

## ii) Elastic Collisions - Two Dimensions

20-13

- Assume special case where target particle is initially at rest,  $\vec{v}_2 = 0$ .
- Particle collide by interacting. For many forces, gravity, electromag., etc. forces act along line joining the particles.
- The initial particle and scattered particles define a common plane - 2-dimensional problem.

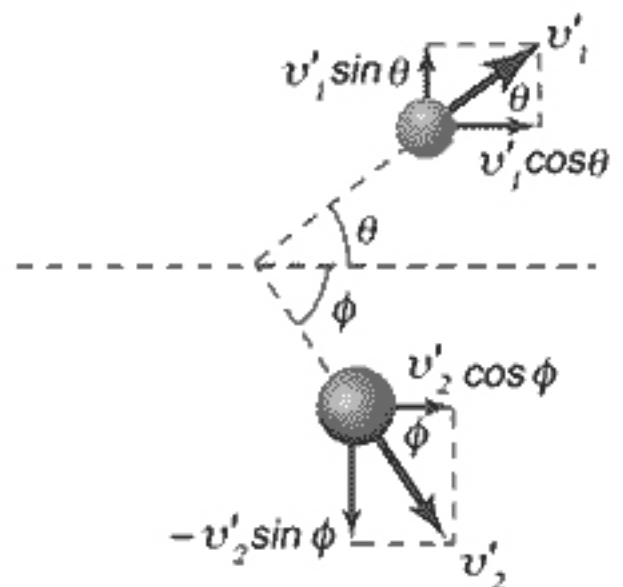
Given initial velocities, we have four unknowns following collision:

$$\left. \begin{array}{l} v'_{1x}, v'_{1y} \\ v'_{2x}, v'_{2y} \end{array} \right\} \text{ x and y velocities of particles -1 and -2.}$$

(a) Before the collision



(b) After the collision



Schematic representation of an elastic glancing collision between two particles: (a) before the collision and (b) after the collision. Note that the impact parameter,  $b$ , must be greater than zero for a glancing collision.

Cons. of Momentum: 2 equations (x, y)  
 Cons. of Energy: 1 equation

$\therefore$  Can only get restrictions on the final motion  
 or at least one other quantity must be known.

Cons of Momentum:

$$m_1 \vec{v}_1 = m_1 \vec{v}_1' + m_2 \vec{v}_2'$$

$$m_1 v_1 = m_1 v_1' \cos \theta + m_2 v_2' \cos \phi$$

$$0 = m_1 v_1' \sin \theta - m_2 v_2' \sin \phi$$

Cons. of Energy:

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

Special Case:  $m_1 = m_2$

$$v_1^2 = v_1'^2 + v_2'^2 \quad (1)$$

$$v_2' \cos \phi = v_1 - v_1' \cos \theta \quad (2)$$

$$v_2' \sin \phi = v_1' \sin \theta \quad (3)$$

$$\textcircled{2}^2 + \textcircled{3}^2 \quad v_2'^2 = v_1^2 - 2v_1v_1'\cos\theta + v_1'^2$$

Use Eq. ① to eliminate  $v_2'$

$$v_1' = v_1 \cos\theta$$

$$\frac{1}{2} m v_1'^2 = \frac{1}{2} m v_1^2 \cos^2\theta$$

KE of deflected  
incident projectile  
after scattering.

We had cons. of momentum :

$$\vec{v}_1 = \vec{v}_1' + \vec{v}_2'$$

$$v_1^2 = v_1'^2 + v_2'^2 + 2v_1'v_2'\cos(\theta+\phi) \quad [\text{square}]$$

Comparing with Eq. ①, we must have that

$$\theta + \phi = \pi/2$$

When particles of equal mass collide, the sum of their scattering angles is  $90^\circ$ .

$$\text{If } m_1 > m_2 \quad 0 \leq \theta \leq \pi/2$$

$$m_1 < m_2 \quad 0 \leq \theta \leq \pi$$

### Example: 2-D Elastic Collision

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$$m_A = 5 \text{ kg} \quad v_{A1} = 4 \text{ m/s}$$

