# Lecture 31 Blackboard #1

## Simple Pendulum

Forces: T and Mg

$Mg\cos\theta = T$ Balanced

$-Mg\sin\theta = F_t$ = Restoring Force

$$F_t = -Mg\sin\theta = m\frac{d^2s}{dt^2}$$

$$s = L\theta$$

$$\frac{d^2s}{dt^2} = L\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \quad \theta \ll 1$$

SHM

$$\theta = \theta_0\cos(\omega t + \delta)$$

$$\omega = \sqrt{g/L}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

More exact:

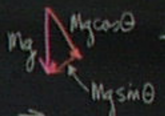
$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \frac{9}{64}\sin^4\frac{\theta_0}{2} + \cdots\right]$$

$\theta_0 = 15^\circ$ Corr = 1.5%

Alternate: Torques about 'O'

$$\tau = -mgL\sin\theta \quad (= \vec{r}\times\vec{F})$$

$$I = mL^2$$




$$\vec{\tau} = I\vec{\alpha}$$

$$I\frac{d^2\theta}{dt^2} = -mgL\sin\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$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## Physical Pendulum

- Any rigid body
- Axis NOT at CM

$$\vec{\tau} = I\vec{\alpha}$$

$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{mgd}{I}\theta = 0 \quad \theta \ll 1$$

$$\theta = \theta_0\cos(\omega t + \delta)$$

$$\omega^2 = \frac{mgd}{I}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

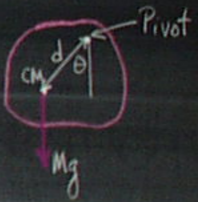
If $I = md^2$

$$T = 2\pi\sqrt{\frac{d}{g}} \Rightarrow \text{Simple Pendulum.}$$

$$T_{\text{Phys Pend}} = T_{\text{Ideal Pend}}$$

If $L = \dfrac{I}{md}$

$\Rightarrow$ Same Oscillation Frequency!

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Lecture 31 Blackboard #1

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## Physical Pendulum

- Any rigid body
- Axis NOT at CM

$$\vec{\tau} = I\vec{\alpha}$$

$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{mgd}{I}\theta = 0 \qquad \theta \ll 1$$

$$\theta = \theta_0\cos(\omega t + \delta)$$

$$\omega^2 = \frac{mgd}{I}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

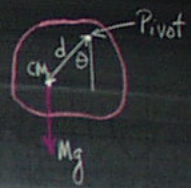
If $I = md^2$

$T = 2\pi\sqrt{\frac{d}{g}} \Rightarrow$ Simple Pendulum.

$$T_{\text{Phys Pend}} = T_{\text{Ideal Pend}}$$

If $L = \frac{I}{md}$

$\Rightarrow$ Same Oscillation Frequency!

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## Physical Pendulum

- Oscillates from axis-O.
- Corresponds to ideal pendulum of length L with all its mass at O'. OO' = L through CM.

O + O' ⇒ Conjugate points.

Measure $T_O$ and calculate L.

## Pendulum about O'

$$T_{O'} = 2\pi\sqrt{\frac{I'}{mg(L-d)}}$$

↳ O' → CM

$$L - d = \frac{I}{md} - d = \frac{I - md^2}{md} = \frac{I_{CM}}{md}$$

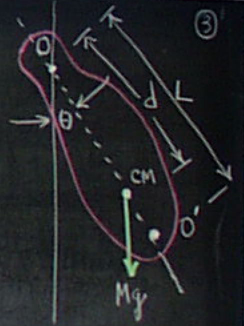
$$I = I_c + md^2$$

$$I' = I_c + m(L-d)^2$$

$$T_{O'} = 2\pi\sqrt{\frac{I_c + m(L-d)^2}{mg(L-d)}} = 2\pi\sqrt{\frac{I_c + m\left(\frac{I_c}{md}\right)^2}{mg\left(\frac{I_c}{md}\right)}}$$

$$T_{O'} = 2\pi\sqrt{\frac{md^2 + I_c}{mgd}} = 2\pi\sqrt{\frac{I}{mgd}}$$

$\therefore T_{O'} \equiv T_O$ Periods are the same!!




