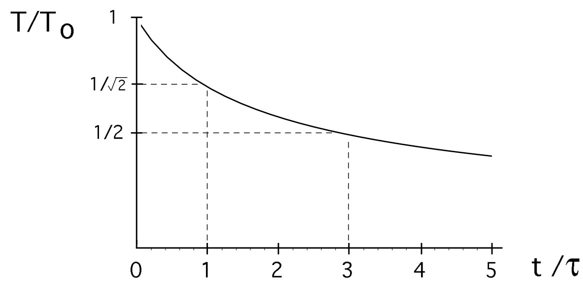
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2003

Exam #4

Problem 1 (35 points) Cooling of a White Dwarf Star



Just after a white dwarf star is formed it begins a long slow radiational cooling process which will eventually reduce it to a cold dark ember. In this problem you will find its temperature as a function of time. You may assume that

- There are no longer any heat sources in the star,
- The thermal conductivity is so high that the temperature T is essentially uniform throughout the star,
- The heat capacity is that of the nearly degenerate ($kT << \epsilon_f$) electron gas and has the form $C_V = \gamma V T^n$ where V is the volume and γ is a constant,
- The surface of the star is a perfect absorber of radiation at all frequencies.

a) What is the value of the exponent n in the expression for the heat capacity?

b) Find an expression for the derivative of the total energy of the star with respect to temperature, dE/dT.

c) Find an expression for the derivative of the total energy of the star with respect to time, dE/dt.

d) Find the differential equation which determines the time evolution of the temperature. Give your result in terms of γ, the radius of the star R, and any physical constants which you think necessary. Check to see that the equation is consistent with your common sense expectation for $T(t)$.

Hint: The correct differential equation is not hard to solve. By solving it you can check your result against the behavior plotted in the figure.

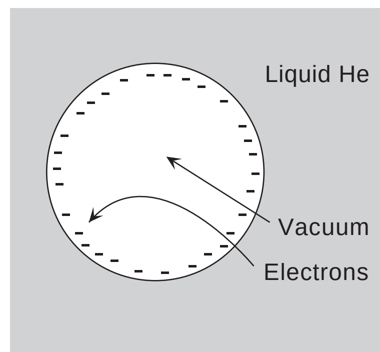
Problem 2 (35 points) Two-Dimensional Electron Gas

Consider a gas of non-interacting, spin 1/2 electrons confined to move in two dimensions. For a rectangular sample with dimensions L_x and L_y, the wavevectors allowed by periodic boundary conditions are $\vec{k} = (2\pi/L_x)m\,\hat{x} + (2\pi/L_y)n\,\hat{y}$ where m and n can take on all positive and negative integer values.

a) Find $D(\vec{k})$, the density of allowed wavevectors as a function of $\vec{k}$.

b) Find $D(\epsilon)$, the density of single particle states as a function of their energy ϵ. Make a carefully labeled sketch of your result.

Researchers have proposed a novel system for studying the properties of a highly degenerate two-dimensional electron gas. The electrons can be accumuated on the inside surface of a spherical bubble in liquid helium, as shown in the figure.



The equilibrium radius of the bubble is found by minimizing its energy $E(R)$ with respect to its radius R.

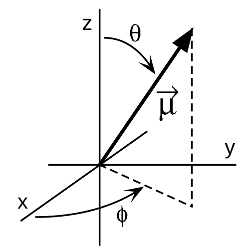
$$E(R) = E_{\text{Coulomb}} + E_{\text{Surface Tension}} + E_{pdV} + E_{\text{Electron Gas}}$$

c) Find at $T = 0$ the contribution to this sum from the kinetic energy of the two-dimesional electron gas, $E_{\text{Electron Gas}}$, as a function of R and N, the number of electrons in the bubble.

d) Find at $T = 0$ the total magnetic moment of the bubble, $M(H)$, as a function of the applied magnetic field H. Make a carefully labeled sketch of $M(H)$ for all positive values of the field. [Recall that the manetic moment of a single electron is quantized, $\mu_z = \pm\mu_0$, and its contribution to the energy is $-\mu_z H_z$]

Problem 3 (30 points) Classical Paramagnet

Consider a collection of N non-interacting classical magnetic moments. Each moment $\vec{\mu}$ has a fixed magnitude μ_0 and its orientation is specified by the two angles $0 \leq \theta < \pi$ and $0 \leq \phi < 2\pi$. In the presence of a magnetic field of magnitude H pointing in the z direction, the energy of an individual moment is given by $\epsilon = -\vec{\mu} \cdot \vec{H} = -\mu_0 H \cos(\theta)$.