$$m_B = 3 \text{ kg} \quad v_{B1} = 0 \text{ [stationary]}$$

Assume after collision,  $v_{A2} = 2 \text{ m/s}$

Find  $v_{B2}$ ,  $\theta$  and  $\phi$

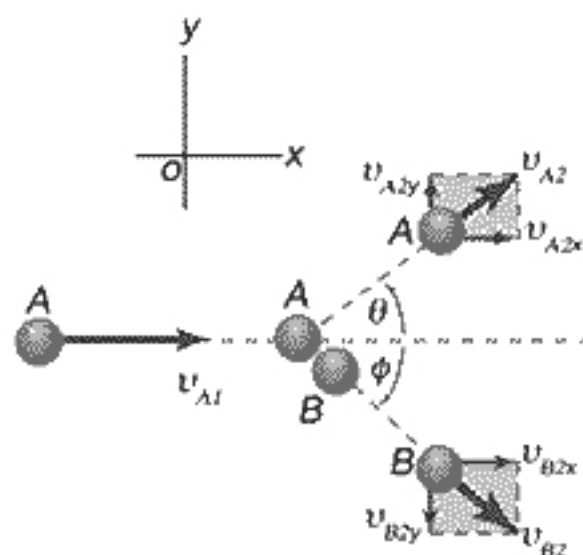
Since collision is elastic

$$KE_f = KE_i$$

$$\frac{1}{2} (5 \text{ kg})(4 \text{ m/s})^2 = \frac{1}{2} (5 \text{ kg})(2 \text{ m/s})^2 + \frac{1}{2} (3 \text{ kg})v_{B2}^2$$

Solving,

$$v_{B2} = 4.47 \text{ m/s}$$



Conservation of x- and y-components of momentum gives,

$$\textcircled{1} \quad (5 \text{ kg})(4 \text{ m/s}) = (5 \text{ kg})(2 \text{ m/s}) \cos \theta + (3 \text{ kg})(4.47 \text{ m/s}) \cos \phi$$

$$\textcircled{2} \quad 0 = (5 \text{ kg})(2 \text{ m/s}) \sin \theta - (3 \text{ kg})(4.47 \text{ m/s}) \sin \phi$$

$$\left. \begin{array}{l} \text{Solve Eq. ① for } \cos \phi \\ \text{Solve Eq. ② for } \sin \phi \end{array} \right\} \text{Square and add} \quad \sin^2 \phi + \cos^2 \phi = 1$$

Then solve for  $\theta$  and finally  $\phi$ :

$$\theta = 36.9^\circ$$

$$\phi = 26.6^\circ$$

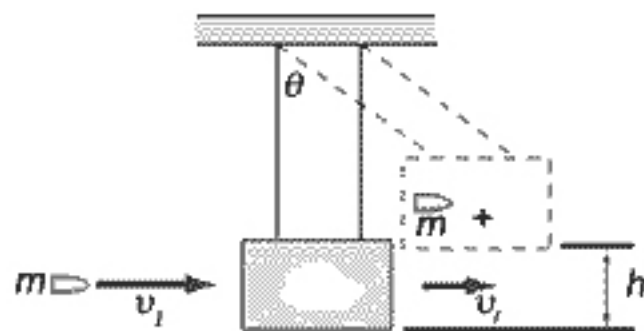


Diagram of a ballistic pendulum. Note that  $v_f$  is the velocity of the system right after the perfectly inelastic collision.

A method used to measure the speed of a projectile such as a bullet.

Bullet: mass =  $m$

speed =  $v_i$  (initially)

Block: mass =  $M \gg m$

speed = 0 (initially)

After collision, mass  $(m+M)$  moves up a height  $h$ .

Collision in two parts:

i) collision and bullet stopped in block.

ii) motion of block and bullet to maximum height  $h$ .

- Recoil -

i) collision:

- collision time is short

- block does not move during collision

- no net external force during collision which is perfectly inelastic and momentum is conserved.

- the velocity right after the collision is given by:

$$mv_i = (m+M)v_f \quad (1)$$

(i) Recoil :

- after collision the system  $(m+M)$  has a Kinetic Energy.
- Energy is conserved.
- KE at the bottom is transformed to PE in the block and the bullet at the height  $h$ .

$$\frac{1}{2}(m+M)v_f^2 = (m+M)gh$$

$$\therefore v_f = \sqrt{2gh} \quad (2)$$

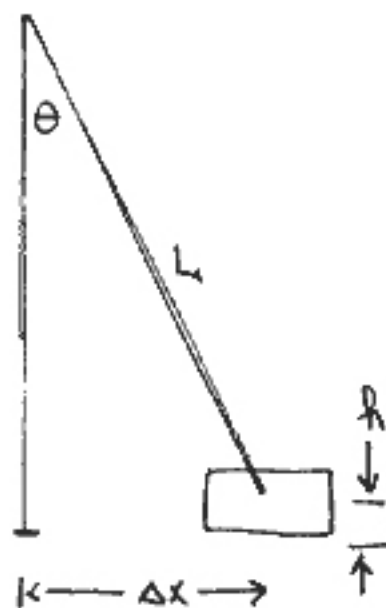
From (1)  $v_i = \frac{(m+M)}{m} \sqrt{2gh}$

