Solutions to Problem Set #4

Problem 1: Heat Capacity at Constant Pressure in a Simple Fluid

Start with the first law of thermodynamics.

$$\dbar Q = dU + P\,dV$$

The relation for C_P we are looking for involves $(\partial U/\partial T)_P$ so it is natural to try to do our expansion in terms of the variables T and P. We expand both dU and dV in terms of dT and dP.

$$dU = \left(\frac{\partial U}{\partial T}\right)_P dT + \left(\frac{\partial U}{\partial P}\right)_T dP$$

$$dV = \left(\frac{\partial V}{\partial T}\right)_P dT + \left(\frac{\partial V}{\partial P}\right)_T dP$$

$$\dbar Q = \left(\left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P\right) dT + \left(\left(\frac{\partial U}{\partial P}\right)_T + P\left(\frac{\partial V}{\partial P}\right)_T\right) dP$$

Since we need the derivative at constant P the second term in the above expression will drop out.

$$
\begin{aligned}
C_P &\equiv \left.\frac{\dbar Q}{dT}\right|_P \\[2mm]
&= \left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P \\[2mm]
&= \underline{\left(\frac{\partial U}{\partial T}\right)_P + \alpha V P}
\end{aligned}
$$

Problem 2: Heat Supplied to a Gas

To find C_P we proceed as follows. Rearrange the first law to isolate $\dbar Q$.

$$\dbar Q = dU + P\,dV$$

Expand the differential of the energy in terms of dT and dV.

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$$

Substitute into the expression for $\dbar Q$.

$$\dbar Q = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\left(\frac{\partial U}{\partial V}\right)_T + P\right) dV$$

Form the derivative with respect to T by dividing by dT and specifying the path as one of constant P.

$$\frac{dQ}{dT}\bigg|_P \equiv C_P = \underbrace{\left(\frac{\partial U}{\partial T}\right)_V}_{C_V} + \left(\left(\frac{\partial U}{\partial V}\right)_T + P\right)\left(\frac{\partial V}{\partial T}\right)_P$$

Now use the given information $(\partial U/\partial V)_T = 0$, $PV = NkT$, and $C_V = (5/2)Nk$ in the equation above to find that

$$C_P = (7/2)Nk.$$

a) It is easy to find the heat along the two rectangular paths by integration.

$$
\begin{aligned}
\Delta Q(acb) &= \int_a^c C_V\,dT + \int_c^b C_P\,dT \\[2mm]
&= \int_{P_1}^{P_2} C_V \frac{V_1}{Nk}\,dP + \int_{V_1}^{V_2} C_P \frac{P_2}{Nk}\,dV \\[2mm]
&= \underline{(19/2)NkT_1}
\end{aligned}
$$

$$
\begin{aligned}
\Delta Q(adb) &= \int_a^d C_P\,dT + \int_d^b C_V\,dT \\[2mm]
&= \int_{V_1}^{V_2} C_P \frac{P_1}{Nk}\,dV + \int_{P_1}^{P_2} C_V \frac{V_2}{Nk}\,dP \\[2mm]
&= \underline{(17/2)NkT_1}
\end{aligned}
$$

Before we compute the heat along the diagonal path, it is useful to find the difference in internal energy between b and a. Since the internal energy is a state function, it does not matter what path we use to find it. We already know the heat input ΔQ along the path adb and the work ΔW is easy to find.

$$\Delta W(adb) = \int -P\,dV = -\int_{V_a}^{V_d} P_1\,dV = -NkT_1$$

Then the path-independent result for ΔU can be computed along this particular path.

$$\Delta U = \Delta Q + \Delta W = (17/2 - 2/2)NkT_1 = (15/2)NkT_1$$

The work along the diagonal path ab can be calculated by integration using the V dependence of P: $P = (P_1/V_1)V$.

$$
\begin{aligned}
\Delta W(ab) &= -\int_a^b P\,dV = -(P_1/V_1)\int_{V_1}^{V_2} V\,dV \\[2mm]
&= -(1/2)(P_1/V_1)(V_2^2 - V_1^2) = -(3/2)P_1V_1 = -(3/2)NkT_1
\end{aligned}
$$

