\text{eff}} \cdot \vec{S} t\right) |\psi(0)\rangle$$

$$\vec{B}_{\text{eff}} = B_0 \left(1 - \frac{\omega}{\omega_0}\right) \vec{e}_z + B_1 \vec{e}_x$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega \left(\hat{N} + \frac{1}{2} \right), \quad \hat{N} = \hat{a}^\dagger \hat{a}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right), \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i\hat{p}}{m\omega} \right),$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}} (\hat{a}^\dagger - \hat{a}),$$

$$[\hat{x}, \hat{p}] = i\hbar, \quad [\hat{a}, \hat{a}^\dagger] = 1.$$

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$$

$$\hat{H}|n\rangle = E_n|n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle, \quad \hat{N}|n\rangle = n|n\rangle, \quad \langle m|n\rangle = \delta_{mn}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle.$$

$$\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar} x^2 \right).$$

$$x_H(t) = \hat{x} \cos \omega t + \frac{\hat{p}}{m\omega} \sin \omega t$$

$$p_H(t) = \hat{p} \cos \omega t - m\omega \hat{x} \sin \omega t$$

- Coherent states and squeezed states

$$T_{x_0} \equiv e^{-\frac{i}{\hbar}\hat{p}x_0}, \quad T_{x_0}|x\rangle = |x+x_0\rangle$$

$$|\tilde{x}_0\rangle \equiv T_{x_0}|0\rangle = e^{-\frac{i}{\hbar}\hat{p}x_0}|0\rangle,$$

$$|\tilde{x}_0\rangle = e^{-\frac{1}{4}\frac{x_0^2}{d^2}} e^{\frac{x_0}{\sqrt{2}d}\hat{a}^\dagger} |0\rangle, \quad \langle x|x_0\rangle = \psi_0(x-x_0), \quad d^2 = \frac{\hbar}{m\omega}$$

$$|\alpha\rangle \equiv D(\alpha)|0\rangle = e^{\alpha\hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle, \quad D(\alpha) \equiv \exp\left(\alpha\hat{a}^\dagger - \alpha^* \hat{a} \right), \quad \alpha = \frac{\langle \hat{x} \rangle}{\sqrt{2}d} + i \frac{\langle \hat{p} \rangle d}{\sqrt{2}\hbar} \in \mathbb{C}$$

$$|\alpha\rangle = e^{\alpha\hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger} |0\rangle, \quad \hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad |\alpha, t\rangle = e^{-i\omega t/2} |e^{-i\omega t}\alpha\rangle$$

$$|0_\gamma\rangle = S(\gamma)|0\rangle, \quad S(\gamma) = \exp\left(-\frac{\gamma}{2}(\hat{a}^\dagger \hat{a}^\dagger - \hat{a}\hat{a}) \right), \quad \gamma \in \mathbb{R}$$

$$|0_\gamma\rangle = \frac{1}{\sqrt{\cosh \gamma}} \exp\left(-\frac{1}{2} \tanh \gamma \hat{a}^\dagger \hat{a}^\dagger \right) |0\rangle$$

$$S^\dagger(\gamma) \hat{a} S(\gamma) = \cosh \gamma \hat{a} - \sinh \gamma \hat{a}^\dagger, \quad D^\dagger(\alpha) \hat{a} D(\alpha) = \hat{a} + \alpha$$

$$|\alpha, \gamma\rangle \equiv D(\alpha) S(\gamma) |0\rangle$$

- Orbital angular momentum operators

$$\hat{L}_i = \epsilon_{ijk} \hat{x}_j \hat{p}_k$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

$$\hat{L}^2 = -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

$$\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}; \quad \hat{L}_{\pm} = \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right)$$

- Spherical Harmonics

$$Y_{\ell,m}(\theta, \phi) \equiv \langle \theta, \phi | \ell, m \rangle$$

$$Y_{0,0}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}; \quad Y_{1,\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta \exp(\pm i\phi); \quad Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{2,\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta \exp(\pm 2i\phi); \quad Y_{2,\pm 1}(\theta, \phi) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta \exp(\pm i\phi);$$

$$Y_{2,0}(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)$$

- Algebra of angular momentum operators \vec{J} (orbital or spin, or sum)

