Chapter 4

Fluid Description of Plasma

The single particle approach gets to be horribly complicated, as we have seen.

Basically we need a more statistical approach because we can't follow each particle separately. If the details of the distribution function in velocity space are important we have to stay with the Boltzmann equation. It is a kind of particle conservation equation.

4.1 Particle Conservation (In 3-d Space)

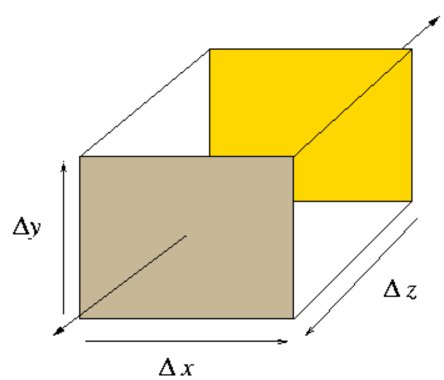

Figure 4.1: Elementary volume for particle conservation

Number of particles in box $\Delta x \Delta y \Delta z$ is the volume, $\Delta V = \Delta x \Delta y \Delta z$, times the density n. Rate of change of number is is equal to the number flowing across the boundary per unit time, the flux. (In absence of sources.)

$$-\frac{\partial}{\partial t}[\Delta x \Delta y \Delta z \; n] = \text{ Flow Out across boundary.} \tag{4.1}$$

Take particle velocity to be $\mathbf{v}(\mathbf{r})$ [no random velocity, only flow] and origin at the center of the box refer to flux density as $n\mathbf{v} = \mathbf{J}$.

$$\text{Flow Out } = [J_z(0,0,\Delta z/2) - J_z(0,0,-\Delta z/2)] \; \Delta x \Delta y \; + \; x \; + \; y \; . \tag{4.2}$$

Expand as Taylor series

$$J_z(0,0,\eta) = J_z(0) + \frac{\partial}{\partial z} J_z \ . \ \eta \tag{4.3}$$

So,

$$\begin{aligned} \text{flow out} \ &\simeq \ \frac{\partial}{\partial z}(nv_z)\Delta z \Delta x \Delta y \ + \ x \ + y \\ &= \ \Delta V \ \nabla \ . \ (n\mathbf{v}). \end{aligned} \tag{4.4}$$

Hence *Particle Conservation*

$$-\frac{\partial}{\partial t} n = \nabla.(n\mathbf{v}) \tag{4.5}$$

Notice we have essential proved an elementary form of Gauss's theorem

$$\int_v \nabla.\mathbf{A} d^3\mathbf{r} \ = \ \int_{\partial\gamma} \mathbf{A}.\mathbf{dS}. \tag{4.6}$$

The expression: *'Fluid Description'* refers to any simplified plasma treatment which does *not* keep track of v-dependence of f detail.

1. Fluid Descriptions are essentially 3-d ($\mathbf{r}$).
2. Deal with quantities averaged over velocity space (e.g. density, mean velocity, ...).
3. Omit some important physical processes (but describe others).
4. Provide tractable approaches to many problems.
5. Will occupy most of the rest of my lectures.

Fluid Equations can be derived mathematically by taking moments[1] of the Boltzmann Equation.

$$\begin{aligned} 0^{th} \text{ moment} \quad & \textstyle\int \quad d^3\mathbf{v} && (4.7) \\ 1st \text{ moment} \quad & \textstyle\int \quad \mathbf{v} d^3 v && (4.8) \\ 2nd \text{ moment} \quad & \textstyle\int \quad \mathbf{vv} d^3 v && (4.9) \end{aligned}$$

These lead, respectively, to (0) Particle (1) Momentum (2) Energy conservation equations.

We shall adopt a more direct 'physical' approach.

[1] They are therefore sometimes called 'Moment Equations.'

4.2 Fluid Motion

The motion of a fluid is described by a vector velocity field $\mathbf{v}(\mathbf{r})$ (which is the mean velocity of all the individual particles which make up the fluid at $\mathbf{r}$). Also the particle density $n(r)$ is required. We are here discussing the motion of fluid of a *single type* of particle of mass/charge, m/q so the charge and mass density are qn and mn respectively.

The particle conservation equation we already know. It is also sometimes called the 'Continuity Equation'

$$\frac{\partial}{\partial t}\, n + \nabla.(n\mathbf{v}) = 0 \tag{4.10}$$

It is also possible to expand the $\nabla.$ to get:

$$\frac{\partial}{\partial t} n + (\mathbf{v}.\nabla)n + n\nabla.\mathbf{v} = 0 \tag{4.11}$$

The significance, here, is that the first two terms are the 'convective derivative" of n

$$\frac{D}{Dt} \equiv \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}.\nabla \tag{4.12}$$

so the continuity equation can be written

$$\frac{D}{Dt} n = -n\nabla.\mathbf{v} \tag{4.13}$$

4.2.1 Lagrangian & Eulerian Viewpoints

There are essentially 2 views.

1. Lagrangian. Sit on a fluid element and move with it as fluid moves.

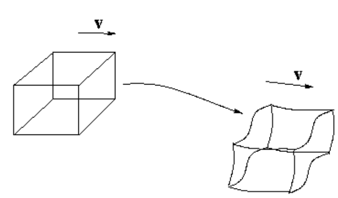

Figure 4.2: Lagrangean Viewpoint

2. Eulerian. Sit at a fixed point in space and watch fluid move through your volume element: "identity" of fluid in volume continually changing

   $\frac{\partial}{\partial t}$ means rate of change at *fixed* point (Euler).

   $\frac{D}{Dt} \equiv \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}.\nabla$ means rate of change at *moving* point (Lagrange).

   $\mathbf{v}.\nabla = \frac{dx}{\partial t}\frac{\partial}{\partial x} + \frac{dy}{\partial t}\frac{\partial}{\partial y} + \frac{dz}{\partial t}\frac{\partial}{\partial z}$: change due to motion.

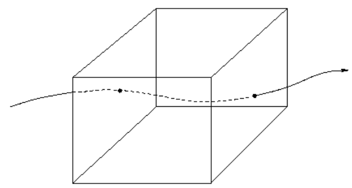

Figure 4.3: Eulerian Viewpoint

Our derivation of continuity was Eulerian. From the Lagrangian view

$$\frac{D}{Dt}\,n = \frac{d}{dt}\,\frac{\Delta N}{\Delta V} = -\frac{\Delta N}{\Delta V^2}\,\frac{d}{dt}\,\Delta V = -n\,\frac{1}{\Delta V}\,\frac{d\Delta V}{dt} \tag{4.14}$$

since total number of particles in volume element (ΔN) is constant (we are moving with them). ($\Delta V = \Delta x \Delta y \Delta z$.)

$$\text{Now } \frac{d}{dt}\Delta V = \frac{d\Delta x}{dt}\,\Delta y\Delta z + \frac{d\Delta y}{dt}\Delta z\Delta x + \frac{d\Delta z}{dt}\Delta y\Delta x \tag{4.15}$$

$$= \Delta V\left[\frac{1}{\Delta x}\,\frac{d\Delta x}{dt} \;+\; \frac{1}{\Delta y}\,\frac{d\Delta y}{dt} + \frac{1}{\Delta x}\,\frac{d\Delta z}{dt}\right] \tag{4.16}$$

$$\text{But } \frac{d\,(\Delta x)}{dt} = v_x\,(\Delta x/2) - v_x\,(-\Delta x/2) \tag{4.17}$$

$$\simeq \Delta x\,\frac{\partial v_x}{\partial x} \quad \text{etc.} \quad \ldots y \quad \ldots z \tag{4.18}$$

Hence

$$\frac{d}{dt}\Delta V = \Delta V\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] = \Delta V \nabla.\mathbf{v} \tag{4.19}$$

and so

$$\frac{D}{Dt}\,n = -n\nabla.\mathbf{v} \tag{4.20}$$

Lagrangian Continuity. Naturally, this is the same equation as Eulerian when one puts $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}.\nabla$.

The quantity $-\nabla.\mathbf{v}$ is the rate of (Volume) compression of element.

4.2.2 Momentum (Conservation) Equation

Each of the particles is acted on by the Lorentz force $q[\mathbf{E}+\mathbf{u}_i\wedge\mathbf{B}]$ ($\mathbf{u}_i$ is individual particle's velocity).

Hence total force on the fluid element due to E-M fields is

$$\sum_i\,(q\,[\mathbf{E}+\mathbf{u}_i\wedge\mathbf{B}]) = \Delta N\;q\,(\mathbf{E}+\mathbf{v}\wedge\mathbf{B}) \tag{4.21}$$

(Using mean: $\mathbf{v} = \sum_i \mathbf{u}/\Delta N$.)

E-M Force density (per unit volume) is:

$$\mathbf{F}_{EM} = nq(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}). \tag{4.22}$$

The total momentum of the element is

$$\sum_i m\mathbf{u}_i = m\ \Delta N\ \mathbf{v} = \Delta V\ mn\mathbf{v} \tag{4.23}$$

so *Momentum Density* is $mn\mathbf{v}$.

If no other forces are acting then clearly the equation of motion requires us to set the time derivative of $mn\mathbf{v}$ equal to $\mathbf{F}_{EM}$. Because we want to retain the identity of the particles under consideration we want D/Dt i.e. the convective derivative (Lagrangian picture).

In general there are additional forces acting.
(1) Pressure (2) Collisional Friction.

4.2.3 Pressure Force

In a gas $p(= nT)$ is the force per unit area arising from thermal motions. The surrounding fluid exerts this force on the element:

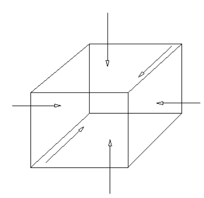

Figure 4.4: Pressure forces on opposite faces of element.

Net force in x direction is

$$\begin{aligned} -\quad & p\,(\Delta x/2)\,\Delta y \Delta z + p\,(-\Delta x/2)\,\Delta y \Delta z && (4.24)\\ \simeq\quad & -\Delta x \Delta y \Delta z\ \frac{\partial p}{\partial x} = -\Delta V\ \frac{\partial p}{\partial x} = -\Delta V\ (\nabla p)_x && (4.25)\end{aligned}$$

So (isotropic) pressure *force density* (/unit vol)

$$\mathbf{F}_p = -\nabla p \tag{4.26}$$

How does this arise in our picture above?
Answer: Exchange of momentum by particle thermal motion across the element boundary.