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Finally we can compute the heat supplied along the diagonal path.

$$\Delta Q = \Delta U - \Delta W = \underline{9NkT_1}$$

b) Examine the constraint along the diagonal path ab.

$$\dcancel{d}Q \quad = \quad dU + P\,dV$$

$$= \quad \underbrace{\left(\frac{\partial U}{\partial T}\right)_V}_{C_V} dT + \left(\underbrace{\left(\frac{\partial U}{\partial V}\right)_T}_{0} + P\right) dV$$

$$C_{ab} \quad \equiv \quad \frac{\dcancel{d}Q}{dT}\bigg|_{ab} = C_V + P\frac{dV}{dT}\bigg|_{ab}$$

Along the path ab

$$P = \frac{P_1}{V_1}V = \frac{NkT}{V} \quad \Rightarrow \quad V^2 = \frac{V_1}{P_1}NkT.$$

So along ab we can construct an expression relating dV to dT by taking the derivative of this expression.

$$2V\,dV \quad = \quad \frac{V_1}{P_1}Nk\,dT$$

$$\Rightarrow \quad \left(\frac{\partial V}{\partial T}\right)_{ab} = \frac{V_1}{P_1}\frac{Nk}{2V}$$

Finally

$$C_{ab} = C_V + \frac{V_1}{P_1}\frac{Nk}{2}\underbrace{\frac{P}{V}}_{P_1/V_1} = C_V + (1/2)Nk = \underline{3Nk}.$$

As a check we can integrate this heat capacity along the path.

$$\Delta Q(ab) = \int_a^b C_{ab}\,dT = 3Nk(T_b - T_a) = 9NkT_1$$

This is identical to the result we found above in part a).

Comment: $C_V = (5/2)Nk$ is an approximation to a diatomic gas where the rotational degrees of freedom are contributing to C_V but the vibrational degrees of freedom are not (they are frozen out; we will understand why later in the course). If we had used the monatomic result $C_V = (3/2)Nk$ we would have found $C_P = (5/2)Nk$, $\Delta Q(acb) = (13/2)NkT_1$, $\Delta Q(adb) = (11/2)NkT_1$, $\Delta U = (9/2)NkT_1$, $\Delta Q(ab) = 6NkT_1$, and $C_{ab} = 2Nk$.

Problem 3: Thermodynamics of a Curie Law Paramagnet

All the manipulations we perform here for a simple magnetic system mirror those we carried out in lecture for a simple hydrostatic system.

a) Heat capacities for the generic magnet.

$$dU = \left(\frac{\partial U}{\partial T}\right)_M dT + \left(\frac{\partial U}{\partial M}\right)_T dM \qquad \text{expansion}$$

$$\dbar Q = dU - \dbar W = dU - H\, dM \qquad \text{first law}$$

$$= \left(\frac{\partial U}{\partial T}\right)_M dT + \left(\left(\frac{\partial U}{\partial M}\right)_T - H\right) dM \qquad \text{substitution}$$

$$C_M \equiv \left.\frac{\dbar Q}{dT}\right|_M = \left(\frac{\partial U}{\partial T}\right)_M \qquad \text{from line above}$$

$$C_H \equiv \left.\frac{\dbar Q}{dT}\right|_H = \left(\frac{\partial U}{\partial T}\right)_M + \left(\left(\frac{\partial U}{\partial M}\right)_T - H\right)\left(\frac{\partial M}{\partial T}\right)_H \qquad \text{from } \dbar Q$$

$$C_H - C_M = \underline{\left(\left(\frac{\partial U}{\partial M}\right)_T - H\right)\left(\frac{\partial M}{\partial T}\right)_H}$$

$$\left(\frac{\partial U}{\partial M}\right)_T = \frac{C_H - C_M}{\left(\frac{\partial M}{\partial T}\right)_H} + H \qquad \text{by rearrangement}$$