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k; \quad [J^2, J_i] = 0$$

$$J^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle; \quad J_z |jm\rangle = \hbar m |jm\rangle, \quad m = -j, \dots, j.$$

$$J_{\pm} = J_x \pm iJ_y, \quad (J_{\pm})^{\dagger} = J_{\mp} \quad J_x = \frac{1}{2}(J_+ + J_-), \quad J_y = \frac{1}{2i}(J_+ - J_-)$$

$$[J_z, J_{\pm}] = \pm \hbar J_{\pm}, \quad [J^2, J_{\pm}] = 0; \quad [J_+, J_-] = 2\hbar J_z$$

$$J^2 = J_+ J_- + J_z^2 - \hbar J_z = J_- J_+ + J_z^2 + \hbar J_z$$

$$J_{\pm} |jm\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle$$

- Radial equation

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right) u_{\nu\ell}(r) = E_{\nu\ell} u_{\nu\ell}(r) \quad (\text{bound states})$$

$$u_{\nu\ell}(r) \sim r^{\ell+1}, \quad \text{as } r \rightarrow 0.$$

$$E > 0 \text{ and } \lim_{r \rightarrow \infty} V(r) = 0, \quad \lim_{r \rightarrow \infty} u_{\ell}(r) = \sin\left(kr - \ell \frac{\pi}{2} + \delta_{\ell}(E)\right), \quad k = \frac{\sqrt{2mE}}{\hbar}$$

- Hydrogen atom

$$E_n = -\frac{e^2}{2a_0} \frac{1}{n^2}, \quad \psi_{n,\ell,m}(\vec{x}) = \frac{u_{n\ell}(r)}{r} Y_{\ell,m}(\theta, \phi)$$

$$n = 1, 2, \dots, \quad \ell = 0, 1, \dots, n-1, \quad m = -\ell, \dots, \ell$$

$$a_0 = \frac{\hbar^2}{me^2}, \quad \alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}, \quad \hbar c \simeq 200 \text{ MeV-fm}$$

$$u_{1,0}(r) = \frac{2r}{a_0^{3/2}} \exp(-r/a_0)$$

$$u_{2,0}(r) = \frac{2r}{(2a_0)^{3/2}} \left(1 - \frac{r}{2a_0}\right) \exp(-r/2a_0)$$

$$u_{2,1}(r) = \frac{1}{\sqrt{3}} \frac{1}{(2a_0)^{3/2}} \frac{r^2}{a_0} \exp(-r/2a_0)$$

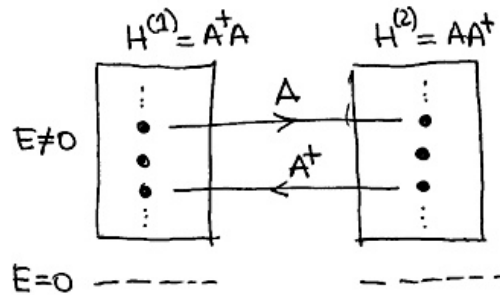
- Factorization method

$$A = \frac{d}{dx} + W(x), \quad A^\dagger = -\frac{d}{dx} + W(x)$$

$$H^{(1)} = A^\dagger A = -\frac{d^2}{dx^2} + W^2 - W'$$

$$H^{(2)} = AA^\dagger = -\frac{d^2}{dx^2} + W^2 + W'$$

$$\phi_0^{(1)} = N \exp\left(-\int^x W(x') dx'\right)$$



- Addition of Angular Momentum $\vec{J} = \vec{J}_1 + \vec{J}_2$

Uncoupled basis : $|j_1 j_2; m_1 m_2\rangle$ CSCO : $\{J_1^2, J_2^2, J_{1z}, J_{2z}\}$

Coupled basis : $|j_1 j_2; j m\rangle$ CSCO : $\{J_1^2, J_2^2, J^2, J_z\}$

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \dots \oplus |j_1 - j_2|$$

$$|j_1 j_2; j m\rangle = \sum_{m_1+m_2=m} |j_1 j_2; m_1 m_2\rangle \underbrace{\langle j_1 j_2; m_1 m_2 | j_1 j_2; j m \rangle}_{\text{Clebsch-Gordan coefficient}}$$

$$\vec{J}_1 \cdot \vec{J}_2 = \frac{1}{2}(J_{1+}J_{2-} + J_{1-}J_{2+}) + J_{1z}J_{2z}$$

Combining two spin 1/2 : $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$

$$|1, 1\rangle = |\uparrow\uparrow\rangle,$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |0, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$|1, -1\rangle = |\downarrow\downarrow\rangle.$$

1. **True or false questions** [20 points] No explanations required. Just indicate T or F for true or false, respectively.