Although in Lagrangian picture we move with the element (as defined by mean velocity $\mathbf{v}$) individual particles also have thermal velocity so that the additional velocity they have is

$$\mathbf{w}_i = \mathbf{u}_i - \mathbf{v} \qquad \text{'peculiar' velocity} \tag{4.27}$$

Because of this, some cross the element boundary and exchange momentum with outside. (Even though there is no net change of number of particles in element.) Rate of exchange of momentum due to particles with peculiar velocity $\mathbf{w}, d^3\mathbf{w}$ across a surface element $\mathbf{ds}$ is

$$\underbrace{f(\mathbf{w})m\mathbf{w}\ d^3\mathbf{w}}_{\text{mom}^m \text{ density at } \mathbf{w}} \quad \times \quad \underbrace{\mathbf{w}\ .\ \mathbf{ds}}_{\text{flow rate across } \mathbf{ds}} \tag{4.28}$$

Integrate over distrib function to obtain the total momentum exchange rate:

$$\mathbf{ds}.\int m\mathbf{w}\mathbf{w}f(\mathbf{w})d^3\mathbf{w} \tag{4.29}$$

The thing in the integral is a tensor. Write

$$\mathsf{p} \ = \int m\mathbf{w}\mathbf{w}f(\mathbf{w})d^3\mathbf{w} \qquad \text{(Pressure Tensor)} \tag{4.30}$$

Then momentum exchange rate is

$$\mathsf{p}\ .\ \mathbf{ds} \tag{4.31}$$

Actually, if $f(\mathbf{w})$ is isotropic (e.g. Maxwellian) then

$$p_{xy} \ = \int m\ w_x\ w_y\ f(\mathbf{w})d^3\mathbf{w} = 0 \quad \text{etc.} \tag{4.32}$$

$$\text{and } p_{xx} \ = \int m w_x^2\ f(w)d^3\mathbf{w} \equiv nT(= p_{yy} = p_{zz} = \text{‘}p\text{’}) \tag{4.33}$$

So then the exchange rate is $p\mathbf{ds}$. (Scalar Pressure).

Integrate $\mathbf{ds}$ over the whole ΔV then x component of mom^m exchange rate is

$$p\left(\frac{\Delta x}{2}\right)\Delta y\Delta z - -p\left(\frac{-\Delta x}{2}\right)\Delta y\Delta z = \Delta V\ (\nabla p)_x \tag{4.34}$$

and so

Total momentum loss rate due to exchange across the boundary per unit volume is

$$\nabla p \qquad (= -\mathbf{F}_p) \tag{4.35}$$

In terms of the momentum equation, either we put ∇p on the momentum derivative side or $\mathbf{F}_p$ on force side. The result is the same.

Ignoring Collisions, Momentum Equation is

$$\frac{D}{Dt}(mn\Delta V\mathbf{v}) = [\mathbf{F}_{EM} + \mathbf{F}_p]\,\Delta V \tag{4.36}$$

Recall that $n\Delta V = \Delta N$; $\frac{D}{Dt}(\Delta N) = 0$; so

$$L.H.S. = mn\Delta V\ \frac{D\mathbf{v}}{dt} \quad . \tag{4.37}$$

Thus, substituting for $\mathbf{F}'s$:

Momentum Equation.

$$mn\frac{D\mathbf{v}}{Dt} = mn\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v}\right) = qn\,(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}) - \nabla p \tag{4.38}$$

4.2.4 Momentum Equation: Eulerian Viewpoint

Fixed element in space. Plasma flows through it.

1. E.M. force on element (per unit vol.)

$$\mathbf{F}_{EM} = nq(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \qquad \text{as before.} \tag{4.39}$$

2. Momentum flux across boundary (per unit vol)

$$= \nabla . \int m(\mathbf{v} + \mathbf{w})(\mathbf{v} + \mathbf{w}) \ f(\mathbf{w}) \ d^3\mathbf{w} \tag{4.40}$$

$$= \nabla . \{ \int m(\mathbf{vv} + \underbrace{\mathbf{vw} + \mathbf{wv}}_{\text{integrates to } 0} + \mathbf{ww}) \ f(\mathbf{w}) \ d^3\mathbf{w} \} \tag{4.41}$$

$$= \nabla . \{ mn\mathbf{vv} + \mathsf{p} \} \tag{4.42}$$

$$= mn(\mathbf{v}.\nabla)v + m\mathbf{v} \left[\nabla . (n\mathbf{v}) \right] + \nabla p \tag{4.43}$$

(Take isotropic p.)

3. Rate of change of momentum within element (per unit vol)

$$= \frac{\partial}{\partial t}(mn\mathbf{v}) \tag{4.44}$$

Hence, total momentum balance:

$$\frac{\partial}{\partial t}(mn\mathbf{v}) + mn(\mathbf{v}.\nabla)\mathbf{v} + m\mathbf{v} \left[\nabla . (n\mathbf{v}) \right] + \nabla p = \mathbf{F}_{EM} \tag{4.45}$$

Use the continuity equation:

$$\frac{\partial n}{\partial t} + \nabla . (n\mathbf{v}) = 0 \quad , \tag{4.46}$$

to cancel the third term and part of the 1st:

$$\frac{\partial}{\partial t}(mn\mathbf{v}) + m\mathbf{v} \left(\nabla . \left(n\mathbf{v} \right) \right) = m\mathbf{v} \{ \frac{\partial n}{\partial t} + \nabla . \left(n\mathbf{v} \right) \} + mn\frac{\partial \mathbf{v}}{\partial t} = mn\frac{\partial \mathbf{v}}{\partial t} \tag{4.47}$$

Then take ∇p to RHS to get final form:

Momentum Equation:

$$mn \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla) \mathbf{v} \right] = nq \left(\mathbf{E} + \mathbf{v} \wedge \mathbf{B} \right) - \nabla p \ . \tag{4.48}$$

As before, via Lagrangian formulation. (Collisions have been ignored.)

4.2.5 Effect of Collisions

First notice that *like* particle collisions *do not* change the total momentum (which is averaged over all particles of that species).

Collisions between *unlike* particles *do* exchange momentum between the species. Therefore once we realize that any quasi-neutral plasma consists of at least two different species (electrons and ions) and hence two different interpenetrating fluids we may need to account for another momentum loss (gain) term.

The rate of momentum density loss by species 1 colliding with species 2 is:

$$\nu_{12} n_1 m_1 (\mathbf{v}_1 - \mathbf{v}_2) \tag{4.49}$$

Hence we can immediately generalize the *momentum equation* to

$$m_1 n_1 \left[\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1 . \nabla) \, \mathbf{v}_1 \right] = n_1 q_1 \left(\mathbf{E} + \mathbf{v}_1 \wedge \mathbf{B} \right) - \nabla p_1 - \nu_{12} n_1 m_1 \left(\mathbf{v}_1 - \mathbf{v}_2 \right) \tag{4.50}$$

With similar equation for species 2.

4.3 The Key Question for Momentum Equation:

What do we take for p?

Basically $p = nT$ is determined by energy balance, which will tell how T varies. We could write an energy equation in the same way as momentum. However, this would then contain a term for heat flux, which would be unknown. In general, the k^{th} moment equation contains a term which is a $(k+1)^{th}$ moment.

Continuity, 0^{th} equation contains $\mathbf{v}$ determined by
Momentum, 1^{st} equation contains p determined by
Energy, 2^{nd} equation contains Q determined by ...

In order to get a sensible result we have to *truncate* this hierarchy. Do this by some sort of assumption about the heat flux. This will lead to an
Equation of State:

$$p n^{-\gamma} = \text{const.} \tag{4.51}$$

The value of γ to be taken depends on the heat flux assumption and on the isotropy (or otherwise) of the energy distribution.

<u>Examples</u>

1. Isothermal: $T = \text{const.}$: $\gamma = 1$.
2. Adiabatic/Isotropic: 3 degrees of freedom $\gamma = \frac{5}{3}$.
3. Adiabatic/1 degree of freedom $\gamma = 3$.

4. Adiabatic/2 degrees of freedom $\quad \gamma = 2$.

In general, $n(\ell/2)\delta T = -p(\delta V/V)$ $\quad$ (Adiabatic ℓ degrees)

$$\frac{\ell}{2}\frac{\delta T}{T} = \frac{-\delta V}{V} \quad = \quad +\frac{\delta n}{n} \tag{4.52}$$

So

$$\frac{\delta p}{p} \quad = \quad \frac{\delta n}{n} + \frac{\delta T}{T} = \left(1 + \frac{2}{\ell}\right)\frac{\delta n}{n}, \tag{4.53}$$

i.e.

$$pn^{-\left(1+\frac{2}{\ell}\right)} = \text{const.} \tag{4.54}$$

In a normal gas, which 'holds together' by collisions, energy is rapidly shared between 3 space-degrees of freedom. Plasmas are often rather collisionless so compression in 1 dimension often stays confined to 1-degree of freedom. Sometimes heat transport is so rapid that the isothermal approach is valid. It depends on the exact situation; so let's leave γ undefined for now.

4.4 Summary of Two-Fluid Equations

Species j

Plasma Response

1. Continuity:

$$\frac{\partial n_j}{\partial t} + \nabla.(n_j \mathbf{v}_j) = 0 \tag{4.55}$$

2. Momentum:

$$m_j n_j \left[\frac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j.\nabla)\,\mathbf{v}_j\right] = n_j q_j\,(\mathbf{E} + \mathbf{v}_j \wedge \mathbf{B}) - \nabla p_j - \bar{\nu}_{jk} n_j m_j\,(\mathbf{v}_j - \mathbf{v}_k) \tag{4.56}$$

3. Energy/Equation of State:

$$p_j n_j^{-\gamma} = \text{const..} \tag{4.57}$$

(j = electrons, ions).