Lecture 31 Blackboard #2

## Torsional Pendulum

- Body suspended by thin wire.
- Twist wire through $\theta$.
- Wire has Hooke's Law restoring torque.

$$\tau = -\kappa\theta = I\alpha$$

$\kappa$: Torsion const of wire

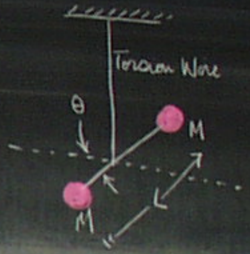
$$-\kappa\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0$$

$$\omega^2 = \frac{\kappa}{I}$$

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

- balance wheel of mech. watches
- galvanometers
- Cavendish Exp.

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## Percussion

- Suspended body
- Impulse at $O'$

$$\vec{J} = \Delta(m v_c) = \int \vec{F}\,dt$$

$$v_c = \frac{J}{M} \qquad \omega = \frac{J r_c'}{I_c}$$

Impulse changes $\vec{L}$ about CM

$$\Delta L = \Delta(I_c\omega) = \tau\,dt = J r_c'$$

Body Translates + Rotates about CM.

Any pt. $\vec{v} = \vec{v}_c + \vec{v}_t$ (rel to cm)

At $O'$: velocities add.
At $O$: velocities opposite. ] $OO'$ Conj. Points

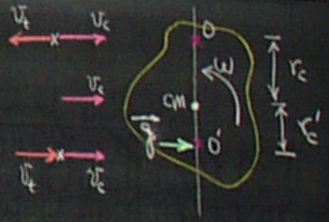
$$v_O = v_c - v_t = v_c - \omega r_c$$

$$= \frac{J}{m} - \omega r_c = \frac{J}{m}\left(1 - \frac{m r_c r_c'}{I_c}\right)$$

But $r_c r_c' = \frac{I_c}{m}$ (See Phy. Pend)

$$\therefore v_O = 0\ !!!$$

$$v_c = \omega r_c.$$

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Lecture 31 Blackboard #3

## Percussion

· Suspended body

· Impulse at O'

$$\vec{\mathcal{J}} = \Delta(M v_c) = \int \vec{F}\,dt$$

$$v_c = \frac{\vec{\mathcal{J}}}{M} \qquad \omega = \frac{\mathcal{J} r_c'}{I_c}$$

Impulse changes $\vec{L}$ about CM

$$\Delta L = \Delta(I_c \omega) = \tau\, dt = \mathcal{J} r_c'$$

Body Translates + Rotates about CM.

Any pt: $\vec{v} = \vec{v}_c + \vec{v}_t$ (rel to cm)

At O': velocities add.

At O: velocities opposite. ] OO' Conj. Points

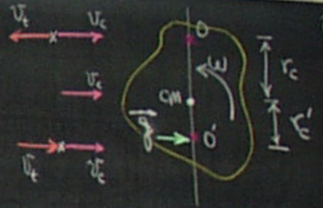
$$v_O = v_c - v_t = v_c - \omega r_c$$

$$= \frac{\mathcal{J}}{m} - \omega r_c = \frac{\mathcal{J}}{m}\left(1 - \frac{m r_c r_c'}{I_c}\right)$$

But $r_c r_c' = \frac{I_c}{m}$ (See Phy. Pend)

$\therefore v_O = 0$ !!!

$v_c = \omega r_c$.

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## Example Potential Well

· Sphere, mass m, rolling on surface of radius R.

