DEEPTO CHAKRABARTY: A pivoted rod held horizontal, parallel to the ground, and released from rest will simply fall, rotating about the pivot point. In particular, suppose we have a mass attached to one end of a pivoted rod. So here is my pivot. Here's my pivoted rod, which we'll assume is massless. And I have-- and it has a length d, and I attach a mass m to one end. If I let this go from rest, it will simply fall, rotating about the pivot point, which I'll call s. And it's easy to understand that in terms of the action of gravity which is acting downward and the resulting torque about point s.

If I replace this point mass with a wheel of the same mass, so this disk is a wheel with, let's say, a radius r and the same mass m attached to a pivoted rod. The rod is massless and has length d. If I hold this horizontal, parallel to the ground, and release it from rest, it will still just fall to the ground. Not a surprising result.

What is surprising is if I then spin up the wheel, so if I have the wheel rotating about its axle with some large angular velocity, little omega, and then I hold it horizontal, parallel to the ground, and release it from rest, then remarkably this wheel plus axle will not fall. It will remain horizontal, parallel to the ground. But the center of mass of the wheel will execute a small circular orbit about the vertical axis through the pivot point.

This remarkable and very non-intuitive motion is called precession and the system that's undergoing precession is called a gyroscope. Let's see if we can understand this behavior in terms of the angular momentum of the system. So what I've drawn here is a side view of the system. So let's define some coordinates. So we have r-hat in the radial outward direction, k-hat in the direction of the z-axis, the vertical axis. And then we'll define theta-hat pointing into the screen. OK, so that's a side view.

I'd like to now draw a top view. So let's say we're looking down along the z-axis from the top on the system. So now here is my pivot point, and here is a top view of my wheel. Again, this is a distance d. Now this is still the r-hat direction. This is the theta-hat direction. And the k-hat direction. is pointing out of the screen.

Let's draw the forces acting on our diagram here. So the weight is acting at the center of mass of the wheel. That's mg downward. And there's a normal force acting upwards at the pivot point. The torque is just given by r cross f, and so relative to point s at the distance d times the weight mg. So the torque is mgd. And by doing r cross f with the right hand rule, we see that it's directed in the plus theta-hat direction.

Now again, let's suppose that I'm holding the wheel horizontal and release it from rest. If the wheel is not spinning-- so this is not rotating, this is a stationary wheel that I'm holding horizontal, and I release it from rest, then its initial angular momentum, with respect to point s, is 0.

Over a short time interval, delta t, the torque which is acting in the theta-hat direction, will cause an angular impulse or change in the angular momentum. That change in angular momentum, delta l vector, is equal to the torque times that short time interval, delta t. And it will act in the theta-hat direction. The change in angular momentum will be in the theta-hat direction because that's the direction that the torque is in.

So for this case-- and by the way, this view here, I should have labeled this. This is a top view. So this is the side view. This is the top view of the same system. So the torque is mgd in the theta-hat direction. So in the side view, that's going into the screen. And in the top view, that's going pointing upward. That's the theta-hat direction.

So in the case where the wheel is not spinning, if we think about the top view, the initial angular momentum is 0. Right? So I'll just draw that as a dot. So that's l initial equals 0. And then I add a small torque, or small angular impulse due to the torque, delta l, which is in the theta-hat direction. So that's pointing in the theta-hat direction. So that's my delta l.

And when I sum those together, I start out with 0. I add a small delta l. So my final angular momentum, a time delta t later, is just equal to my delta l. So this is l final. And that's pointing in the theta-hat direction.

So a torque in the theta-hat direction, into the screen for the side view, is consistent with the wheel falling down. So with theta-hat pointing into the screen, the wheel will basically fall this way. It's rotating about point s. And that's consistent with the torque pointing in the theta-hat direction. Now as the wheel falls, the torque continues to point in the theta-hat direction. And so the angular acceleration will increase.

Now what happens if this wheel is not stationary, but instead is spinning rapidly? In that case, the torque remains the same. It's still mgd in the theta-hat direction. But now the initial angular momentum is not 0. Rather it is a very large vector pointing along the spin axis. Let's choose the sense of rotation such that l points in the plus r-hat direction. That's actually the way I've drawn it here. So in that case, the angular momentum vector initially points in the plus r-hat direction.