Maxwell's Equations

$$\nabla.\mathbf{B} \quad = \quad 0 \qquad\qquad \nabla.\mathbf{E} = \rho/\epsilon_o \tag{4.58}$$

$$\nabla \wedge \mathbf{B} \quad = \quad \mu_o \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \wedge \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t} \tag{4.59}$$

With

$$\rho = q_e n_e + q_i n_i = e\left(-n_e + Z n_i\right) \tag{4.60}$$

$$\mathbf{j} = q_e n_e \mathbf{v}_e + q_i n_i \mathbf{v}_i = e\left(-n_e \mathbf{v}_e + Z n_i \mathbf{v}_i\right) \tag{4.61}$$

$$= -e n_e \left(\mathbf{v}_e - \mathbf{v}_i r\right) \qquad \text{(Quasineutral)} \tag{4.62}$$

<u>Accounting</u>

Unknowns		Equations	
n_e, n_i	2	Continuity e, i	2
$\mathbf{v}_e, \mathbf{v}_i$	6	Momentum e, i	6
p_e, p_i	2	State e, i	2
$\mathbf{E}, \mathbf{B}$	6	Maxwell	8
	16		18

but 2 of Maxwell ($\nabla.$ equs) are redundant because can be deduced from others: e.g.

$$\nabla.\left(\nabla \wedge \mathbf{E}\right) = 0 = -\frac{\partial}{\partial t}\left(\nabla.\mathbf{B}\right) \tag{4.63}$$

$$\text{and } \nabla.\left(\nabla \wedge \mathbf{B}\right) = 0 = \mu_o \nabla.\mathbf{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\nabla.\mathbf{E}\right) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{-\rho}{\epsilon_o} + \nabla.\mathbf{E}\right) \tag{4.64}$$

So 16 equs for 16 unknowns.

Equations still very difficult and complicated mostly because it is *Nonlinear*

In some cases can get a tractable problem by 'linearizing'. That means, take some known equilibrium solution and suppose the deviation (perturbation) from it is small so we can retain only the 1st linear terms and not the others.

4.5 Two-Fluid Equilibrium: Diamagnetic Current

Slab: $\frac{\partial}{\partial x} \neq 0 \qquad \frac{\partial}{\partial y}, \frac{\partial}{\partial z} = 0.$

Straight B-field: $\mathbf{B} = \mathbf{B}\hat{\mathbf{z}}.$

Equilibrium: $\frac{\partial}{\partial t} = 0 \qquad (E = -\nabla\phi)$

Collisionless: $\nu \rightarrow 0.$

Momentum Equation(s):

$$m_j n_j (\mathbf{v}_j.\nabla)\mathbf{v}_j = n_j q_j (\mathbf{E} + \mathbf{v}_j \wedge \mathbf{B}) - \nabla p_j \tag{4.65}$$

Drop j suffix for now. Then take x, y components:

$$mn\, v_x \frac{d}{dx} v_x = nq(E_x + v_y B) - \frac{dp}{dx} \tag{4.66}$$

$$mn\, v_x \frac{d}{dx} v_y = nq(0 - v_x B) \tag{4.67}$$

Eq 4.67 is satisfied by taking $v_x = 0$. Then 4.66 $\rightarrow$

$$nq(E_x + v_y B) - \frac{dp}{dx} = 0. \tag{4.68}$$

i.e.

$$v_y = \frac{-E_x}{B} + \frac{1}{nqB}\frac{dp}{dx} \tag{4.69}$$

or, in vector form:

$$\mathbf{v} = \underbrace{\frac{\mathbf{E}\wedge\mathbf{B}}{B^2}}_{\mathbf{E}\wedge\mathbf{B}\ \text{drift}} - \underbrace{\frac{\nabla p}{nq}\wedge\frac{\mathbf{B}}{B^2}}_{\text{Diamagnetic Drift}} \tag{4.70}$$

Notice:

- In magnetic field ($\perp$) fluid velocity is determined by component of momentum equation *orthogonal* to it (and to **B**).
- Additional drift (diamagnetic) arises in standard $\mathbf{F}\wedge\mathbf{B}$ form from pressure force.
- Diagmagnetic drift is opposite for opposite signs of charge (electrons vs. ions).

Now restore species distinctions and consider electrons plus single ion species i. Quasineutrality says $n_i q_i = -n_e q_e$. Hence adding solutions

$$n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i = \frac{\mathbf{E}\wedge\mathbf{B}}{B^2}\underbrace{(n_i q_i + n_e q_e)}_{=0} - \nabla(p_e + p_i)\wedge\frac{\mathbf{B}}{B^2} \tag{4.71}$$

Hence current density:

$$\mathbf{j} = -\nabla(p_e + p_i)\wedge\frac{\mathbf{B}}{B^2} \tag{4.72}$$

This is the diamagnetic current. The electric field, **E**, disappears because of quasineutrality. (General case $\sum_j q_j n_j v_j = -\nabla(\sum p_j)\wedge\mathbf{B}/B^2$).

4.6 Reduction of Fluid Approach to the Single Fluid Equations

So far we have been using fluid equations which apply to electrons and ions *separately.* These are called *'Two Fluid'* equations because we always have to keep track of both fluids separately.

A further simplification is possible and useful sometimes by combining the electron and ion equations together to obtain equations governing the plasma viewed as a *'Single Fluid'.*

Recall 2-fluid equations:

$$\text{Continuity (C}_\text{j}\text{)} \qquad \frac{\partial n_j}{\partial t} + \nabla.(n_j \mathbf{v}_j) = 0. \tag{4.73}$$

$$\text{Momentum (M}_\text{j}\text{)} \qquad m_j n_j \left(\frac{\partial}{\partial t} + \mathbf{v}_j.\nabla\right)\mathbf{v}_j = n_j q_j \left(\mathbf{E} + \mathbf{v}_j \wedge \mathbf{B}\right) - \nabla p_j + \mathbf{F}_{jk} \tag{4.74}$$

(where we just write $\mathbf{F}_{jk} = -\overline{\mathbf{v}}_{jk} n_j m_j \left(\mathbf{v}_j - \mathbf{v}_k\right)$ for short.)

Now we rearrange these 4 equations (2×2 species) by adding and subtracting appropriately to get new equations governing the new variables:

$$\begin{array}{llcll}
\text{Mass Density} & \rho_m & = & n_e m_e + n_i m_i & (4.75)\\
\text{C of M Velocity} & \mathbf{V} & = & \left(n_e m_e \mathbf{v}_e + n_i m_i \mathbf{v}_i\right)/\rho_m & (4.76)\\
\text{Charge density} & \rho_q & = & q_e n_e + q_i n_i & (4.77)\\
\text{Electric Current Density} & \mathbf{j} & = & q_e n_e \mathbf{v}_e + q_i n_i \mathbf{v}_i & (4.78)\\
 & & = & q_e n_e \left(\mathbf{v}_e - \mathbf{v}_i\right) \text{ by quasi neutrality} & (4.79)\\
\text{Total Pressure} & p & = & p_e + p_i & (4.80)
\end{array}$$

1st equation: take $m_e \times C_e + m_i \times C_i \rightarrow$

$$(1) \qquad \frac{\partial \rho_m}{\partial t} + \nabla.\left(\rho_m \mathbf{V}\right) = 0 \qquad \text{Mass Conservation} \tag{4.81}$$

2nd take $q_e \times C_e + q_I \times C_i \rightarrow$

$$(2) \qquad \frac{\partial \rho_q}{\partial t} + \nabla.\mathbf{j} = 0 \qquad \text{Charge Conservation} \tag{4.82}$$

3rd take $M_e + M_i$. This is a bit more difficult. RHS becomes:

$$\sum n_j q_j \left(\mathbf{E} + \mathbf{v}_j \wedge \mathbf{B}\right) - \nabla_{pj} + F_{jk} = \rho_q \mathbf{E} + \mathbf{j} \wedge \mathbf{B} - \nabla\left(p_e + p_i\right) \tag{4.83}$$

(we use the fact that $F_{ei} - F_{ie}$ so no *net* friction). LHS is

$$\sum_j m_j n_j \left(\frac{\partial}{\partial t} + \mathbf{v}_j.\nabla\right) v_j \tag{4.84}$$

The difficulty here is that the convective term is non-linear and so does not easily lend itself to reexpression in terms of the new variables. But note that since $m_e << m_i$ the contribution from electron momentum is usually much less than that from ions. So we ignore it in this equation. To the same degree of approximation $\mathbf{V} \simeq \mathbf{v}_i$: the CM velocity is the ion velocity. Thus for the LHS of this momentum equation we take

$$\sum_j m_i n_i \left(\frac{\partial}{\partial t} + \mathbf{v}_j.\nabla\right)\mathbf{v}_j \simeq \rho_m \left(\frac{\partial}{\partial t} + \mathbf{V}.\nabla\right)\mathbf{V} \tag{4.85}$$

so:

$$\text{(3)} \qquad \rho_m \left(\frac{\partial}{\partial t} + \mathbf{V}.\nabla \right) \mathbf{V} = \rho_q \mathbf{E} + \mathbf{j} \wedge \mathbf{B} - \nabla p \tag{4.86}$$

Finally we take $\frac{q_e}{m_e} M_e + \frac{q_i}{m_i} M_i$ to get:

$$\sum_j n_j q_j \left[\frac{\partial}{\partial t} + (\mathbf{v}_j.\nabla) \right] \mathbf{v}_j = \sum_j \{ \frac{n_j q_j^2}{m_j} (\mathbf{E} + \mathbf{v}_j \wedge \mathbf{B}) - \frac{q_j}{m_j} \nabla p_j + \frac{q_j}{m_j} \mathbf{F}_{jk} \} \tag{4.87}$$

Again several difficulties arise which it is not very profitable to deal with rigorously. Observe that the LHS can be written (using quasineutrality $n_i q_i + n_e q_e = 0$) as $\rho_m \frac{\partial}{\partial t} \left(\frac{\mathbf{j}}{\rho_m} \right)$ provided we discard the term in $(\mathbf{v}.\nabla)\mathbf{v}$. (Think of this as a linearization of this question.) [The $(\mathbf{v}.\nabla)\mathbf{v}$ convective term is a term which is not satisfactorily dealt with in this approach to the single fluid equations.]