For  $\theta \ll 1$

$$\Delta x = L \sin \theta \approx L\theta \Rightarrow \theta \approx \frac{\Delta x}{L}$$

$$h = L(1 - \cos \theta) \\ \approx \frac{L\theta^2}{2} \approx \frac{(\Delta x)^2}{2L}$$

$$v_i = \left(\frac{m+M}{m}\right) \sqrt{2g \left(\frac{\Delta x^2}{2L}\right)} \\ = \left(\frac{m+M}{m}\right) \sqrt{\frac{g}{L}} (\Delta x)$$



$$m = 2.7 \text{ gm} \quad L = 1.14 \text{ m} \\ M = 3840 \text{ gm} \quad \Delta x = 6.5 \text{ cm}$$

$$v_i = 293 \text{ m/s}$$

## Ballistic Pendulum / Kinetic Energies

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$$K_i = \frac{1}{2} m v_i^2$$

$$K_f = \frac{1}{2} (m+M) v_f^2 = \frac{1}{2} (m+M) \left( \frac{m}{m+M} \right)^2 v_i^2 = \frac{m^2}{2(m+M)} v_i^2$$

$$\frac{K_f}{K_i} = \frac{m}{m+M} \lll 1$$

Most of the initial KE is lost.

### Collision Time

Assume bullet decelerates uniformly in a distance of 0.10 m.

$$v_f^2 - v_0^2 = 2as \quad v_f \equiv 0.0$$

$$a = -\frac{v_0^2}{2s} \approx -\frac{(300)^2}{2 \times 0.10} \text{ m/s}^2$$

Also  $v_f = v_0 + at$

$$t \sim \frac{-v_0}{a} = \frac{-300}{-\frac{(300)^2}{2 \times 0.10}} = .00067 \text{ s}$$

Period of pendulum:  $T = 2\pi \sqrt{\frac{L}{g}} = 2\pi \sqrt{\frac{1.14}{9.81}} = 2.14 \text{ s}$

$t \lll T$  [ $\therefore$  good approx.]

## COLLISIONS

### 1. Conditions:

An event is a collision if  $\Delta t \ll \Delta T$

- time can be separated into before, during + after
- $\Delta t$  collision time
- $\Delta T$  observation time

An event is a collision if  $|I_{\text{ext}}| \ll |I_{\text{coll}}|$  -  
the impulse of external forces can be neglected  
and momentum is conserved.

### 2. Collision Classifications:

Elastic - KE is conserved

Inelastic - KE is not conserved

Completely inelastic - particles stick together after

### 3. Notation

$m_1, m_2$  - masses of the two particles

$\vec{v}_{1i}, \vec{v}_{2i}$  - initial (before collision) velocities of parts 1, 2.

$\vec{v}_{1f}, \vec{v}_{2f}$  - final (after collision) velocities of particles 1, 2.

### 4. Equations

Conservation of momentum - valid for all collisions:

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$$

Conservation of kinetic energy - valid only for elastic collisions.

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$



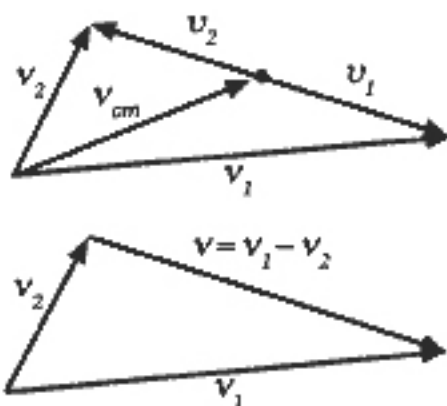
## Center-of-Mass Frame / Collisions / KE

- 2 Particle System

$\vec{u}_1, \vec{u}_2$  : CM velocities

$\vec{v}_1, \vec{v}_2$  : Lab velocities

$\vec{v}_{cm}$  : Velocity of CM.



We had for the Kinetic Energy

$$K = \frac{1}{2} M v_{cm}^2 + K_{INT}$$

$$= \frac{1}{2} M v_{cm}^2 + \underbrace{\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2}_{\text{Internal Energy}}$$

$u_1, u_2$  : velocities of particles relative to CM frame

$\frac{1}{2} M v_{cm}^2$   $\rightarrow$  Translational motion of CM. When no external forces act must be conserved.

$$\vec{u}_1 = \vec{v}_1 - \vec{v}_{cm} = \vec{v}_1 - \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} = \frac{m_2 (\vec{v}_1 - \vec{v}_2)}{m_1 + m_2}$$

$$\vec{u}_2 = \vec{v}_2 - \vec{v}_{cm} = \vec{v}_2 - \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} = -\frac{m_1 (\vec{v}_1 - \vec{v}_2)}{m_1 + m_2}$$

$$\therefore K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2)^2$$

$\uparrow$  Relative velocity.