Assume $\theta \ll 1$

$x = R\theta$

$U(\theta) = mgR(1-\cos\theta)$

$\simeq mgR\frac{\theta^2}{2}$

$U(0) = 0$

Let $\theta_0$ = max disp

$x_0 = R\theta_0$ (max x)

$$\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 + mgR\frac{\theta^2}{2} = mgR\frac{\theta_0^2}{2}$$

$$\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 + mg\frac{x^2}{2R} = \frac{mg}{2R}x_0^2$$

Compare with

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2$$

Rewrite

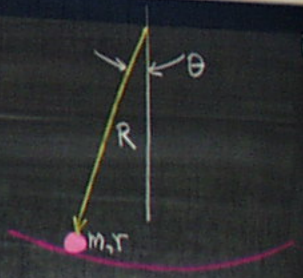
$$\frac{1}{2}\beta\frac{mr^2}{r^2}(r\omega)^2 + \frac{1}{2}mv^2 + \frac{mg}{2R}x^2 = \frac{1}{2}\frac{mg}{R}x_0^2$$

$$\frac{1}{2}mv^2[\beta+1] + \frac{mg}{2R}x^2 = \frac{1}{2}\frac{mg}{2R}x_0^2$$

$$\omega^2 = \frac{k}{m} = \frac{g}{R}\frac{1}{\beta+1}$$

$$T = 2\pi\sqrt{\frac{R}{g}(\beta+1)} \qquad R =$$

$$T_{sphere} = 2\pi\sqrt{\frac{7}{5}\frac{R}{g}}$$

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Lecture 31 Blackboard #4

## Damped Oscillations

- No Friction: Amp. cons. with time.

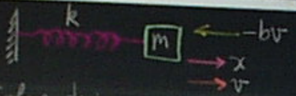
Real Systems: Decay + Stop

Assume Friction ~ velocity

$F = -bv$

← strength of damping

$-kx - bv = m\frac{d^2x}{dt^2}$




Solve de:

$x = A_0 e^{-\frac{b}{2m}t}\cos(\omega' t + \varphi)$

$= A(t)\cos(\omega' t + \varphi)$

$\omega'^2 = \frac{k}{m} - \frac{b^2}{4m^2}$

$\omega'^2 < \omega_0^2 = \frac{k}{m}$

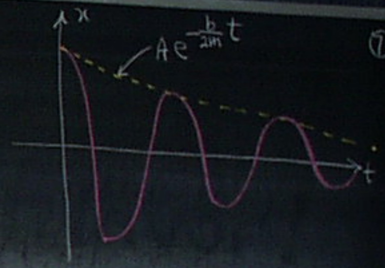
- Ampl decreases with time.

$E = \frac{1}{2}kA^2$ SHM.

$E(t) = \frac{1}{2}kA_0^2 e^{-b/m\,t}$

$E(t) = E_0 e^{-\frac{b}{m}t}$

$\gamma = \frac{b}{m}$ measures time for energy to reach 1/e initial value

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## Critical Damping

Suppose $\frac{k}{m} - \frac{b^2}{4m^2} = 0$

$b_c = 2\sqrt{km}$

Recall: $\omega'^2 = \omega_0^2 - \frac{\gamma^2}{4}$

If $b > b_c$, $\omega'^2 < 0$ !!

∴ NO Oscillation.

New solution:

$x(t) = A_1 e^{-(\frac{\gamma}{2}+\beta)t} + A_2 e^{-(\frac{\gamma}{2}-\beta)t}$

where $\beta^2 = \frac{\gamma^2}{4} - \omega_0^2$

When $b = b_c = 2\sqrt{km}$

$x(t) = (A + Bt)e^{-\gamma t/2}$

Critical damping important for mech. systems.

Shock applied to a critically damped system results in NO overshoot or oscillation.

→ gentle motion to new equilibrium.

Note:

$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

$\frac{dE}{dt} = mv\frac{dv}{dt} + kx\frac{dx}{dt}$

$= v(ma) + kx$

$= -bv^2$ with damping.

If $b = 0$ $\frac{dE}{dt} = 0$ ; E conserved

If $b \neq 0$ E decreases with time.

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Lecture 31 Blackboard #5