In the R.H.S. we use quasineutrality again to write

$$\sum_j \frac{n_j q_j^2}{m_j} \mathbf{E} = n_e^2 q_e^2 \left(\frac{1}{n_e m_e} + \frac{1}{n_i m_i} \right) \mathbf{E} = n_e^2 q_e^2 \frac{m_i n_i + m_e n_e}{n_e m_e n_i m_i} \mathbf{E} = -\frac{q_e q_i}{m_e m_i} \rho_m \mathbf{E}, \tag{4.88}$$

$$\begin{aligned} \sum \frac{n_j q_q^2}{m_j} \mathbf{v}_j &= \frac{n_e q_e^2}{m_e} \mathbf{v}_e + \frac{n_i q_i^2}{m_i} \mathbf{v}_i \\ &= \frac{q_e q_i}{m_e m_i} \{ \frac{n_e q_e m_i}{q_i} \mathbf{v}_e + \frac{n_i q_i m_e}{q_e} \mathbf{v}_i \} \\ &= -\frac{q_e q_i}{m_e m_i} \{ n_e m_e \mathbf{v}_e + n_i m_i \mathbf{v}_i - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) (q_e n_e \mathbf{v}_e + q_i n_i \mathbf{v}_i) \} \\ &= -\frac{q_e q_i}{m_e m_i} \{ \rho_m \mathbf{V} - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) \mathbf{j} \} \end{aligned} \tag{4.89}$$

Also, remembering $\mathbf{F}_{ei} = -\overline{\nu}_{ei} n_e m_i (\mathbf{v}_e - \mathbf{v}_i) = -\mathbf{F}_{ie}$,

$$\begin{aligned} \sum_j \frac{q_j}{m_j} \mathbf{F}_{jk} &= -\overline{\nu}_{ei} \left(n_e q_e - n_e q_i \frac{m_e}{m_i} \right) (\mathbf{v}_e - \mathbf{v}_i) \\ &= -\overline{\nu}_{ei} \left(1 - \frac{q_e}{q_i} \frac{m_e}{m_i} \right) \mathbf{j} \end{aligned} \tag{4.90}$$

So we get

$$\begin{aligned} \rho_m \frac{\partial}{\partial t} \left(\frac{j}{\rho_m} \right) &= -\frac{q_e q_i}{m_e m_i} \left[\rho_m \mathbf{E} + \left\{ \rho_m \mathbf{V} - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) \mathbf{j} \right\} \wedge \mathbf{B} \right] \\ &- \frac{q_e}{m_e} \nabla p_e - \frac{q_i}{m_i} \nabla p_i - \left(1 - \frac{q_e}{q_i} \frac{m_e}{m_i} \right) \overline{\nu}_{ei} \mathbf{j} \end{aligned} \tag{4.91}$$

Regroup after multiplying by $\frac{m_e m_i}{q_e q_i \rho_m}$:

$$\begin{aligned} \mathbf{E} + \mathbf{V} \wedge \mathbf{B} &= -\frac{m_e m_i}{q_e q_i} \frac{\partial}{\partial t} \left(\frac{\mathbf{j}}{\rho_m} \right) + \frac{1}{\rho_m} \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) \mathbf{j} \wedge \mathbf{B} \\ &- \left(\frac{q_e}{m_e} \nabla p_e + \frac{q_i}{m_i} \nabla p_i \right) \frac{m_e m_i}{\rho_m q_e q_i} - \left(1 - \frac{q_e}{q_i} \frac{m_e}{m_i} \right) \frac{m_e m_i}{q_e q_i \rho_m} \overline{\nu}_{ei} \mathbf{j} \end{aligned} \tag{4.92}$$

Notice that this is an equation relating the Electric field in the frame moving with the fluid (L.H.S.) to things depending on current $\mathbf{j}$ i.e. this is a generalized form of *Ohm's Law*.

One essentially never deals with this full generalized Ohm's law. Make some approximations recognizing the physical significance of the various R.H.S. terms.

$$\frac{m_e m_i}{q_e q_i}\frac{\partial}{\partial t}\left(\frac{j}{\rho_m}\right) \text{ arises from } \textit{electron inertia}.$$

it will be negligible for low enough frequency.

$$\frac{1}{\rho_m}\left(\frac{m_i}{q_i}+\frac{m_e}{q_e}\right)\mathbf{j}\wedge\mathbf{B} \text{ is called the } \textit{Hall Term}.$$

and arises because current flow in a B-field tends to be diverted across the magnetic field. It is also often dropped but the justification for doing so is less obvious physically.

$$\frac{q_i}{m_i}\nabla p_i \text{ term } \ll \frac{q_e}{m_e}\nabla p_e \text{ for comparable pressures,}$$

and the latter is $\sim$ the Hall term; so ignore $q_i\nabla p_i/m_i$.

Last term in $\mathbf{j}$ has a coefficient, ignoring m_e/m_i c.f. 1 which is

$$\frac{m_e m_i \overline{\nu}_{ei}}{q_e q_i\left(n_i m_i\right)}=\frac{m_e\overline{\nu}_{ei}}{q_e^2 n_e}=\eta \quad \text{the resistivity.} \tag{4.93}$$

Hence dropping electron inertia, Hall term and pressure, the simplified Ohm's law becomes:

$$\mathbf{E}+\mathbf{V}\wedge\mathbf{B}=\eta\mathbf{j} \tag{4.94}$$

Final equation needed: state:

$$p_e n_e^{-\gamma_e}+p_i n_i^{-\gamma_i}=\text{constant.}$$

Take quasi-neutrality $\Rightarrow n_e \propto n_i \propto \rho_m$. Take $\gamma_e=\gamma_i$, then

$$p\rho_m^{-\gamma}=\text{const.} \tag{4.95}$$

4.6.1 Summary of Single Fluid Equations: M.H.D.

$$\text{Mass Conservation}: \quad \frac{\partial\rho_m}{\partial t}+\nabla\left(\rho_m\mathbf{V}\right)=0 \tag{4.96}$$

$$\text{Charge Conservation}: \quad \frac{\partial\rho_q}{\partial t}+\nabla.\mathbf{j}=0 \tag{4.97}$$

$$\text{Momentum}: \quad \rho_m\left(\frac{\partial}{\partial t}+\mathbf{V}.\nabla\right)\mathbf{V}=\rho_q\mathbf{E}+\mathbf{j}\wedge\mathbf{B}-\nabla p \tag{4.98}$$

$$\text{Ohm's Law}: \quad \mathbf{E}+\mathbf{V}\wedge\mathbf{B}=\eta\mathbf{j} \tag{4.99}$$

$$\text{Eq. of State}: \quad p\rho_m^{-\gamma}=\text{const.} \tag{4.100}$$

4.6.2 Heuristic Derivation/Explanation

Mass Charge: Obvious.

$$Mom^m \qquad \underbrace{\rho_m \left(\frac{\partial}{\partial t} + \mathbf{V}.\nabla\right)\mathbf{V}}_{\substack{\text{rate of change of} \\ \text{total momentum density}}} = \underbrace{\rho_q \mathbf{E}}_{\substack{\text{Electric} \\ \text{body force}}} + \underbrace{\mathbf{j}\wedge\mathbf{B}}_{\substack{\text{Magnetic Force} \\ \text{on current}}} - \underbrace{\nabla p}_{\text{Pressure}} \tag{4.101}$$

Ohm's Law

The electric field 'seen' by a moving (conducting) fluid is $\mathbf{E} + \mathbf{V}\wedge\mathbf{B} = \mathbf{E}_V$ electric field in frame in which fluid is at rest. This is equal to 'resistive' electric field $\eta\mathbf{j}$:

$$\mathbf{E}_V = \mathbf{E} + \mathbf{V}\wedge\mathbf{B} = \eta\mathbf{j} \tag{4.102}$$

The $\rho_q E$ term is generally dropped because it is much smaller than the $\mathbf{j}\wedge\mathbf{B}$ term. To see this, take orders of magnitude:

$$\nabla.\mathbf{E} = \rho_q/\epsilon_0 \quad \text{so} \quad \rho_q \sim E\epsilon_0/L \tag{4.103}$$

$$\nabla\wedge\mathbf{B} = \mu_0\mathbf{j}\left(+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right) \quad \text{so} \quad \sigma E = j \sim B/\mu_0 L \tag{4.104}$$

Therefore

$$\frac{\rho_q E}{jB} \sim \frac{\epsilon_0}{L}\left(\frac{B}{\mu_0\sigma L}\right)^2 \frac{L\mu_0}{B^2} \sim \frac{L^2/c^2}{(\mu_0\sigma L^2)^2} = \left(\frac{\text{light transit time}}{\text{resistive skin time}}\right)^2 . \tag{4.105}$$

This is generally a very small number. For example, even for a small cold plasma, say $T_e = 1$ eV ($\sigma \approx 2\times 10^3$ mho/m), $L = 1$ cm, this ratio is about 10^{-8}.

Conclusion: the $\rho_q E$ force is *much* smaller than the $\mathbf{j}\wedge\mathbf{B}$ force for essentially all practical cases. Ignore it.

Normally, also, one uses MHD only for low frequency phenomena, so the Maxwell displacement current, $\partial\mathbf{E}/c^2\partial t$ can be ignored.

Also we shall not need Poisson's equation because that is taken care of by quasi-neutrality.

4.6.3 Maxwell's Equations for MHD Use

$$\nabla.\mathbf{B} = 0 \;\; ; \;\; \nabla\wedge\mathbf{E} = \frac{-\partial\mathbf{B}}{\partial t} \;\; ; \;\; \nabla\wedge\mathbf{B} = \mu_o\mathbf{j} \;\; . \tag{4.106}$$

The MHD equations find their major use in studying macroscopic magnetic confinement problems. In Fusion we want somehow to confine the plasma pressure away from the walls of the chamber, using the magnetic field. In studying such problems MHD is the major tool.

On the other hand if we focus on a small section of the plasma as we do when studying short-wavelength waves, other techniques: 2-fluid or kinetic are needed. Also, plasma is approx. uniform.

'Macroscopic' Phenomena MHD

'Microscopic' Phenomena 2-Fluid/Kinetic

4.7 MHD Equilibria

Study of how plasma can be 'held' by magnetic field. Equilibrium $\Rightarrow$ $\mathbf{V} = \frac{\partial}{\partial t} = 0$. So equations reduce. Mass and Faraday's law are $\sim$ automatic. We are left with

$$(Mom^m) \rightarrow \text{'Force Balance'} \quad 0 = \mathbf{j} \wedge \mathbf{B} - \nabla p \tag{4.107}$$

$$\text{Ampere} \qquad \nabla \wedge \mathbf{B} = \mu_o \mathbf{j} \tag{4.108}$$

Plus $\nabla.\mathbf{B} = 0$, $\nabla.\mathbf{j} = 0$.