Now substitute into the general expansion of dU to arrive at

$$dU(T,M) = C_M(T,M)\, dT + \left[\frac{C_H(T,M) - C_M(T,M)}{\left(\frac{\partial M}{\partial T}\right)_H} + H(T,M)\right] dM$$

b) Now find the results specific to the Curie law paramagnet.

$$C_M = bT, \qquad \left(\frac{\partial U}{\partial M}\right)_T = 0 \qquad \qquad \text{given}$$

$$dU = \underbrace{\left(\frac{\partial U}{\partial T}\right)_M}_{C_M} dT + \underbrace{\left(\frac{\partial U}{\partial M}\right)_T}_{0} dM \qquad \qquad \text{expansion}$$

$$U(T,M) = \int_0^T bT' \, dT' = (1/2)bT^2 + f(M) \qquad \qquad \text{integration}$$

$$\left(\frac{\partial U}{\partial M}\right)_T = f'(M) = 0 \text{ (given)} \Rightarrow f(M) = \text{ constant } = U(T=0)$$

$$U(T) = (1/2)bT^2 + \overbrace{U(T=0)}^{\text{0 by assumption}}$$

$$\left(\frac{\partial M}{\partial T}\right)_H = \frac{\partial}{\partial T}\left(\frac{aH}{T}\right)_H = -\frac{aH}{T^2} \qquad \qquad \text{from eq. of state}$$

$$C_H - C_M = (0 - H)(-\frac{aH}{T^2}) = \frac{aH^2}{T^2} = \frac{M^2}{a} \qquad \qquad \text{from a)}$$

$$C_H(T,M) = bT + \frac{M^2}{a}$$

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c) Here we practice working with an alternative pair of independent variables.

$$dU \;=\; \left(\frac{\partial U}{\partial H}\right)_M dH + \left(\frac{\partial U}{\partial M}\right)_H dM \qquad\qquad \text{expansion}$$

$$đQ \;=\; dU - H\,dM = \left(\frac{\partial U}{\partial H}\right)_M dH + \left(\left(\frac{\partial U}{\partial M}\right)_H - H\right)dM \qquad \text{first law and substitution}$$

$$C_M \;\equiv\; \left.\frac{đQ}{dT}\right|_M = \left(\frac{\partial U}{\partial H}\right)_M \left(\frac{\partial H}{\partial T}\right)_M \qquad\qquad \text{dividing the above by } dT$$

$$\left(\frac{\partial H}{\partial T}\right)_M \;=\; \frac{-1}{\left(\frac{\partial T}{\partial M}\right)_H \left(\frac{\partial M}{\partial H}\right)_T} = -\frac{\left(\frac{\partial M}{\partial T}\right)_H}{\chi_T} \qquad\qquad \text{chain rule}$$

$$C_M \;=\; -\frac{1}{\chi_T}\left(\frac{\partial U}{\partial H}\right)_M \left(\frac{\partial M}{\partial T}\right)_H \qquad\qquad \text{substitution}$$

$$\left(\frac{\partial U}{\partial H}\right)_M \;=\; -\frac{C_M\,\chi_T}{\left(\frac{\partial M}{\partial T}\right)_H} \qquad\qquad \text{rearrangement of above}$$

$$C_H \;\equiv\; \left.\frac{đQ}{dT}\right|_H = \left(\left(\frac{\partial U}{\partial M}\right)_H - H\right)\left(\frac{\partial M}{\partial T}\right)_H \qquad\qquad \text{from } đQ \text{ expression}$$

$$\left(\frac{\partial U}{\partial M}\right)_H \;=\; \frac{C_H}{\left(\frac{\partial M}{\partial T}\right)_H} + H \qquad\qquad \text{rearrangement of above}$$

Now substitute into the general expansion of $dU(H, M)$ to arrive at

$$dU(H, M) = -\frac{C_M(H, M)\,\chi_T(H, M)}{\left(\frac{\partial M}{\partial T}\right)_H}\,dH + \left[\frac{C_H(H, M)}{\left(\frac{\partial M}{\partial T}\right)_H} + H\right]dM$$

Note that the coefficient of the dM term in the expansion of $dU(H, M)$ found here is different from the coefficient of the dM term in the expansion of $dU(T, M)$ found in part a).