So then what happens over a short time, delta t? So now let's consider the case where the wheel is spinning. So now my initial angular momentum is a large vector pointing in the r-hat direction. That's l initial. I'm adding a small perpendicular vector, delta l, in the theta-hat direction. And so the sum of those two things, if this is the original r-hat direction, what's happened is that my new vector is at a small angle with respect to the original r-hat direction. I'll call that angle delta theta.

So what I've done is I've rotated my initial angular momentum vector by a small angle without changing its length. Notice the two very different situations. In one case, I start out with 0 angular momentum. And all the angular momentum I end up with comes from the angular impulse due to the torque.

In the second case, where the wheel is spinning, I start out with a very large initial angular momentum. I then add a small angular impulse, small compared to my initial angular momentum, in the perpendicular direction. And that causes not a change in the length of the vector, but a change in its direction. Which means that the angular momentum vector rotates. That's why the system processes when the wheel is rotating rapidly.

Before we can understand precession more carefully, it'll be useful to review the mathematics of rotating vectors. So first, suppose we have a vector that I'll call r1. And I'm going to add a vector delta r to this. And I'm going to have the condition that the length of delta r is much, much smaller than the length of my original vector r1, and that delta r is perpendicular to R1. So here's my delta r.

And so if I add these two vectors, let's say r2 is the sum of r1 plus delta r, so that's my vector r2. And I'm going to call this angle delta theta.

And now few things to note. First of all, since this is a right triangle, notice that the length r2 is equal to r1 divided by the cosine of delta theta. But if that angle is small, if delta theta is a small angle, then cosine theta is well approximated by 1, and so this is just my original length r1. And since r1 is equal to r2, I'm just going to call that-- call them both r. And this is for small delta theta.

So that tells us that the vector just rotates. When I have a large vector, and I add a small perpendicular vector, the result is to rotate the original vector without changing its length. As long as the angle is small or, equivalently, as long as delta r is very small compared to my original vector length, I'll get a pure rotation.

In addition, again looking at this triangle, notice that delta r is equal to [AUDIO OUT], which I can describe as r, times the sine of delta theta. And again, if the angle is small, if delta theta's a small angle, then this is well approximated by r times the angle delta theta in radians. And again, this is for small delta theta.

So now if I divide this last equation by delta t, small time interval in which this rotation is happening, then I write this as delta r vector divided by delta t. And I take the magnitude of that. That is equal to r delta theta divided by delta t.

And if I go to the limit as delta t gets small, as delta t approaches 0, then I can write this as the derivative of the vector r, the time derivative, I should say the magnitude of the time derivative of vector r. And that's equal to the length r times the time derivative of the angle d theta dt.

But d theta dt is just the angular velocity. So I could write that as r times capital omega for the angular velocity. And this is just our familiar result that for circular motion, for the rotation of a position vector, the velocity is equal to the radius of the circle times the angular velocity of the rotation.

Another way of thinking of that is that the magnitude of the rate of change of the position vector, which we call the velocity, is equal to the length of the rotating vector r times the angular velocity of the rotation. But there's nothing special about the position vector that I used in order to do this analysis.

So if instead of a position vector I considered any vector, so let's say this is my rotational motion of a vector that I'll call A. So that's A at time t. And then at some later time, that's A of t plus delta t. This vector is my delta A. I'll call this angle delta theta. And so the vector A is rotating in that direction with an angular velocity capital omega.

Now in the example I just did, my vector A is actually r of t. But so everywhere where I have an r here, I can just write an A. This is now just an arbitrary vector in space that is rotating at an angular velocity capital omega. And what we see, using the same analysis, we would find that the magnitude of the time rate of change of the vector, of the rotating vector, is equal to the length of the rotating vector, which is A, times d theta dt.

Or in other words, the length of the rotating vector times the angular velocity of the rotation. That's true for any vector. OK? This is a general result. v equals r omega is just a special case of this general rule where, in that case, my rotating vector is a position vector. But this is true for any vector that's rotating in space.

In particular for a rotating angular momentum-- for a rotating angular momentum vector, I have that the magnitude of the time derivative of the rotating angular momentum vector is just equal to the length of the angular momentum vector, the magnitude of the angular momentum, times the angular velocity of rotation.

Now in addition, in the particular case of a rotating angular momentum vector or of any angular momentum vector rather, we know that the time derivative of the angular momentum vector is also equal to the torque vector. So I can set these two things equal. In the case of a rotating angular momentum vector, the magnitude of the torque is given by the magnitude of the rotating angular momentum vector times the angular speed of rotation. And that's just using the general behavior of a rotating vector in space.