Notice that provided we don't ask questions about Ohm's law. $\mathbf{E}$ doesn't come into MHD equilibrium.

These deceptively simple looking equations are the subject of much of Fusion research. The hard part is taking into account complicated geometries.

We can do some useful calculations on simple geometries.

4.7.1 θ-pinch

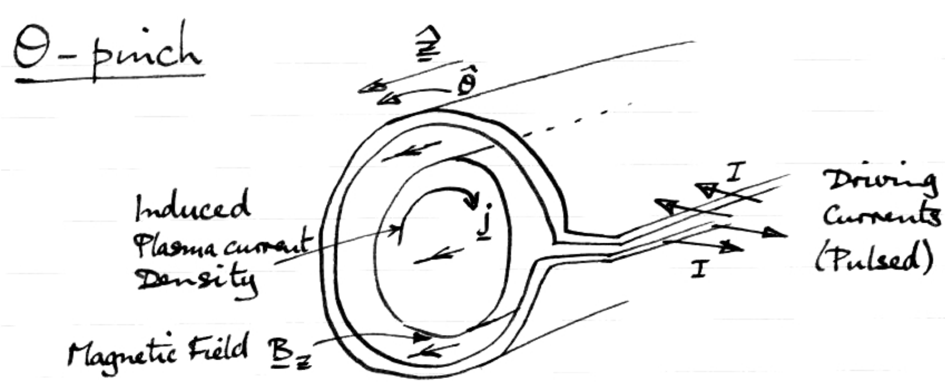

Figure 4.5: θ-pinch configuration.

So called because plasma currents flow in θ-direction.

<u>Use MHD Equations</u>

Take to be ∞ length, uniform in z-dir.

By symmetry $\mathbf{B}$ has only z component.

By symmetry $\mathbf{j}$ has only θ- comp.

By symmetry ∇p has only r comp.

So we only need

$$\begin{aligned}
\text{Force} \qquad (\mathbf{j} \wedge \mathbf{B})_r &- (\nabla p)_r = 0 && (4.109)\\
\text{Ampere} \qquad (\nabla \wedge \mathbf{B})_\theta &= (\mu_o \mathbf{j})_\theta && (4.110)\\
\text{i.e.} \qquad j_\theta B_z &- \frac{\partial}{\partial r} p = 0 && (4.111)\\
-\frac{\partial}{\partial r} B_z &= \mu_o j_\theta && (4.112)\\
\text{Eliminate } j: \quad -\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{\partial p}{\partial r} &= 0 && (4.113)
\end{aligned}$$

i.e.

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{B_z^2}{2\mu_o} + p\right) &= 0 && (4.114)\\
\text{Solution} \quad \frac{B_z^2}{2\mu_o} + p &= \text{const.} && (4.115)
\end{aligned}$$

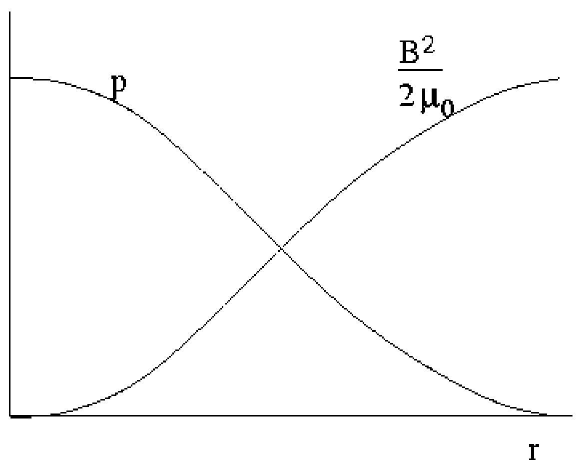

Figure 4.6: Balance of kinetic and magnetic pressure

$$\frac{B_z^2}{2\mu_o} + p = \frac{B_{z\ ext}^2}{2\mu_o} \tag{4.116}$$

[Recall Single Particle Problem]

Think of these as a pressure equation. Equilibrium says *total* pressure = const.

$$\underbrace{\frac{B_z^2}{2\mu_o}}_{\text{magnetic pressure}} + \underbrace{p}_{\text{kinetic pressure}} = \text{const.} \tag{4.117}$$

Ratio of kinetic to magnetic pressure is plasma 'β'.

$$\beta = \frac{2\mu_o p}{B_z^2} \tag{4.118}$$

measures 'efficiency' of plasma confinement by B. Want large β for fusion but limited by instabilities, etc.

4.7.2 Z-pinch

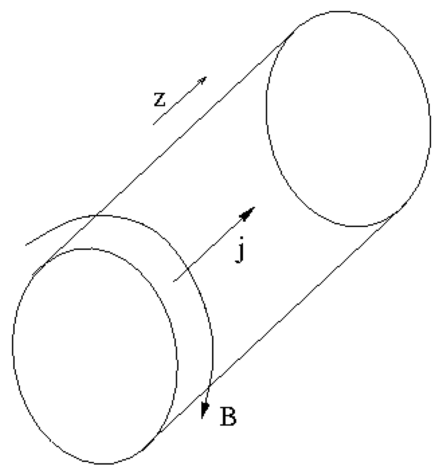

Figure 4.7: Z-pinch configuration.

so called because $\mathbf{j}$ flows in z-direction. Again take to be ∞ length and uniform.

$$\mathbf{j} = \mathbf{j}_z \hat{\mathbf{e}}_z \qquad \mathbf{B} = \mathbf{B}_\theta \hat{\mathbf{e}}_\theta \tag{4.119}$$

$$\text{Force} \quad (\mathbf{j} \wedge \mathbf{B})_r - (\nabla p)_r = -j_z B_\theta - \frac{\partial p}{\partial r} = 0 \tag{4.120}$$

$$\text{Ampere} \quad (\nabla \wedge \mathbf{B})_z - (\mu_o \mathbf{j})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \mu_o j_z = 0 \tag{4.121}$$

Eliminate j:

$$\frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial p}{\partial r} = 0 \tag{4.122}$$

or

$$\underbrace{\frac{B_\theta^2}{\mu_o r}}_{\text{Extra Term}} + \frac{\partial}{\partial r}\underbrace{\left(\frac{B_\theta^2}{2\mu_o} + p\right)}_{\text{Magnetic+Kinetic pressure}} = 0 \tag{4.123}$$

Extra term acts like a magnetic *tension* force. Arises because B-field lines are *curved*.

Can integrate equation

$$\int_a^b \frac{B_\theta^2}{\mu_o} \frac{dr}{r} + \left[\frac{B_\theta^2}{2\mu_o} + p(r)\right]_a^b = 0 \tag{4.124}$$

If we choose b to be edge ($p(b) = 0$) and set $a = r$ we get

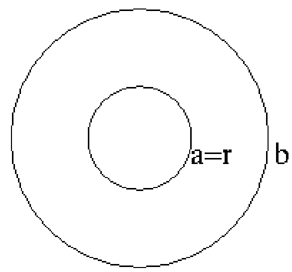

Figure 4.8: Radii of integration limits.

$$p(r) = \frac{B_\theta^2\,(b)}{2\mu_o} - \frac{B_\theta^2\,(r)}{2\mu_o} + \int_r^b \frac{B_\theta^2}{\mu_o} \frac{dr'}{r'} \tag{4.125}$$

Force balance in z-pinch is somewhat more complicated because of the tension force. We can't choose $p(r)$ and $j(r)$ independently; they have to be self consistent.

Example $\quad j = \text{const.}$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right) = \mu_o j_z \Rightarrow B_\theta = \frac{\mu_o j_z}{2}\; r \tag{4.126}$$

Hence

$$p(r) = \frac{1}{2\mu_o}\left(\frac{\mu_o j_z}{2}\right)^2 \{b^2 - r^2 + \int_r^b 2r' dr'\} \tag{4.127}$$

$$= \frac{\mu_o j_z^2}{4}\{b^2 - r^2\} \tag{4.128}$$

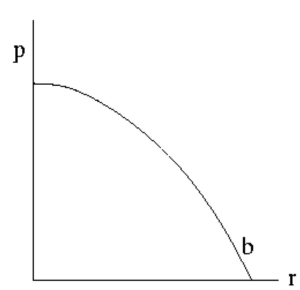

Figure 4.9: Parabolic Pressure Profile.

Also note $B_\theta(b) = \frac{\mu_o j_z b}{2}$ so

$$p = \frac{B_{\theta b}^2}{2\mu_o}\frac{2}{b^2}\{b^2 - r^2\} \tag{4.129}$$

4.7.3 'Stabilized Z-pinch'

Also called 'screw pinch', $\theta - z$ pinch or sometimes loosely just 'z-pinch'.
Z-pinch with some additional B_z as well as B_θ

$$\text{(Force)}_r \quad j_\theta B_z - j_z B_\theta - \frac{\partial}{\partial r} = 0 \tag{4.130}$$

$$\text{Ampere :} \quad \frac{\partial}{\partial r} B_z = \mu_o j_\theta \tag{4.131}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r b_\theta) = \mu_o j_z \tag{4.132}$$

Eliminate j:

$$-\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(r B_\theta) - \frac{\partial p}{\partial r} = 0 \tag{4.133}$$

or

$$\underbrace{\frac{B_\theta^2}{\mu_o r}}_{\substack{\text{Mag Tension}\\ \theta \text{ only}}} + \frac{\partial}{\partial r}\underbrace{\left(\frac{B^2}{2\mu_o} + p\right)}_{\text{Mag }(\theta+\text{z})+\text{Kinetic pressure}} = 0 \tag{4.134}$$

4.8 Some General Properties of MHD Equilibria

4.8.1 Pressure & Tension

$$\mathbf{j} \wedge \mathbf{B} - \nabla p = 0 \quad : \quad \nabla \wedge \mathbf{B} = \mu_o \mathbf{j} \tag{4.135}$$