- (1) The anti-commutator of two hermitian operators is hermitian.
- (2) The Heisenberg Hamiltonian and the Schrödinger Hamiltonian are equal if the Schrödinger Hamiltonians at different times commute.
- (3) The length scale $a_0 \cdot \alpha$ is the classical electron radius (a_0 is the Bohr radius and α the fine structure constant).
- (4) The length scale $a_0 \cdot \alpha^2$ is the Compton wavelength of the electron.
- (5) In the factorization method either $H^{(1)}$ or $H^{(2)}$ *must* have a normalizable zero energy eigenstate.
- (6) If \vec{J}_1 and \vec{J}_2 are sets of operators each satisfying the algebra of angular momentum the combination $\vec{J}_1 - \vec{J}_2$ also satisfies the algebra of angular momentum.
- (7) Let \vec{J}_1 and \vec{J}_2 be angular momentum operators. The operator $\vec{J}_1 \cdot \vec{J}_2$ commutes with $J_{1x} + J_{2x}$.
- (8) The traces of J_x , J_y , and J_z are all zero for any representation ($j = 1/2, 1, 3/2, \dots$).
- (9) The states on the first excited level of the spherically symmetric harmonic oscillator (potential $V(r) = \frac{1}{2}m\omega r^2$) fit into an $\ell = 1$ multiplet of angular momentum.
- (10) The states on the second excited level of the spherically symmetric harmonic oscillator fit into an $\ell = 2$ multiplet of angular momentum.

2. **A short problem on Harmonic Oscillator** [10 points]

Associated with the annihilation and creation (Schrödinger) operators \hat{a} and \hat{a}^\dagger there are Heisenberg operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$. Calculate these time-dependent Heisenberg operators in terms of \hat{a} , \hat{a}^\dagger and other physical constants. How is the Heisenberg counterpart $\hat{N}_H(t)$ of the number operator \hat{N} related to \hat{N} ?

3. **Deriving and testing an inequality** [15 points]

- (a) Let u denote a Hermitian operator whose eigenvalues are all non-negative so that the following expectation value is non-negative:

$$\left\langle \frac{1}{u}(u - \langle u \rangle)^2 \right\rangle \geq 0.$$

Use the above to prove an inequality relating $\langle \frac{1}{u} \rangle$ and $\frac{1}{\langle u \rangle}$.

- (b) Verify your inequality for the operator $u = r$ and the ground state of the hydrogen atom by direct calculation of $\langle r \rangle$ and $\langle \frac{1}{r} \rangle$. (The second can be obtained without doing integrals by using the virial theorem which tells you that the $\langle T \rangle = -\frac{1}{2}\langle V \rangle$, where T and V are, respectively, the kinetic and potential energies).

Possibly useful integral: $\int_0^\infty dx x^n e^{-x} = n!$ for $n = 1, 2, 3, \dots$

4. **A curious rewriting of the Hydrogen Hamiltonian** [15 points]

Consider the hydrogen atom Hamiltonian

$$H = \frac{\vec{p}^2}{2m} - \frac{e^2}{r}.$$

We will write it as

$$H = \frac{1}{2m} \sum_{i=1}^3 \left(\hat{p}_i + i\beta \frac{\hat{x}_i}{r} \right) \left(\hat{p}_i - i\beta \frac{\hat{x}_i}{r} \right) + \gamma,$$

where \hat{p}_i and \hat{x}_i are, respectively, the Cartesian components of the momentum and position operators, and β and γ are real constants to be adjusted so that the two Hamiltonians are the same.