d) Now we use the equation of state associated with a Curie law paramagnet.

$$\chi_T \equiv \left(\frac{\partial M}{\partial H}\right)_T = \frac{\partial}{\partial H}\left(\frac{aH}{T}\right)_T = \frac{a}{T}$$

$$\left(\frac{\partial M}{\partial T}\right)_H = -\frac{aH}{T^2}$$

$$dU(H,M) = \underbrace{\left(-\frac{bT\frac{a}{T}}{-\frac{aH}{T^2}}\right) dH}_{bT^2/H} + \underbrace{\left(\frac{bT + a\frac{H^2}{T^2}}{-\frac{aH}{T^2}} + H\right) dM}_{-bT^2/M}$$

$$= \frac{bT^2}{H} dH - \frac{bT^2}{M} dM$$

$$= \underline{\frac{ba^2 H}{M^2} dH - \frac{ba^2 H^2}{M^3} dM} \qquad \text{by eliminating } T$$

$$U(H,M) = \frac{ba^2}{2}\frac{H^2}{M^2} + f(M) \qquad \text{integration}$$

$$\left(\frac{\partial U}{\partial M}\right)_H = -\frac{ba^2 H^2}{M^3} + f'(M)$$

Comparison with the coefficient of dM in the differential form $dU(H,M)$ above shows that $f'(M) = 0$ which, when integrated, gives $f = $ constant. Thus we can write

$$U(H,M) = \frac{ba^2}{2}\frac{H^2}{M^2} + \text{constant}$$

There is no reason to carry around a constant term in the internal energy which never responds to any change in the independent variables, so we are free to set the constant equal to zero.

$$\underline{U(H,M) = \frac{ba^2}{2}\frac{H^2}{M^2}}$$

By using the equation of state, $M = aH/T$, we see that this reduces to the same result obtained in b), that is $U = (b/2)T^2$.

e) We are looking for the adiabatic constraint on dT and dM.

$$\require{cancel}\cancel{d}Q \;=\; \underbrace{\left(\frac{\partial U}{\partial T}\right)_M}_{C_M} dT + \left(\left(\frac{\partial U}{\partial M}\right)_T - H\right) dM = 0 \qquad \text{for an adiabatic path}$$

$$C_M\, dT \;=\; -\left(\left(\frac{\partial U}{\partial M}\right)_T - H\right) dM \qquad\qquad \text{by rearrangement}$$

$$\Rightarrow \quad \left.\frac{dT}{dM}\right|_{\Delta Q=0} = -\frac{\left(\left(\frac{\partial U}{\partial M}\right)_T - H\right)}{C_M}$$

This gives the slope of an adiabatic path for any magnetic system.

f) Next we specialize to the case of a Curie law paramagnet.

$$\left(\frac{\partial U}{\partial M}\right)_T \;=\; 0, \qquad C_M = bT \qquad\qquad\qquad \text{Curie law paramagnet}$$

$$\left.\frac{dT}{dM}\right|_{\Delta Q=0} \;=\; -\frac{(0-H)}{bT} = \frac{H}{bT} = \frac{M}{ab} \qquad\qquad \text{using the general result from e)}$$

$$dT \;=\; \frac{1}{ab} M\, dM \qquad\qquad\qquad \text{after rearrangement}$$

$$(T - T_0) \;=\; \frac{1}{2ab}(M^2 - M_0^2) \qquad\qquad\qquad \text{integration}$$

$$(T - T_0) \;=\; \frac{1}{2ab}(M - M_0)(M + M_0)$$

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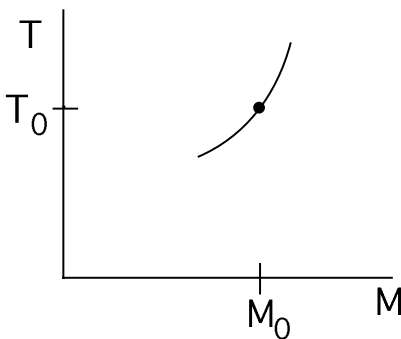
g) An isothermal path in the H M plane is easy to picture from the given equation of state, $M = aH/T$. It is a straight line going through the origin with slope $(\partial M/\partial H)_T = a/T$. This is shown in the figure accompanying the statement of the problem. In part f) we found the relation which must hold between dT and dM along an adiabatic path: $dT = (1/ab)M\,dM$. In order to explore an adiabatic path in the H M plane we must express dT in terms of dM and dH.