We can eliminate $\mathbf{j}$ in the *general* case to get

$$\frac{1}{\mu_o}(\nabla \wedge \mathbf{B}) \wedge \mathbf{B} = \nabla p. \tag{4.136}$$

Expand the vector triple product:

$$\nabla p = \frac{1}{\mu_o}(\mathbf{B}.\nabla)\mathbf{B} - \frac{1}{2\mu_o}\nabla B^2 \tag{4.137}$$

put $\mathbf{b} = \frac{\mathbf{B}}{|B|}$ so that $\nabla \mathbf{B} = \nabla B \mathbf{b} = B \nabla \mathbf{b} + \mathbf{b} \nabla B$. Then

$$\begin{aligned} \nabla p &= \frac{1}{\mu_o}\{B^2(\mathbf{b}.\nabla)\mathbf{b} + B\mathbf{b}(\mathbf{b}.\nabla)B\} - \frac{1}{2\mu_o}\nabla B^2 && (4.138)\\ &= \frac{B^2}{\mu_o}(\mathbf{b}.\nabla)\mathbf{b} - \frac{1}{2\mu_o}(\nabla - \mathbf{b}(\mathbf{b}.\nabla))B^2 && (4.139)\\ &= \frac{B^2}{\mu_o}(\mathbf{b}.\nabla)\mathbf{b} - \nabla_\perp\left(\frac{B^2}{2\mu_p}\right) && (4.140) \end{aligned}$$

Now $\nabla_\perp \left(\frac{B^2}{2\mu_o}\right)$ is the perpendicular (to $\mathbf{B}$) derivative of *magnetic pressure* and $(\mathbf{b}.\nabla)\mathbf{b}$ is the *curvature* of the magnetic field line giving *tension*.

$|\ (\mathbf{b}.\nabla)\mathbf{b}\ |$ has value $\frac{1}{R}$. R: radius of curvature.

4.8.2 Magnetic Surfaces

$$0 = \mathbf{B}.\left[\mathbf{j} \wedge \mathbf{B} - \nabla p\right] = -\mathbf{B}.\nabla p \tag{4.141}$$

*Pressure is constant on a field line (in MHD situation).

(Similarly, $0 = \mathbf{j}.[\mathbf{j} \wedge \mathbf{B} - \nabla p] = \mathbf{j}.\nabla p$.)

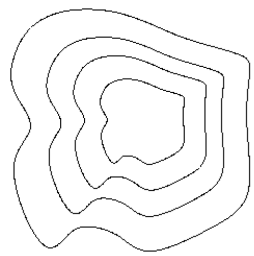

Figure 4.10: Contours of pressure.

Consider some arbitrary volume in which $\nabla p \neq 0$. That is, some plasma of whatever shape. Draw contours (surfaces in 3-d) on which $p =$ const. At any point on such an isoberic surface ∇p is perp to the surface. But $\mathbf{B}.\nabla p = 0$ implies that $\mathbf{B}$ is also perp to ∇p.

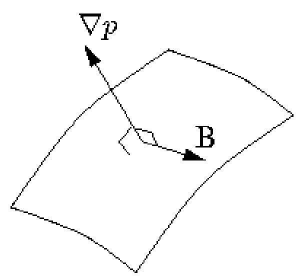

Figure 4.11: $\mathbf{B}$ is perpendicular to ∇p and so lies in the isobaric surface.

Hence

$\mathbf{B}$ lies in the surface $p =$ const.

In equilibrium *isobaric surfaces* are *'magnetic surfaces'*.

[This argument does not work if $p =$ const. i.e. $\nabla p = 0$. Then there need be no magnetic surfaces.]

4.8.3 'Current Surfaces'

Since $\mathbf{j}.\nabla p = 0$ in equilibrium the same argument applies to current density. That is

$\mathbf{j}$ lies in the surface $p =$ const.

Isobaric Surfaces are 'Current Surfaces'.

Moreover it is clear that

'Magnetic Surfaces' are 'Current Surfaces'.

(since both coincide with isobaric surfaces.)

[It is important to note that the existence of magnetic surfaces is guaranteed only in the MHD approximation when $\nabla p \neq 0 >$ Taking account of corrections to MHD we may not have magnetic surfaces even if $\nabla p \neq 0$.]

4.8.4 Low β equilibria: Force-Free Plasmas

In many cases the ratio of kinetic to magnetic pressure is small, $\beta << 1$ and we can approximately *ignore* ∇p. Such an equilibrium is called 'force free'.

$$\mathbf{j} \wedge \mathbf{B} = 0 \tag{4.142}$$

implies $\mathbf{j}$ and $\mathbf{B}$ are parallel.

i.e.

$$\mathbf{j} = \mu(\mathbf{r})\mathbf{B} \tag{4.143}$$

Current flows *along* field lines *not across*. Take divergence:

$$\begin{aligned} 0 &= \nabla.\mathbf{j} = \nabla.\left(\mu\left(\mathbf{r}\right)\mathbf{B}\right) = \mu\left(\mathbf{r}\right)\nabla.\mathbf{B} + \left(\mathbf{B}.\nabla\right)\mu & (4.144)\\ &= \left(\mathbf{B}.\nabla\right)\mu. & (4.145) \end{aligned}$$

The ratio $j/B = \mu$ is *constant along field lines.*

μ is constant on a magnetic surface. If there are no surfaces, μ is constant *everywhere.*

Example: Force-Free Cylindrical Equil.

$$\begin{aligned} j \wedge \mathbf{B} &= \quad \Leftrightarrow \quad \mathbf{j} = \mu(r)\mathbf{B} & (4.146)\\ \nabla \wedge \mathbf{B} &= \mu_o\mathbf{j} = \mu_o\mu(r)\mathbf{B} & (4.147) \end{aligned}$$

This is a somewhat more convenient form because it is linear in $\mathbf{B}$ (for specified $\mu(r)$).

$$\text{Constant-}\mu: \qquad \nabla \wedge \mathbf{B} = \mu_o\mu\mathbf{B} \tag{4.148}$$

leads to a Bessel function solution

$$B_z = B_o J_o(\mu_o \mu r) \tag{4.149}$$
$$B_\theta = B_o J_1(\mu_o \mu r) \tag{4.150}$$

for $\mu_o \mu r >$ 1st zero of J_o the toroidal field reverses. There are plasma confinement schemes with $\mu \simeq$ const. 'Reversed Field Pinch'.

4.9 Toroidal Equilibrium

Bend a z-pinch into a torus

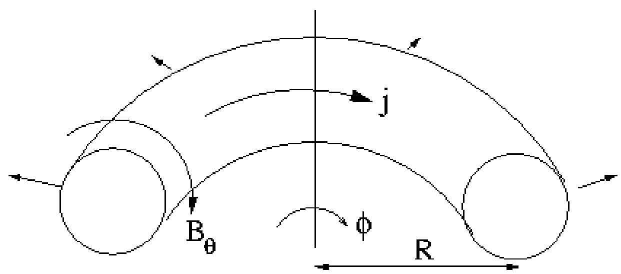

Figure 4.12: Toroidal z-pinch

B_θ fields due to current are stronger at small R side $\Rightarrow$ Pressure (Magnetic) Force *outwards*.

Have to balance this by applying a *vertical field* $\mathbf{B}_v$ to push plasma back by $\mathbf{j}_\phi \wedge \mathbf{B}_v$.

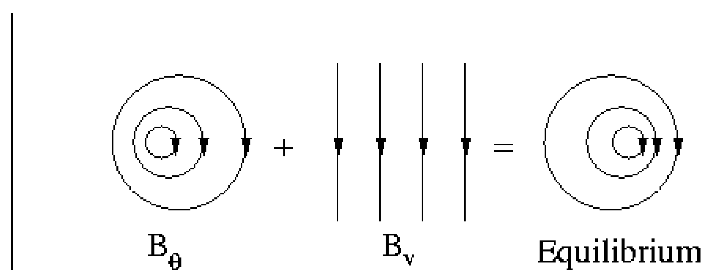

Figure 4.13: The field of a toroidal loop is not an MHD equilibrium. Need to add a vertical field.

Bend a θ-pinch into a torus: $B\phi$ is stronger at small R side $\Rightarrow$ outward force.

Cannot be balanced by $\mathbf{B}_v$ because no j_ϕ. No equilibrium for a toroidally symmetric θ-pinch.

Underlying Single Particle reason:

Toroidal θ-pinch has B_ϕ only. As we have seen before, curvature drifts are uncompensated in such a configuration and lead to rapid *outward* motion.

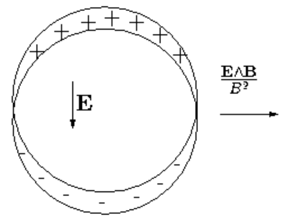

Figure 4.14: Charge-separation giving outward drift is equivalent to the lack of MHD toroidal force balance.

We know how to solve this: Rotational Transform: get some B_θ. Easiest way: add $\mathbf{j}_\phi$. From MHD viewpoint this allows you to push the plasma back by $\mathbf{j}_\phi \wedge \mathbf{B}_v$ force. Essentially, this is Tokamak.

4.10 Plasma Dynamics (MHD)

When we want to analyze *non*-equilibrium situations we must retain the momentum terms. This will give a dynamic problem. Before doing this, though, let us analyse some purely Kinematic Effects.

<u>'Ideal MHD'</u> $\Leftrightarrow$ Set $eta = 0$ in Ohm's Law.
A good approximation for high frequencies, i.e. times shorter than resistive decay time.

$$\mathbf{E} + \mathbf{V} \wedge \mathbf{B} = 0. \qquad \text{Ideal Ohm's Law.} \tag{4.151}$$

Also

$$\nabla \wedge \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's Law.} \tag{4.152}$$

Together these two equations imply constraints on how the magnetic field can change with time: Eliminate $\mathbf{E}$:

$$+\nabla \wedge (\mathbf{V} \wedge \mathbf{B}) = +\frac{\partial \mathbf{B}}{\partial t} \tag{4.153}$$

This shows that the changes in $\mathbf{B}$ are completely determined by the flow, $\mathbf{V}$.

4.11 Flux Conservation

Consider an arbitrary closed contour C and spawning surface S in the fluid.