Applying the canonical ensemble one finds that

$$p(\theta, \phi) = Z_1^{-1} \exp(\frac{\mu_0 H}{kT} \cos(\theta)) \qquad \text{where} \qquad \int p(\theta, \phi) \sin(\theta)\, d\theta\, d\phi = 1.$$

The partition function for a single moment Z_1 can be expressed in terms of the dimensionless combination of parameters $\eta = \mu_0 H / kT$:

$$Z_1(\eta) = \int_0^{2\pi} d\phi \int_0^{\pi} e^{\eta \cos\theta} \sin(\theta)\, d\theta = \frac{4\pi}{\eta} \sinh(\eta).$$

a) Using the information given above, find an expression for the magnetic moment of the sample, $M = N < \mu_z >$, in terms of an appropriate derivative of the single moment partition function.

b) Find an analytic expression for M in terms of μ_0, N, and η.

c) Find expressions for the temperature dependence of M in both the high temperature and low temperature regimes. Use these results to make a careful sketch of M as a function of T for a fixed value of H.

d) Two concepts used to explain the properties of the *quantum* paramagnet were Curie Law behavior and energy gap behavior.

   i) Does the classical paramagnet exhibit Curie Law behavior? Explain.

   ii) Does the classical paramagnet exhibit energy gap behavior? Explain.

e) In the classical paramagnet, as in the quantum paramagnet, the link between statistical mechancs and thermodynamics is provided by the Gibbs free energy: $-kT \ln Z = G(T, H)$. Find an expression for the entropy of the system in terms of $Z_1(\eta)$ and its derivatives. You do not have to carry out any derivatives of $Z_1(\eta)$ that might be involved.

Work in simple systems

Hydrostatic system	$-PdV$
Surface film	$\mathcal{S}dA$
Linear system	$\mathcal{F}dL$
Dielectric material	$\mathcal{E}d\mathcal{P}$
Magnetic material	HdM

Thermodynamic Potentials when work done on the system is $dW = Xdx$

Energy	E	$dE = TdS + Xdx$
Helmholtz free energy	$F = E - TS$	$dF = -SdT + Xdx$
Gibbs free energy	$G = E - TS - Xx$	$dG = -SdT - xdX$
Enthalpy	$H = E - Xx$	$dH = TdS - xdX$

Results from hyperbolic trigonometry

$$\sinh(u) = (e^u - e^{-u})/2 \qquad \cosh(u) = (e^u + e^{-u})/2$$

$$\tanh(u) = \sinh(u)/\cosh(u) \qquad \coth(u) = 1/\tanh(u)$$

$$\frac{d}{dx}(\sinh\ u) = (\cosh\ u)\frac{du}{dx} \qquad \frac{d}{dx}(\cosh\ u) = (\sinh\ u)\frac{du}{dx}$$

Limiting behavior of	as $u \to 0$	as $u \to \infty$
$\sinh(u)$	u	$e^u/2$
$\cosh(u)$	$1 + u^2/2$	$e^u/2$
$\tanh(u)$	u	1
$\coth(u)$	$1/u + \frac{1}{3}u$	1

Radiation laws

Kirchoff's law: $e(\omega, T)/\alpha(\omega, T) = \frac{1}{4}c\,u(\omega, T)$ for all materials where $e(\omega, T)$ is the emissive power and $\alpha(\omega, T)$ the absorptivity of the material and $u(\omega, T)$ is the universal blackbody energy density function.

Stefan-Boltzmann law: $e(T) = \sigma T^4$ for a blackbody where $e(T)$ is the emissive power integrated over all frequencies. ($\sigma = 56.9 \times 10^{-9}$ watt-m^{-2}K^{-4})