$$T \;=\; \frac{aH}{M} \qquad\qquad\qquad\text{equation of state}$$

$$dT \;=\; \frac{a}{M}\,dH - \frac{aH}{M^2}\,dM \qquad\qquad\qquad\text{differential of above}$$

Substitute this general expression for dT into the adiabatic path derivative and separate the dH and dM terms.
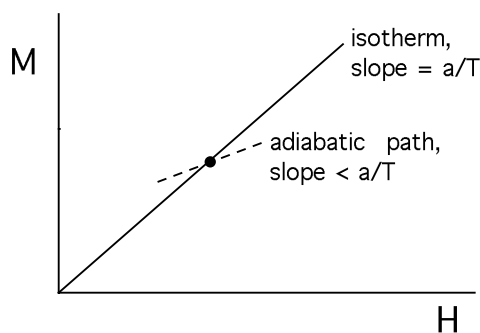
$$(1/ab)M\,dM \;=\; \frac{a}{M}\,dH - \frac{aH}{M^2}\,dM$$

$$dH \;=\; \left(\frac{1}{a^2 b}M^2 + \frac{H}{M}\right)dM = \left(\frac{M^2}{a^2 b} + \frac{T}{a}\right)dM = \frac{T}{a}\left(1 + \frac{M^2}{abT}\right)dM$$

This allows us to find the slope of an adiabatic line in the H M plane in terms of the quantity a/T which is the slope of an isotherm.

$$\frac{dM}{dH}\bigg|_{\Delta Q=0} = \frac{a}{T}\left(\frac{1}{1 + \frac{M^2}{abT}}\right) \quad < \quad \frac{a}{T}$$

Note that the slope of the adiabatic path is less than that of the isothermal path going through the same point.

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Problem 4: Classical Magnetic Moments

a)

$$-\mu \le \quad m_i \quad \le \mu$$
$$-\mu N \le \quad M \quad \le \mu N$$
$$-N(\mu H) \le \quad E \quad \le N(\mu H)$$

b) There are $2N$ microscopic variables necessary to specify the state of the system. Some possibilities include the x and z component of each spin, the z component and the angle in the xy plane for each spin, or the polar angles θ and ϕ for each spin.

c)

$$\frac{H}{T} = -\left(\frac{\partial S}{\partial M}\right)_N = -k\frac{1}{\Omega}\left(\frac{\partial \Omega}{\partial M}\right)_N \qquad \text{using } S = -k\ln\Omega$$

$$= -k\frac{1}{\Omega}\left(-\frac{2M}{(2/3)\mu^2 N}\right)\Omega = k\left(\frac{M}{(1/3)N\mu^2}\right)$$

$$\Rightarrow \quad M(H,T) = \left(\frac{N\mu^2}{3k}\right)\frac{H}{T} \qquad\qquad \text{the Curie law result}$$

d) The expression found in c) allows M to grow without bound as $T \to 0$. But $|M|$ is bounded by μN. Thus the expression can only be trusted as an approximation when

$$M(H,T) \quad << \quad \mu N$$
$$\frac{N\mu^2}{3kT} \quad << \quad \mu N$$
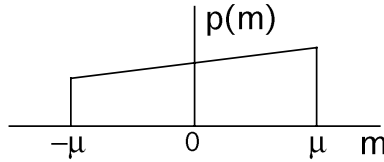$$\Rightarrow \quad kT >> (1/3)\mu H$$

This result says that the "thermal energy", kT, must be much greater than the maximum energy allowable for a single spin.

e) To find Ω' reduce N by 1 and reduce M by m.