Flux linked by C is

$$\Phi = \int_S \mathbf{B}.\mathbf{ds} \tag{4.154}$$

Let C and S move with fluid:

Total rate of change of Φ is given by two terms:

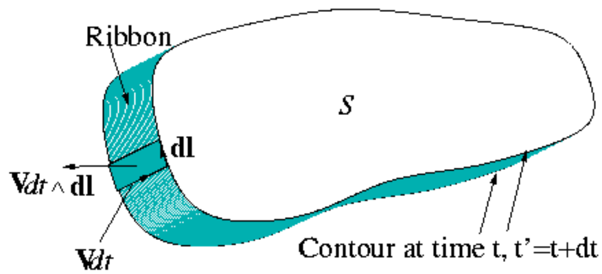

Figure 4.15: Motion of contour with fluid gives convective flux derivative term.

$$\dot{\Phi} = \int_S \underbrace{\frac{\partial \mathbf{B}}{\partial t}.\mathbf{ds}}_{\text{Due to changes in }\mathbf{B}} + \underbrace{\oint_C \mathbf{B}.(\mathbf{V}\wedge \mathbf{dl})}_{\text{Due to motion of C}} \tag{4.155}$$

$$= -\int_S \nabla \wedge \mathbf{E}.\mathbf{ds} - \oint_C (\mathbf{V}\wedge\mathbf{B}).\mathbf{dl} \tag{4.156}$$

$$= -\oint_C (\mathbf{E}+\mathbf{V}\wedge\mathbf{B}).\mathbf{dl} = 0 \quad \text{by Ideal Ohm's Law.} \tag{4.157}$$

Flux through any surface moving with fluid is conserved.

4.12 Field Line Motion

Think of a field line as the intersection of two surfaces both tangential to the field everywhere:

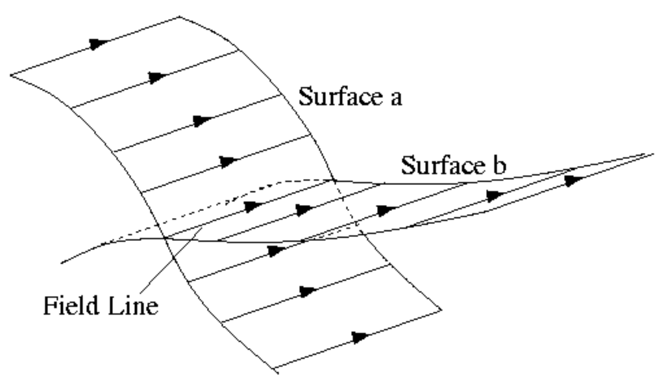

Figure 4.16: Field line defined by intersection of two flux surfaces tangential to field.

Let surfaces move with fluid.

Since all parts of surfaces had zero flux crossing at start, they also have zero after, (by flux conserv.).

Surfaces are tangent after motion

⇒ Their intersection defines a field line after.

We think of the new field line as the same line as the old one (only moved).

Thus:

1. Number of field lines (≡ flux) through any surface is constant. (Flux Cons.)
2. A line of fluid that starts as a field line remains one.

4.13 MHD Stability

The fact that one can find an MHD *equilibrium* (e.g. z-pinch) does not guarantee a useful confinement scheme because the equil. might be unstable. Ball on hill analogies:

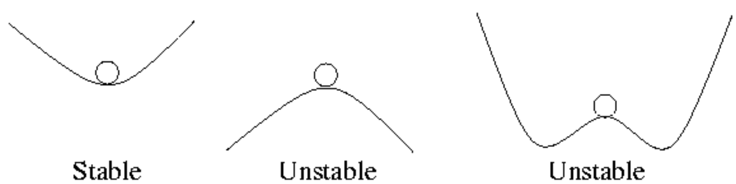

Figure 4.17: Potential energy curves

An equilibrium is *unstable* if the curvature of the 'Potential energy surface' is downward away from equil. That is if $\frac{d^2}{dx^2}\{W_{\rm pot}\} < 0$.

In MHD the potential energy is Magnetic + Kinetic Pressure (usually mostly magnetic).

If we can find *any* type of perturbation which *lowers* the potential energy then the equil is *unstable*. It will not remain but will rapidly be lost.

<u>Example Z-pinch</u>

We know that there is an equilibrium: Is it stable?

Consider a perturbation thus:

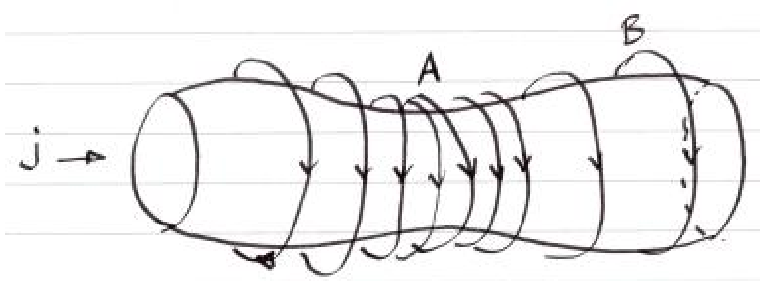

Figure 4.18: 'Sausage' instability

Simplify the picture by taking the current all to flow in a skin. We know that the pressure is supported by the combination of $B^2/2\mu_o$ pressure and $\frac{B^2}{\mu_o r}$ tension forces.

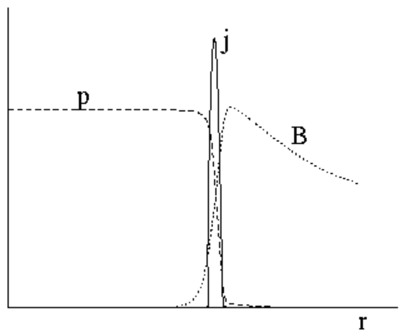

Figure 4.19: Skin-current, sharp boundary pinch.

At the place where it pinches *in* (A)
B_θ and $\frac{1}{r}$ *increase* $\rightarrow$ Mag. pressure & tension increase $\Rightarrow$ inward force no longer balance by $p \Rightarrow$ perturbation grows.

At place where it bulges *out* (B)
B_θ & $\frac{1}{r}$ *decrease* $\rightarrow$ Pressure & tension $\Rightarrow$ perturbation grows.

Conclusion a small perturbation induces a force tending to increase itself. *Unstable* ($\equiv \delta W < 0$).

4.14 General Perturbations of Cylindrical Equil.

Look for things which go like $\exp[i(kz + m\theta)]$. [Fourier (Normal Mode) Analysis].

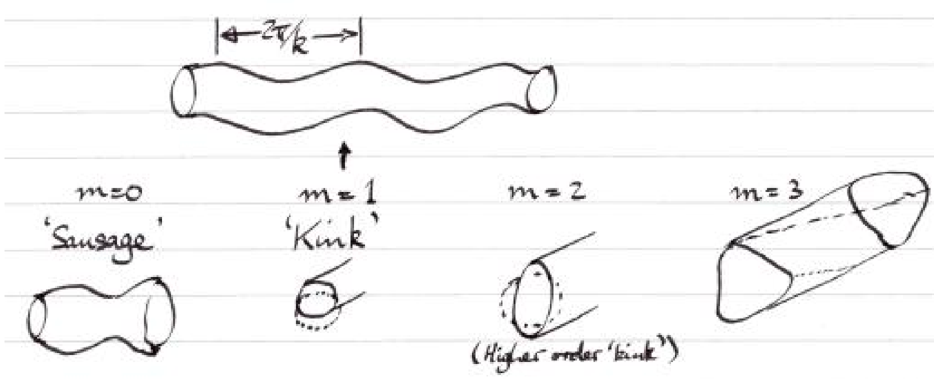

Figure 4.20: Types of kink perturbation.

Generally Helical in form (like a screw thread). Example: $m = 1 \quad k \neq 0$ z-pinch

4.15 General Principles Governing Instabilities

(1) They try *not* to bend field lines. (Because bending takes energy). Perturbation

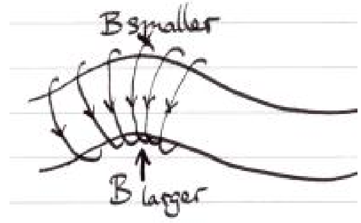

Figure 4.21: Driving force of a kink. Net force tends to increase perturbation. Unstable.

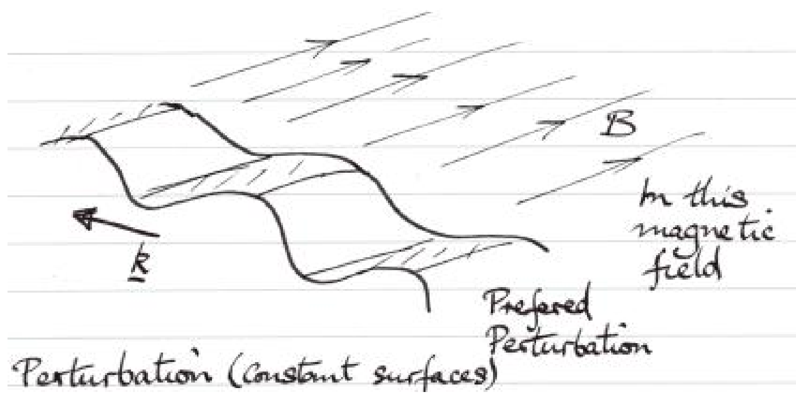

Figure 4.22: Alignment of perturbation and field line minimizes bending energy.

(Constant surfaces) lies *along* magnetic field.

Example: θ-pinch type plasma column:

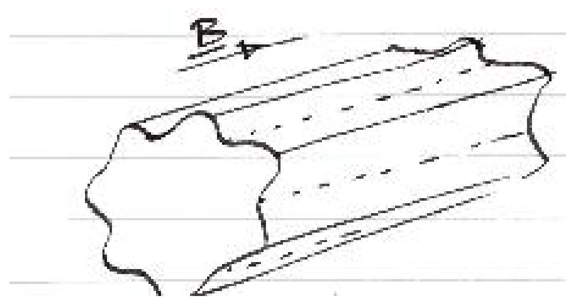

Figure 4.23: 'Flute' or 'Interchange' modes.

Preferred Perturbations are 'Flutes' as per Greek columns $\rightarrow$ 'Flute Instability.' [Better name: 'Interchange Instability', arises from idea that plasma and vacuum change places.]