2nd term: Internal Energy Represents maximum energy available for a totally inelastic collision.

$$\text{let } \left. \begin{aligned} \vec{v}_{\text{rel}} &= \vec{u}_1 - \vec{u}_2 \\ &= \vec{v}_1 - \vec{v}_2 \end{aligned} \right\} \text{relative velocity of two particles}$$

$$K = \frac{1}{2} M v_{\text{cm}}^2 + \frac{1}{2} \mu v_{\text{rel}}^2$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \Rightarrow \text{reduced mass for a 2-particle system.}$$

### Collisions

In any collision, whether elastic or inelastic, when external forces can be neglected, total momentum is conserved.

$$\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 \quad [\text{constant}]$$

Value of  $\vec{P}$  depends on coordinate system, but conservation is true in all frames.

### 2-Particles

$$\vec{r}_{\text{cm}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{v}_{\text{c}} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$



(on line joining,  $\vec{v}_1, \vec{v}_2$ )

Momenta in C-System:

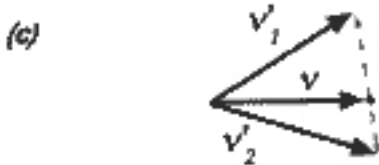
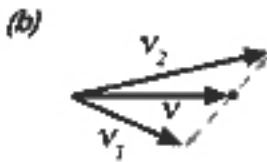
$$\vec{p}_{1c} = m_1 \vec{u}_1 = \frac{m_1 m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2) = \mu \vec{v}_{rel}$$

$$\vec{p}_{2c} = m_2 \vec{u}_2 = -\frac{m_1 m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2) = -\mu \vec{v}_{rel}$$

Total Momentum:

$$\vec{P}_C = \vec{p}_{1c} + \vec{p}_{2c} \equiv 0. \quad [\text{CM-Frame}]$$

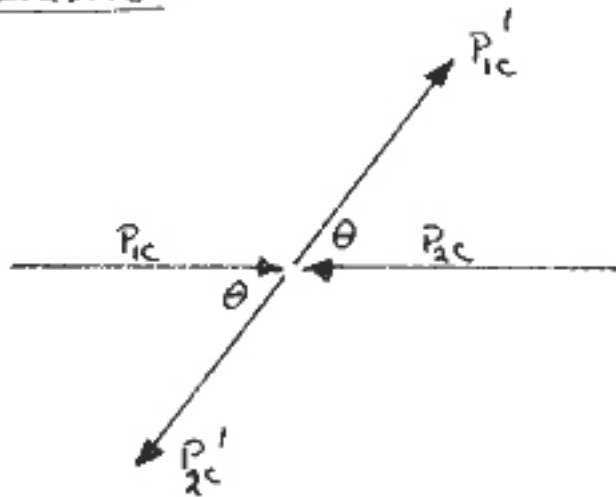
$$\vec{P}_L = \vec{p}_1 + \vec{p}_2 = (m_1 + m_2) \vec{V}_{cm}$$



What does collision look like in CM frame?

- Initial and final velocities determine the scattering plane.
- Each particle is scattered through the same angle  $\theta$ .

Elastic:



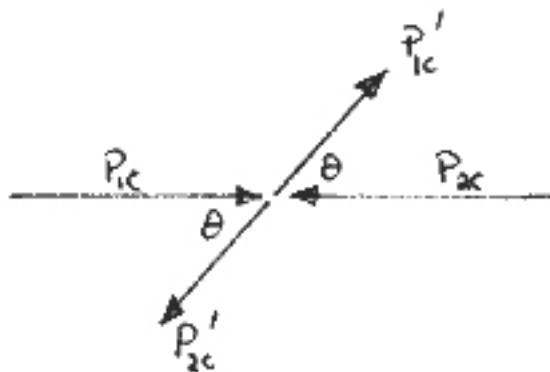
$\xrightarrow{V_{cm}}$   
 [cm is moving at constant speed before, during and after collision]

$$|p_{1c}| = |p'_{1c}|$$

$$|p_{2c}| = |p'_{2c}|$$

lengths of momentum vectors before and after collision are equal. Energy conserved. Collisions are back-to-back.  $\vec{p} = 0$ .

Inelastic



Back-to-Back  
 Momentum magnitudes are reduced following collision.  
 loss in KE  
 Inelastic Collision.