$$\Omega' = (2\mu)^{N-1}\exp\left[-\frac{(M-m)^2}{(2/3)(N-1)\mu^2}\right]$$

f)

$$p(m) = \frac{\Omega'}{\Omega} = \frac{(2\mu)^{N-1}}{(2\mu)^N} \frac{\exp\left[-\frac{(M-m)^2}{(2/3)(N-1)\mu^2}\right]}{\exp\left[-\frac{M^2}{(2/3)N\mu^2}\right]}$$

$$= \frac{1}{2\mu} \exp\left[-\frac{M^2 - 2Mm + m^2}{(2/3)(N-1)\mu^2}\right] \exp\left[\frac{M^2}{(2/3)N\mu^2}\right]$$

$$\approx \frac{1}{2\mu} \exp\underbrace{\left[\frac{3mM}{N\mu^2}\right]}_{\text{small since } M \ll \mu N}$$

$$\approx \frac{1}{2\mu}\left(1 + (\frac{3M}{N\mu^2})m\right) \qquad -\mu \le m \le \mu$$

Now check the normalization.

$$\int_{-\mu}^{\mu} p(m)\, dm = \int_{-\mu}^{\mu} \frac{1}{2\mu}\, dm + \int_{-\mu}^{\mu} \frac{3M}{2N\mu^3} m\, dm$$

$$= 1 + 0 = 1$$

g)

$$<m> = \int p(m)\, m\, dm$$

$$= \underbrace{\int_{-\mu}^{\mu} \frac{1}{2\mu} m\, dm}_{=0} + \int_{-\mu}^{\mu} \frac{3M}{2N\mu^3} m^2\, dm$$

$$= \frac{3M}{2N\mu^3}\left[\int_{-\mu}^{\mu} \frac{1}{3} m^3\right] = \underline{\frac{M}{N}}$$

This tells us that the total magnetization M is N times the average moment of an individual dipole, the result that one would expect on physical grounds.

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Problem 5: A Strange Chain

a) The number of ways of choosing n_+ elements from a total of N is $N!/(N-n_+)!n_+!$. It follows that

$$\Omega(N, n_+) = \frac{N!}{(N-n_+)!n_+!}$$

$$S(N, n_+) = k \ln \Omega$$

$$\approx k\{N \ln N - (N-n_+)\ln(N-n_+) - n_+ \ln n_+ \underbrace{-N + (N-n_+) + n_+}_{=0}\}$$

$$= \underline{k\{N \ln N - (N-n_+)\ln(N-n_+) - n_+ \ln n_+\}}$$

b)

$$\frac{\mathcal{F}}{T} = -\left(\frac{\partial S}{\partial L}\right)_{N,E}$$

$$= -\frac{\partial S}{\partial n_+} \underbrace{\frac{\partial n_+}{\partial L}}_{1/2l}$$

$$= -\frac{k}{2l}\left\{\frac{N-n_+}{N-n_+} + \ln(N-n_+) - \frac{n_+}{n_+} - \ln n_+\right\}$$

$$-\frac{2l\mathcal{F}}{kT} = \ln\left(\frac{N-n_+}{n_+}\right)$$

$$\mathcal{F}(N, T, n_+) = \underline{-\frac{kT}{2l}\ln\left(\frac{N-n_+}{n_+}\right)}$$

c) Now rearrange the last result, take the exponential of both sides and solve for n_+.

$$\exp[-2l\mathcal{F}/kT] = \frac{N - n_+}{n_+}$$

$$n_+ = N\frac{1}{1 + \exp[-2l\mathcal{F}/kT]}$$

Next, use the expression for n_+ to find L.

$$L = l(2n_+ - N) = Nl\left(\frac{2}{1 + \exp[]} - \frac{1 + \exp[]}{1 + \exp[]}\right)$$

$$= Nl\frac{1 - \exp[-2l\mathcal{F}/kT]}{1 + \exp[-2l\mathcal{F}/kT]}$$

$$= Nl\frac{\exp[l\mathcal{F}/kT] - \exp[-l\mathcal{F}/kT]}{\exp[l\mathcal{F}/kT] + \exp[-l\mathcal{F}/kT]}$$

$$= \underline{Nl\tanh(l\mathcal{F}/kT)}$$

For high temperatures, where $kT >> l\mathcal{F}$, $\tanh x \to x$ for small x, so

$$L \approx \left(\frac{Nl^2}{kT}\right)\mathcal{F}.$$

The fact that the length L is proportional to the tension $\mathcal{F}$ shows that Hooke's law applies to this system, at least for high temperatures.

