We've seen previously that if a rope is under tension and we approximate the rope is massless that the tension is then uniform everywhere along the rope, even if the rope is accelerated.

However, if the rope has nonzero mass, then its tension will vary along its length.

We can see that by looking at this example.

Imagine we have a massive rope of length l suspended from a ceiling.

Now, it's easy to see that, at the top of the rope, a little element of rope right at the top has to support the entire weight of the entire rope.

So if I were to examine just a little piece at the top here, the weight of the entire rope would be acting downwards.

And for the rope to remain stationary, there has to be a tension upward.

I'll call this t-top, because this is at the top of the rope.

And so we see immediately from that that at the top of the rope, the tension is just equal to mg, where m is the mass of the entire rope.

Similarly, if we asked what the tension is at the bottom of the rope, an element right at the bottom of the rope isn't supporting any weight, because there is no weight below it.

And so at the bottom, the tension is 0, because there is no weight pulling on the bottom of the rope.

So the tension is going to vary from mg at the top down to 0 at the bottom.

And if we wanted to work out at some distance from the ceiling-- let's call it x-- at some position x-- say here, what the tension is at that point-- one way of figuring that out would just be imagine we cut the rope at this point and then asked what tension force would be necessary to support this bottom part of the rope.

This length is l minus x, so the mass of that fraction of the rope is l minus x over l.

And the mass of that fraction of the rope is that fraction times m.

And so the tension at this point, t at x, is equal to the weight of the length of rope below that point.

So that's this mass times g.

And I can rewrite that as 1 minus x over l times mg.

And this gives me mg if I put in x equals 0, and it gives me 0 if I put in x equals l.

So that's one way of figuring this out.

I'd like to use this same example, though, to introduce another, more elegant way of analyzing what the tension as a function of a position on this rope.

The advantage-- so this is going to be a little bit more complicated, but it's a much more powerful method, and it's one that we can generally use for any continuous distribution of mass as opposed to a point mass.

So let's consider the same example again of a hanging mass of rope of length l and mass m.

Here's the length l and the mass m.

And the approach we're going to use is called differential analysis.

It's a technique from calculus, and it's applicable to any continuous distribution of mass.

What we're going to do is imagine our continuous distribution of mass is made up of a whole bunch of little pieces, little elements, examine f equals ma acting on a single element, and then generalize to the entire mass.

So let's do that here.

What we'll do is we'll examine a piece at some position x.

So I'm measuring x from the top.

And let's say I define a little element here, which is that of position x and which has an extent delta x.

So this thickness is delta x, and we'll call the mass of that little piece, that little element, delta m.

Now, one thing I want to point out to begin with is that we want to pick an element that's somewhere in the middle of the distribution, not at one end or the other.

The endpoints are special cases, so we want to pick a general case.

So choose some x somewhere in the middle of our distribution which has a finite extent.

That extent is delta x.

In this case, we'll assume that that piece delta x has a mass delta m.

So let's blow that up here and analyze it.

So here is my element.

And it's going to have-- let's analyze what the forces acting on it are.

So there's gravity acting downwards, which will be delta m times g.

There is a tension acting upwards, exerted by the rope above it.

I'll write that as t of x, because it's at the location x.

There's also a tension exerted in the downward direction by the remainder of the rope below the mass element.

And I'll call that t of x plus delta x.

We expect the tension to vary along the rope.

And because this element does have a finite extent, the tension at the bottom is going to be slightly different than the tension at the top- so t of x upwards, t of x plus delta x downwards, and then the weight downwards.

So let's now-- that's our free body diagram.

Let's write down Newton's second law, f equals ma for that mass element.

So in the positive direction, which is downward, we have t of x plus delta x plus delta mg.

And then in the upward or minus direction, we have minus t of x.

So those are the combined forces.

Now, this rope is suspended.

It's not moving.

And so mass object acceleration is just 0.

I'm going to rearrange that.

I can write that as t of x plus delta x minus t of x is equal to minus delta m times g.

And by the way, let me remind you-- if this were a massless rope, then delta n would be 0.

And so the right-hand side would be 0, and the tension would be uniform.

We'd have the same tension above and below.

But because the rope does have mass, and in particular, this element has a nonzero mass delta m, there is a difference in the tensions.

OK.

Now, our delta m can be represented in terms of what this length is.

Notice that that mass, delta m-- so note that delta m is just a fraction of the total mass that's in that particular mass element.

Well, the fraction of the total rope is just the length of this element, delta x, divided by the length of the entire rope, which is l.

So that's the fraction, and I multiply that by the total mass.

