Now, we've been discussing steady uniform precession, which is the simplest possible case of a phenomenon that can be much more complicated.

As an example and the case we've been discussing so far where we release a gyroscope from rest when it's horizontal, very careful measurements would show that the initial motion isn't just steady precession but has some additional motion superimposed on it.

The spin axis sort of bounces or nods up and down as it precesses.

However, friction at the pivot point causes the amplitude of the nodding to rapidly decay until it settles into steady precession.

This nodding motion is called nutation.

In our example, it decays so quickly that we don't even notice it.

But if you had a perfectly frictionless pivot point, it would be a more noticeable effect.

And it would persist for much longer.

Now, there's an interesting point about our horizontal gyroscope that I'd like to point out.

So here is my rod of length d.

This is my point s.

Here's my wheel that's rotating with some angular velocity.

And again, there's the r hat direction.

That's the direction that little omega vector's pointing in.

That's the k hat direction.

And that's the theta hat direction.

Now, before I release it, before I release the gyroscope from rest, the angular momentum vector points in the r hat direction, as drawn at this instant.

When I release it from rest, there is a torque acting in the theta hat direction that causes the angular momentum vector to rotate.

And we've seen this in some detail.

But there's something else to keep in mind, which is that now when the system is precessing, that means that the center of mass of the wheel is orbiting around the z-axis, the vertical axis, through this pivot point.

So you can think of that as a point mass with the full mass of the gyroscope moving in a circular path around point s with a radius d.

That implies that there must be a component of angular momentum pointing in the k hat direction, pointing along the z-axis, corresponding to the translational motion of the center of mass, the circular translational motion of the center of mass, of the gyroscope around the z-axis.

But where did that k component come from?

There was no initial z component of the angular momentum.

We said that initially the angular momentum just pointed in the r hat direction.

And there's no torque in the k hat direction.

The only torque is in the theta hat direction.

So how do we end up with some angular momentum pointing in the k hat direction?

This seems to violate the conservation of angular momentum.

The solution to this puzzle is that, in fact, the gyroscope doesn't remain precisely horizontal when I release it.

Instead, it dips by a very small angle.

So that's the horizontal.

After I release it, the gyroscope's actually dipped downward by a small amount.

I've exaggerated it considerably in this drawing.

We'll call that angle delta theta.

That small dip gives the small negative component of the spin angular momentum that's able to balance the plus z angular momentum due to the motion center of mass.

Now, because the spin angular momentum vector has such a large magnitude-- that's my spin angular momentum vector-- only a small angle is necessary in order to get enough of a negative Lz component to balance out the Lz corresponding to the center of mass motion.

OK?

But what we see is that although gyroscopes seem like a remarkable system, they're not magic.

And, in fact, angular momentum is conserved.

The larger little omega is, the faster the spin angular velocity is.

And, therefore, the larger the vector L is, the smaller that angle is and the less of a dip that you get.

So now we can write the exact angular momentum for the gyroscope.

This is my distance d.

This is my pivot point s.

My spin angular velocity looks like that.

Here is the r hat direction, the k hat direction, and the theta hat direction into the screen.

And we're orbiting-- the precession is around the z-axis with an angular speed capital omega.

Now, recall that the total angular momentum with respect to point s can be written in two parts.

There's the angular momentum due to the translational motion of the center of mass.

And we know the center of mass is just orbiting around the z-axis.

So I'll call that, in fact, the orbital angular momentum.

So this is due to the translation of the center of mass with respect to point s.

And then the second term is due to the rotational angular momentum, or spin angular momentum, relative to the center of mass.

So this is due to rotation about the center of mass.

So again, the general angular momentum of a rigid body is equal to the center of mass translational angular momentum plus the angular momentum due to rotation, a pure rotation, about the center of mass.

So let's write each of these terms.

The orbital angular momentum, that is the translational angular momentum of the center of mass, is just due to the motion of the center of mass of the gyroscope, which is at radius d with respect to point s, moving in a circle with angular speed capital omega.

So that's equal to the mass times the center of mass velocity times the radius of the circle.

And that angular momentum is in the k hat direction.

But the center of mass velocity is just equal to-- since it's a circular motion-- is just equal to the radius of the circle, d, times the angular speed of the circular motion of the center of mass, which is capital omega.

So I can write this as m capital omega d squared in the k hat direction.

So that's the angular momentum due to the translational motion of the center of mass around point s, what I'm calling the orbital angular momentum.

Now, the spin angular momentum, we've been talking about the rapid spin of the wheel around its axis.

So that's given by the moment of inertia about that axis times the spin angular velocity.

And that's pointing in the r hat direction.

But there's a subtlety here.

It turns out that is not the only rotation about the center of mass that this wheel is undergoing.

And actually because of that, I'm going to call this I1.

And let me just draw a picture here.

So for my disk rotating around this axis, the relevant moment of inertia is what I'll call I1.

And for a disk, we know that would be 1/2 m r squared.

The other rotation is a subtle one.

Notice in this drawing, suppose I were to draw a dot on the outside face of the wheel.

When this wheel precessed around 180 degrees to the other side, that dot on the outside would be facing in the minus r hat direction now or would be pointing to the left rather than to the right.

And what that means is that this disk has actually rotated about its diameter.

So this disk has rotated around a diameter like this.

And that rotation is at the slower angular speed capital omega.

And it takes one full orbit for it to rotate entirely around.

If that rotation weren't happening-- this wouldn't make sense physically the way I have this set up.

But as an object, if that rotation weren't happening, what that would mean is that this face with a dot on it would always be pointing to the right as the disk moved around.

If it had an independent pivot point right at the center, it would be physically possible for it to do that.

That's not what's happening in this case.

In this case, this face is always pointing outward, radially away from the pivot point.

And that results in a rotation around this diameter.

Now, it turns out that moment of inertia, which I'll call I2, happens to be half the moment of inertia for this axis.

So in this case, it's one 1/4 m r squared.

So that is another kind of rotation that's happening about the center of mass.

And so there's an additional angular momentum term arising from that rotation.

And that's equal to I2 times the angular velocity of that rotation, which is capital omega.

And because the axis there is the z-axis, this is pointing in the k hat direction.

So the total angular momentum of the gyroscope is I1 times omega in the r hat direction plus I2 times capital omega in the k hat direction plus m capital omega d squared in the k hat direction.

The first term, the r hat component, is the only part that's rotating.

So this is a rotating vector.

The k hat terms are constant.

This is the exact expression for the angular momentum of a gyroscope.

And now we see the gyroscopic approximation more precisely is saying that this rotating term dominates over the other two terms.

So it's actually that this term is very large compared to either of these two terms, which, as we can see, is roughly equivalent to saying that little omega is very large compared to big omega.