22.51 Quantum Theory of Radiation Interactions

Problem set # 2

Issued on Monday Sept 24, 2012. Due on Monday Oct. 8, 2012

Problem 1: Neutron Interferometry

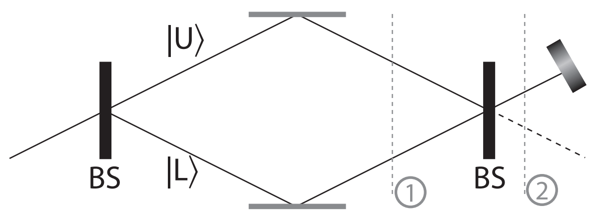

Figure 1: Neutron Interferometer

Consider a neutron interferometer (NI), such as the Mach-Zehnder interferometer in the figure. We send in a beam of neutrons. We assume that the flux of neutrons is so low (neutrons can be very slow) so that only one neutron is present at any time inside the interferometer. The first beamsplitter divides the neutron flux into two parts, that will go into the upper arm or the lower arm. The lower and upper beams are then reflected at the mirrors and recombined at the second beam splitter, after which the neutron flux is measured at one arm. We assume that both beamsplitter works in the same way, delivering an equal flux to each arm (that is, the transmission and reflection are the same).

a) Define the (minimal) Hilbert space describing this problem (e.g. give the basis spanning the space)

b) What is the propagator describing the action of the Beamsplitter?

c) What is the state at the position 1, assuming the neutron was initially traveling upward before entering the NI?

d) What is the probability of observing a neutron in the upper arm detector?

e) We now introduce an object in the lower path. This modifies the momentum of the neutron, and its effect is seen as an added phase to the neutrons passing through the lower path. Write the operator describing this phase shift and calculate again the probability of measuring a neutron at the upper path detector.

f) The usual signal for interferometers is the contrast $C = |(S_U - S_L)/(S_U + S_L)|$, where $S_U(S_L)$ is the signal (# of neutrons) at the upper(lower) detector. Indeed, this is always necessary since we need to calibrate and normalized the signal. What is the contrast for the neutron interferometer if the added phase (see previous question) is $\varphi = \pi/2$? Bonus: what is the operator describing the observable measured by the contrast?

Problem 2: Pure vs. Mixed States

Consider again the NI of Problem 1:. We previously assumed that the beam of neutrons all had the same (upward) momentum $|\psi(0)\rangle = |U\rangle$. However, if the neutrons arrive from a reactor, we might not be able to control their initial state.

b) Assume the neutrons can be represented by a mixed-state, with 50-50 probability of having initially an upward $|U\rangle$ or downward $|L\rangle$ momentum. What is the operator describing this state?

c) For the initial state described above, what is the measured contrast for the same condition as in Problem 1:.f? Compare this result to the answer you found previously: is it possible to distinguish the two initial states from this measurement? If not, propose another measurement that would distinguish the two cases.

e) Consider a more abstract question: For the family of pure states represented by $|\vartheta\rangle = (|+\rangle + e^{i\vartheta}|-\rangle)/\sqrt{2}$, and the non-pure state $\rho = \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|)$ we have $\langle\sigma_x\rangle = 0$. Thus a measurement of σ_x cannot distinguish the two states. How would you differentiate one state from another (with an appropriate measurement)? [On notations: we define the states $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, where $|0\rangle, |1\rangle$ are as usual the eigenstates of the σ_z operator.]

Problem 3: Mixed state on the Bloch sphere

a) The purity of a state ρ is defined as $\mathrm{Tr}\left\{\rho^2\right\}$. Show that $\mathrm{Tr}\left\{\rho^2\right\} \leq 1$, with the equality only if ρ is a pure state.

b) How does this condition translate to the Bloch vector $\vec{n}$ of a two-state system? (recall that a TLS density state can always be expressed as $\rho = \frac{1}{2}(\mathbb{1} + \vec{n}\cdot\vec{\sigma})$.)

c) What are the Bloch vectors for $|1\rangle\langle 1|$, $|-\rangle\langle -|$, $\rho = \begin{pmatrix} \frac{2}{5} & e^{i\pi/3}\frac{\sqrt{3}}{2} \\ e^{-i\pi/3}\frac{\sqrt{3}}{2} & \frac{3}{5} \end{pmatrix}$ and $|\psi\rangle\langle\psi|$ (where $|\psi\rangle = (|1\rangle + |+\rangle)/\sqrt{2+\sqrt{2}}$)? What is $\langle\sigma_y\rangle$ in in each case ?

