Massachusetts Institute of Technology

Physics 8.03 Fall 2016
Homework 5

Problems

Problem 5.1 (20 pts)

In a lab, a graduate student was studying a strong of length L, fixed at both ends.

a. Find the total energy of a vibration of the strong oscillating with its nth normal mode with amplitude A. The tension in the string is T and its total mass is M. (Hint: Consider the integrated kinetic energy at the instant when the string is straight, such that the potential energy due to the vibration is zero).
b. Calculate the total energy of vibration of the same string if it is vibrating in the following superposition of normal modes. Assume that is is the sum of the energies of the two modes taken separately.

$$\psi(x,t) = A_1 \sin\left(\frac{\pi x}{L}\right)\cos(\omega_1 t) + A_3 \sin\left(\frac{3\pi x}{L}\right)\cos(\omega_3 t - \pi/4)$$

Problem 5.2 (20 pts)

A string of length L, tension T, and mass per unit length ρ_L fixed at both ends is distorted as shown in Figure 1. The height of the square pulse is h. The string is constrained to vibrate only in the vertical direction. At $t = 0$, the string is carefully released such that the initial velocity of the string is 0.

a. What is the speed of a progressing wave on this string?
b. Plot to scale the shape of the string at $t = \frac{L}{4}\sqrt{\frac{\rho_L}{T}}$
c. Plot to scale the shape of the string at $t = L\sqrt{\frac{\rho_L}{T}}$

Problem 5.3 (20 pts)

Consider a string of mass density ρ under tension T. A pulse is traveling on the string, causing a vertical displacement given by:

$$y(x,t) = y_0 e^{\left[-\frac{1}{2}\left(\frac{(x-vt)}{\sigma}\right)^2\right]}$$

Energy in the wave is a sum of kinetic energy of the moving string and the potential energy of the string deformation integrated over the whole length of the string.

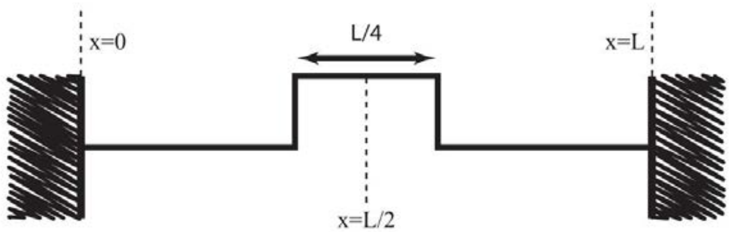


Figure 1: A string

a. For the displacement given above, find an expression for the kinetic energy density of the string at some time t and sketch the result as function of x.

b. Find an expression for the potential energy density of the string. Compare it with the kinetic energy density.

c. Find an expression for the total energy of the pulse in terms of T, y_0 and σ. Check the units. Recall that the energy should vary as a square of the amplitude. (Hint: You may need to use Gaussian integral $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$)

d. Find an expression for the power (Watts) flowing across the point $x = x_0$ as a function of time. Sketch the result.

Problem 5.4 (20 pts)

Consider a string with linear density ρ_L split into two pieces. The two halves are attached to a massless ring which slides vertically without friction on a rod at $x = 0$ as shown in Figure 2. One of the two halves is stretched in the negative x direction with tension T_L, the other is stretched in the positive x direction with tension T_R. Note that the vertical rod is necessary to balance the horizontal forces on the massless string arising from the two string with different tensions.

Assume that there is a traveling wave of amplitude A coming from the negative x direction.

a. Assuming the incident wave has this functional form: $\psi(x,t) = \alpha \cos(kx - \omega t)$, write the general expression for the string displacement $\psi(x,t)$ on both sides of the ring.

b. Write the expressions for boundary conditions at $x = 0$. (Hint: since the ring is massless, the total force on the ring is 0)

c. Find the reflection (R) and transmission (T) coefficients for this system.

d. Show that the energy is conserved at the junction.

e. What are the reflection and transmission coefficients in the following three cases: $T_R = T_L$, $T_R \gg T_L$, and $T_R \ll T_L$? Does your result make sense?

Problem 5.5 (20 pts)

A string of mass density ρ_L under tension T is attached by a massless ring to a wire covered with a viscous grease. The ring experiences a vertical drag force $F_y = -b\frac{\partial y}{\partial t}$ when the end of the string moves.

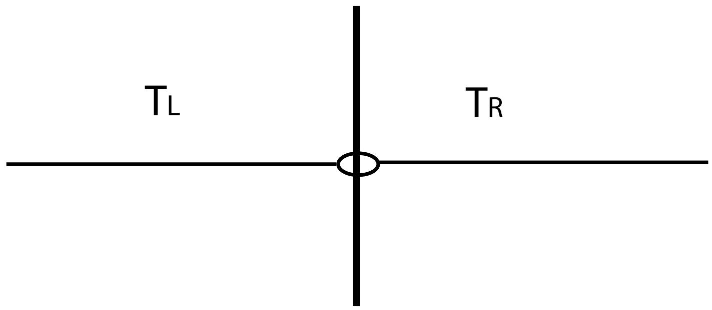
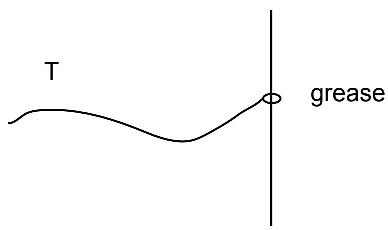


Figure 2: Two string halves with a ring in between



a. Use Newton's Law applied to the ring to find the boundary condition at the end of the string as a relation between the partial derivatives of $y(x,t)$ at that point. (Hint: there are three forces acting on the ring: string tension, normal force from the wire and the vertical drag force)

b. Show that the boundary condition is satisfied by an incident pulse $f(t-\frac{x}{v})$ and reflected pulse $g(t+\frac{x}{v})$. Find g in terms of f.

c. Show that your result has the proper behavior in the limits $b \to 0$ (free slip) and $b \to \infty$ (clamped string).