(2) Occur when a 'heavier' fluid is supported by a 'lighter' (Gravitational analogy).

Why does water fall out of an inverted glass? Air pressure could sustain it but does not because of Rayleigh-Taylor instability.

Similar for supporting a plasma by mag field.

(3) Occur when $\mid B \mid$ decreases *away* from the plasma region.

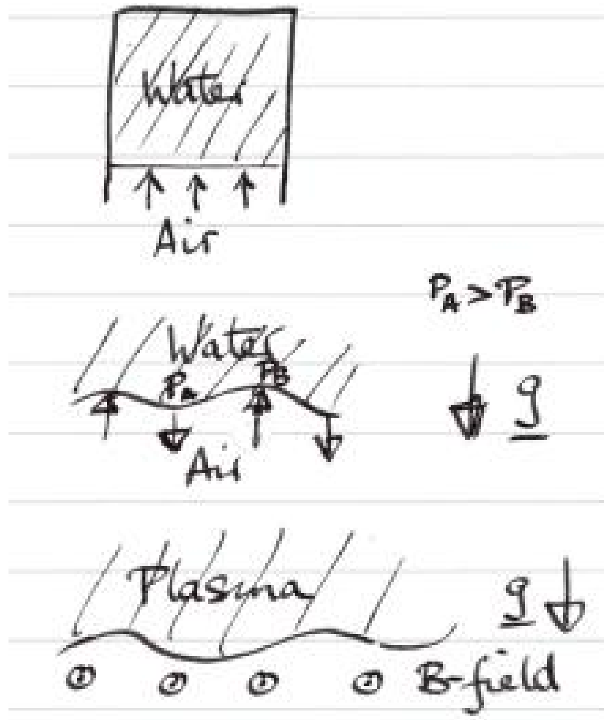

Figure 4.24: Inverted water glass analogy. Rayleigh Taylor instability.

$$\frac{B_A^2}{2\mu_o} < \frac{B_B^2}{2\mu_o} \tag{4.158}$$

$\Rightarrow$ Perturbation Grows.

(4) Occur when field line curvature is *towards* the plasma (Equivalent to (3) because of $\nabla \wedge \mathbf{B} = 0$ in a vacuum).

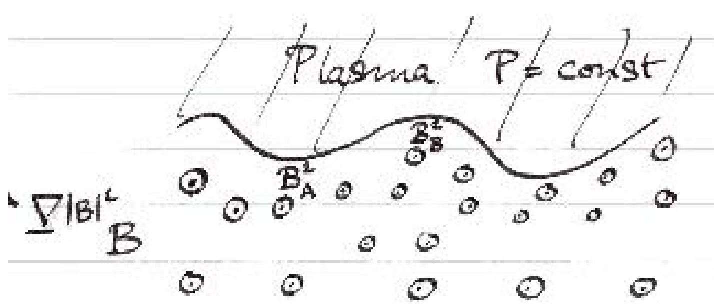

Figure 4.25: Vertical upward field gradient is unstable.

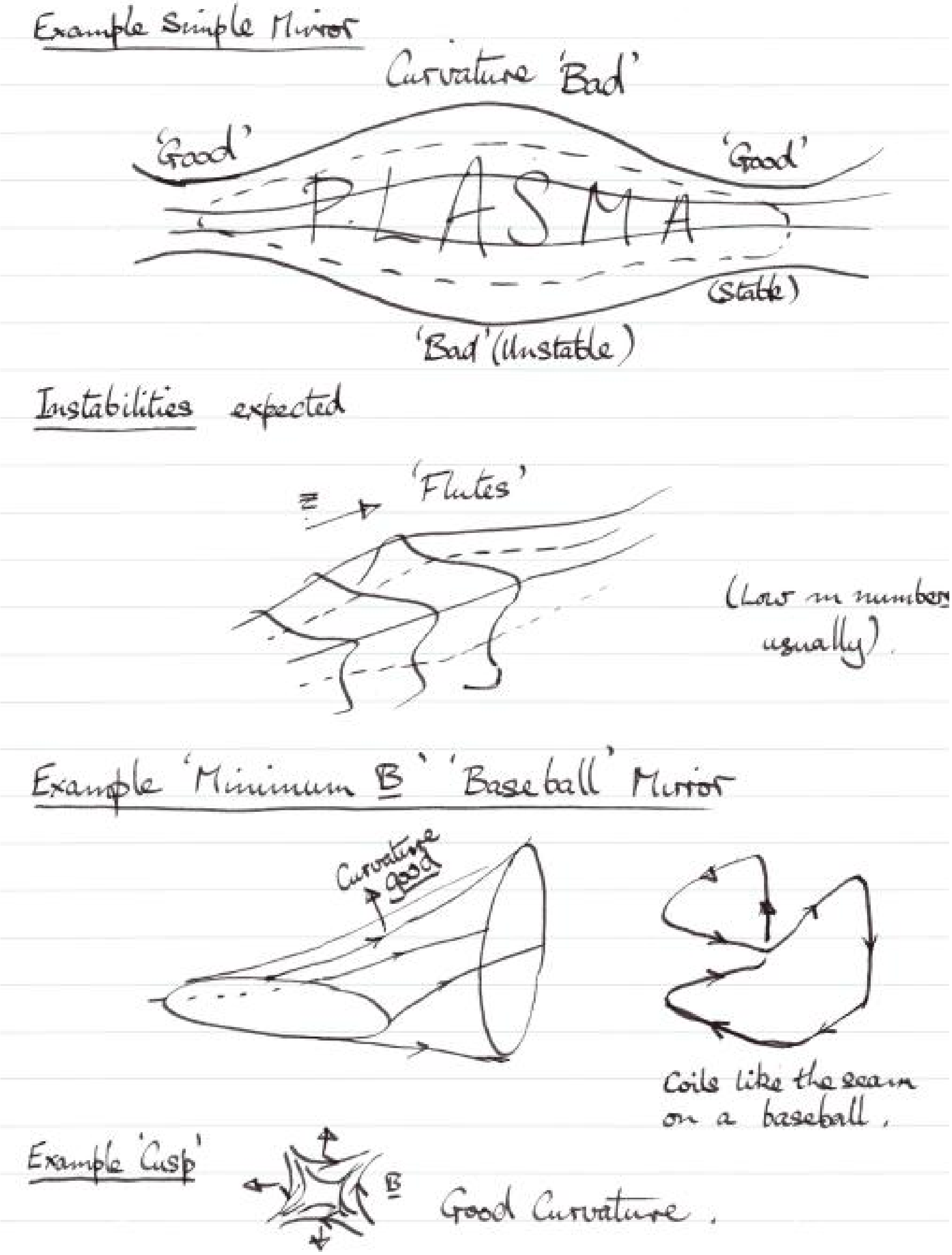

Figure 4.26: Examples of magnetic configurations with good and bad curvature.

4.16 Quick and Simple Analysis of Pinches

θ-pinch $\quad \mid B \mid =$ const. outside pinch
$\equiv$ No field line curvature. *Neutral stability*

z-pinch $\quad \nabla \mid B \mid$ away from plasma outside
$\equiv$ Bad Curvature (Towards plasma) $\Rightarrow$ *Instability.*

Generally it is difficult to get the curvature to be good everywhere. Often it is sufficient to make it good *on average* on a field line. This is referred to as 'Average Minimum B'. Tokamak has this.

General idea is that if field line is only in bad curvature over part of its length then to perturb in that region and not in the good region requires field line bending:

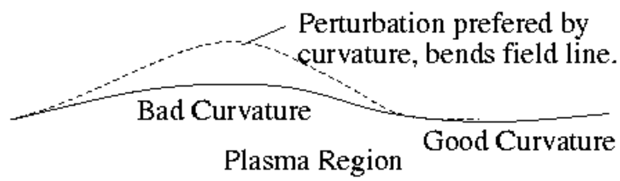

Figure 4.27: Parallel localization of perturbation requires bending.

But bending is *not* preferred. So this may stabilize.

Possible way to stabilize configuration with bad curvature: *Shear*
Shear of Field Lines

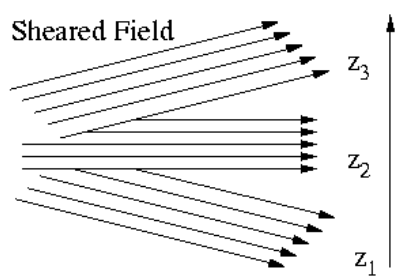

Figure 4.28: Depiction of field shear.

Direction of B changes. A perturbation along B at z_3 is *not* along $\mathbf{B}$ at z_2 or z_1 so it would have to *bend* field there $\rightarrow$ Stabilizing effect.

General Principle: Field line bending is stabilizing.

Example: Stabilized z-pinch

Perturbations (e.g. sausage or kink) *bend* $\mathbf{B}_z$ so the tension in B_z acts as a restoring force to prevent instability. If wave length very *long* bending is less. $\Rightarrow$ Least stable tends to be longest wave length.

Example: 'Cylindrical Tokamak'

Tokamak is in some ways like a periodic cylindrical stabilized pinch. Longest allowable wave length = 1 turn round torus the long way, i.e.

$$kR = 1 : \qquad \lambda = 2\pi R. \tag{4.159}$$

Express this in terms of a toroidal mode number, n (s.t. perturbation $\propto \exp i(n\phi + m\theta)$: $\phi = \frac{z}{R} \quad n = kR$.

Most unstable mode *tends* to be $n = 1$.

[Careful! Tokamak has important toroidal effects and some modes can be localized in the bad curvature region ($n \neq 1$).

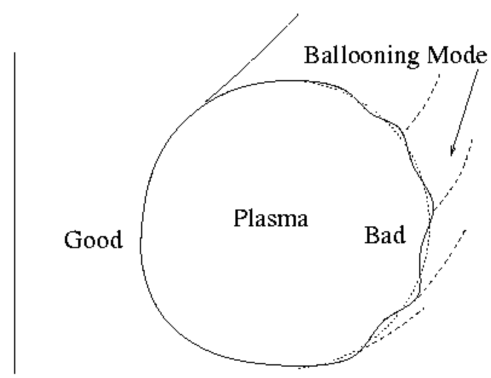

Figure 4.29: Ballooning modes are localized in the outboard, bad curvature region.