d)

$$\alpha \equiv L^{-1}\left(\frac{\partial L}{\partial T}\right)_{\mathcal{F}}$$

$$= \frac{1}{L}\left(-\frac{1}{T}(L)\right)$$

$$= \underline{-\frac{1}{T}}$$

Problem 6: Classical Harmonic Oscillators

a) In x space

$$E = \sum_{i=1}^{2N} x_i^2$$

is a sphere in $2N$ dimensions with radius $\sqrt{E}$. Its volume is $\pi^N E^N/N!$. The corresponding volume in pq space is

$$\Omega(E, N) = \frac{\pi^N}{N!}(\sqrt{2m})^N \left(\sqrt{\frac{2}{m\omega^2}}\right)^N E^N$$

$$= \underline{\left(\frac{2\pi}{\omega}\right)^N \frac{1}{N!} E^N}$$

b)

$$S(E, N) = k \ln \Omega(E, N) = \underline{k \ln \left\{\left(\frac{2\pi}{\omega}\right)^N \frac{1}{N!} E^N\right\}}$$

c)

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N = k\frac{NE^{-1}\{\}}{\{\}} = \frac{Nk}{E}$$

$$\Rightarrow \underline{E = NkT} \qquad \underline{C_N = Nk}$$

d) Let Ω' be the volume in a phase space for $N-1$ oscillators of total energy $E - \epsilon$ where $\epsilon = (1/2m)p_i^2 + (m\omega^2/2)q_i^2$. Since the oscillators are all similar, $< \epsilon >= E/N = kT$.

$$p(p_i, q_i) \;=\; \Omega'/\Omega$$

$$\Omega' \;=\; \left(\frac{2\pi}{\omega}\right)^{N-1} \frac{1}{(N-1)!} (E - \epsilon)^{N-1}$$

$$\frac{\Omega'}{\Omega} \;=\; \left(\frac{2\pi}{\omega}\right)^{-1} \frac{N!}{(N-1)!} \left(\frac{E - \epsilon}{E}\right)^N \frac{1}{E - \epsilon}$$
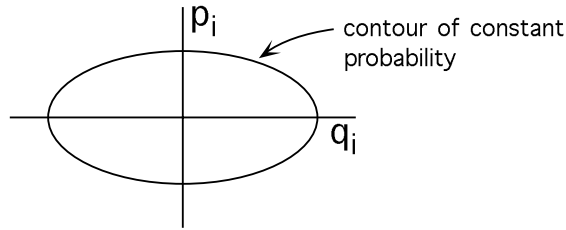
$$=\; \frac{\omega}{2\pi} \underbrace{\frac{N}{E - \epsilon}}_{\approx <\epsilon>^{-1}} \underbrace{\left(1 - \frac{\epsilon}{E}\right)^N}_{\approx \exp[-\epsilon/<\epsilon>]}$$

$$p(p_i, q_i) \;=\; \frac{1}{(2\pi/\omega) < \epsilon >} \exp[-\epsilon/ < \epsilon >]$$

$$=\; \frac{1}{(2\pi/\omega)kT} \exp[-p_i^2/2mkT] \exp[-(m\omega^2/2kT)q_i^2]$$

$$=\; \left(\frac{1}{\sqrt{2\pi mkT}} \exp[-p_i^2/2mkT]\right) \left(\frac{1}{\sqrt{2\pi(kT/m\omega^2)}} \exp[-q_i^2/2(kT/m\omega^2)]\right)$$

$$=\; p(p_i) \times p(q_i) \;\Rightarrow\; p_i \text{ and } q_i \text{ are S.I.}$$

contour of constant probability

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