- (a) Calculate the constants β and γ .
- (b) Explain carefully why for any state $\langle H \rangle \geq \gamma$.
- (c) Find the wavefunction of the state for which the above energy inequality is saturated. You may assume that this wavefunction just depends on the radial coordinate.

5. **Hamiltonian for three spin-1 particles** [15 points]

Consider 3 distinguishable **spin-1** particles, called 1,2, and 3, with spin operators \vec{S}_1, \vec{S}_2 and \vec{S}_3 , respectively. The spins are placed along a circle and the interactions are between nearest neighbors. The hamiltonian takes the form

$$H = \frac{\Delta}{\hbar^2} \left(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1 \right),$$

with $\Delta > 0$ a constant with units of energy. For this problem it is useful to consider the total spin operator $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$.

- What is the dimensionality of the state space of the three combined particles. Write the Hamiltonian in terms of squares of spin operators.
- Determine the energy eigenvalues for H and the degeneracies of these eigenvalues.
- Calculate the ground state, expressing it as a superposition of states of the form

$$|m_1, m_2, m_3\rangle \equiv |1, m_1\rangle \otimes |1, m_2\rangle \otimes |1, m_3\rangle,$$

where $\hbar m_i$ is the eigenvalue of $(S_z)_i$ and applying some suitable constraint. [Hint: The general superposition with arbitrary coefficients has 7 candidate states. Show that the coefficient of $|0, 0, 0\rangle$ is zero and determine all others. Write your answer as a normalized state.]

6. **Factorization method structure** [10 points]

In solving a central potential problem we need to find the spectrum of the set of Hamiltonians H_ℓ , with $\ell \geq 0$.

Assume that with some superpotential W_ℓ we succeed in constructing Hamiltonians $H_\ell^{(1)}$ and $H_\ell^{(2)}$ such that

$$H_\ell = H_\ell^{(1)} + \alpha_\ell,$$

with some constants α_ℓ assumed known and

$$H_\ell^{(2)} = H_{\ell+1}^{(1)} + \beta_\ell,$$

with some constants β_ℓ also assumed known. Since we know the superpotential, the operators A_ℓ and A_ℓ^\dagger are determined as well.

Finally, assume that $H_\ell^{(1)}$ has a zero-energy solution $\phi_{0,\ell}^{(1)}$, known for each ℓ .

- Write expressions for the three lowest energy eigenstates $\phi_{n,\ell}$ of H_ℓ ($n = 0, 1, 2$) and give their energies $E_{n,\ell}$. Your expressions for the states should use the zero energy wavefunctions of $H^{(1)}$ and the A or A^\dagger operators. Your expressions for the energy should use the constants α and β introduced above.
- Generalize to write an expression for the energy eigenstate $\phi_{n,\ell}$ of H_ℓ with arbitrary $n \geq 0$ and find its energy $E_{n,\ell}$.

7. **Time dependent uncertainty on a squeezed vacuum** [15 points]

Recall the definition of the squeezed vacuum state

$$|0_\gamma\rangle = S(\gamma)|0\rangle, \quad \text{with} \quad S(\gamma) = \exp\left(-\frac{\gamma}{2}(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a})\right), \quad \gamma \in \mathbb{C}.$$

- (a) Calculate the action of the squeezing operator S on the position and momentum operators:

$$\begin{aligned} S^\dagger(\gamma)\hat{x}S(\gamma) &= \dots \\ S^\dagger(\gamma)\hat{p}S(\gamma) &= \dots \end{aligned}$$

Write your answers in terms of \hat{x} , \hat{p} , and simple functions of γ .

- (b) Let $|0_\gamma, t\rangle$ denote the state that results from time evolution of the squeezed vacuum $|0_\gamma\rangle$ at time zero. Determine the time dependent uncertainty $(\Delta x(t))^2$ on the state. Your answer should be of the form

$$(\Delta x(t))^2 = \frac{\hbar}{2m\omega} G(\gamma, \omega t),$$

where $G(\gamma, \omega t)$, to be determined, is a unit-free function of γ and ωt .

[A little help with the algebra: $\langle 0 | (\hat{x}\hat{p} + \hat{p}\hat{x}) | 0 \rangle = 0$.]

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8.05 Quantum Physics II
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