So that is my mass delta m in terms of the length.

So now I can rewrite this equation as t of x plus delta x minus t of x is equal to minus delta x m over l times g.

Now, I'm going to rearrange this by dividing both sides by delta x.

I'll do that over here.

So we have t of x plus delta x minus t of x divided by delta x is equal to minus mg over l.

This tells us how the tension is varying over this small but finite-sized mass element, delta x.

Now I'd like to examine what happens if I go to the limit of a small-massed element-- the limit as delta x goes to 0, or in other words, the limit of an infinitesimally small mass element.

So I'm going to take the limit of this equation as delta x approaches 0.

Now, the left-hand side of this equation should look familiar.

It's just an expression for the derivative of the tension t as a function of position.

So I can write that as dt dx, and that's equal to minus mg over l.

This is an example of a differential equation, or an equation that involves a derivative.

This particular differential equation can be solved very simply by a technique called the separation of variables, where I just take each part of the integral-- the dt and the dx-- and put it on different sides of the equation and then integrate both sides of the equation to find the solution.

So in this case, I'll multiply both sides of the equation by dx.

So I get dt on the left-hand side is equal to minus mg over l dx.

And now I want to integrate both sides.

So I'll integrate this side.

And remember, mg over l is a constant, so I can keep it outside the integral.

And I'll integrate this side.

I'm going to do a definite integral over the continuous distribution that I'm studying.

So on the right-hand side, I'm going to start at x equals 0 and go to my position of interest, which is x.

And to avoid confusion, I'm going to call the integration viable dx prime.

This is a dummy variable.

So x prime here represents all the values of position, ranging from my first endpoint 0 up to my other endpoint x.

So x here represents a particular position, whereas dx prime is a placeholder for all the positions between the two endpoints in my infinite sum, which is an integral.

So that's on the right-hand side.

On the left-hand side, I'm integrating the tension t, with respect to the tension t.

My limits need to correspond to the limits on the right-hand side integral.

So on the right-hand side, my lower limit is at x equals 0.

So on the left-hand side, I want to have my lower limit be the tension at that position x equals 0.

So that's t of 0.

And then the upper limit of the integral is the tension at the upper limit of the position integral, so that's t of x.

And again, to avoid confusion, I'm actually going to call the integration variable dt prime.

This is a dummy variable.

It's a placeholder for all the values of tension from the tension at x equals 0 to the tension at my position of interest x.

And so this integral now tells me how the tension is varying in a continuous fashion along this continuous mass distribution.

So now I can evaluate both integrals.

On this side, I have the integral of a constant with respect to the tension.

And so that's just going to give me t of x minus t of 0, and that's equal to minus mg over l.

The interval of dx prime from 0 to x is just x.

So this tells me how the tension changes from the endpoint to some arbitrary position x.

If I want to actually solve for t of x, I need to specify what the tension is at the endpoint.

But we know what the tension is at the endpoint.

We found that earlier.

We solved from the simple argument that at the endpoint, the tension here is just equal to the weight of the entire length of rope below that point.

So we know that t at x equals 0 is just equal to mg.

And therefore, the tension at position x is just-- if I bring [INAUDIBLE] of 0 to this side is just mg minus mg x over l.

And so I can just write that as mg times 1 minus x over l.

So that tells me what the tension is as a function of position.

Note again that if I put in x equals 0, I just get mg.

If I put in x equals l, I get the tension is 0.

And for any point in between, we see that the tension varies smoothly between mg and 0.

This is exactly the same result we found earlier by just cutting the rope at x and asking how do we balance the weight of the remaining rope below that point.

But the advantage of this technique which is a little bit more complicated, is that it's a very powerful technique applicable to any continuous distribution of mass.

So in this specific problem, if instead of the uniform density rope that we had here, imagine we had a clumpy rope where the mass of the rope wasn't distributed smoothly, but there were clumps, little parts of the rope that were heavier than others, and so the density varied with position along the rope.

In that case, we would represent that when we were writing what our mass element delta m is.

Delta m, instead of just being delta x over l times m, where here, delta x over l was the uniform density of the rope, we would have to put in some position-dependent density.

So delta n would depend upon position.

And then when I did this integral, instead of a constant out front with my integral of dx prime, I would have some thing that was a function of x, and I'd get a different value for the integral.

But the technique would still work.

So this differential analysis technique is applicable to any system where we have a continuous mass distribution.

And we're going to actually use this technique over and over again in this course.

We'll see it a number of times.

This is just the first instance we're using it.

We wanted to introduce you carefully to the approach.