Problem 4: Operators on composite systems

Write the operator matrix representations for the operator $\sigma_y^A = \sigma_y^A \otimes 1_B$ and $\sigma_x^A \otimes \sigma_z^B$, where each operator is defined on the respective Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$ (each spanned by the identity plus the Pauli matrices) and the operators you are looking for are defined on the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$.

Problem 5: Creation of entanglement

a) Consider the unitary operator $U(\vartheta,\varphi) = \begin{pmatrix} \cos(\vartheta/2) & e^{-i\varphi}\sin(\vartheta/2) \\ -e^{i\varphi}\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix}$ in the usual basis $|0\rangle,\ |1\rangle$. Which of the following states is entangled and which one is separable?

1. $[U(\vartheta_1,\varphi_1) \otimes U(\vartheta_1,\varphi_1)]\,|00\rangle$
2. $[U(\vartheta_1,\varphi_1) \otimes U(\vartheta_2,\varphi_2)](|00\rangle + |01\rangle)/\sqrt{2}$
3. $[U(\vartheta_1,\varphi_1) \otimes U(\vartheta_2,\varphi_2)](|00\rangle - |11\rangle)/\sqrt{2}$

[Notice that you should be able to give an answer even without making any calculation!]

b) Consider the Hamiltonian $\mathcal{H} = a\sigma_x\sigma_x + (1-a)\sigma_y\sigma_y$.

1. Are its eigenstates entangled? (you can use your favorite math program to diagonalize the matrix).
2. Consider the initial state $|00\rangle$. What is the rate of creation of entanglement by the Hamiltonian above? [Take for example as an entanglement measure the purity of the reduced state. Give an analytical expression and then you calculate the rate using your favorite program (it can also be done by hand)]

Problem 6: Evolution of 2 TLS

a) For the two qubit state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$, calculate the final state $|\psi(t)\rangle$ evolving under the Hamiltonian $\mathcal{H} = \omega\sigma_z^1\sigma_z^2$. Is the state ever entangled?

b) Consider the state $|\psi(t)\rangle$ that you found above. What is the time evolution of its concurrence? What is the purity of the reduced density operator at each instant in time?

c) What if the Hamiltonian is instead $\mathcal{H} = \omega\sigma_z^1\sigma_y^2$? (give again the time evolution of the system and whether entanglement is created, by calculating one entanglement measure.)

Problem 7: Spin-1 precession

Consider the spin-precession problem for a spin-1 system. Denote by $|-1\rangle, |0\rangle, |+1\rangle$ the states with spin angular momentum along z $\{-1, 0, +1\}$ respectively (in units of $\hbar$). The Hamiltonian for a spin 1-system subjected to an external magnetic field $\mathbf{B}$ (setting $\hbar = 1$) is

$$H = \mathbf{S} \cdot \gamma \mathbf{B} + \Delta S_z^2$$

where γ is the gyromagnetic ratio and Δ is the so-called zero-field splitting or it could be a quadrupolar interaction term. Take $\mathbf{B}$ to be static, and we set it along the z-axis direction.

a) Using the Hadamard formula

$$e^{xA} B e^{-xA} = B + [A, B]x + \frac{1}{2!}[A, [A, B]]x^2 + \frac{1}{3!}[A, [A, [A, B]]]x^3 + \dots$$

evaluate $e^{iS_z\varphi} S_x e^{-iS_z\varphi}$ and $e^{iS_z^2\varphi} S_x e^{-iS_z^2\varphi}$.

b) Using the Heisenberg picture, find $\langle S_{x,y,x} \rangle$ as a function of time assuming the spin was initially in the state $|\psi\rangle = (|0\rangle + |+1\rangle)/\sqrt{2}$.

Problem 8: Rabi oscillation for Spin 1

Consider the problem of a spin system in the presence of a sinusoidal oscillating time-dependent potential: in class we have used the interaction picture to solve this problem for a TLS. [This is just a reminder] Consider then a more general problem in which we add a (small) time-dependent magnetic field along the transverse direction (e.g. x-axis):

$$\vec{B}(t) = B_z \hat{z} + 2B_1 \cos(\omega t)\hat{x} = B_z \hat{z} + B_1 \left[(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}) + (\cos(\omega t)\hat{x} - \sin(\omega t)\hat{y})\right],$$

where B_1 is the strength of the radio-frequency (for nuclei) or microwave (for electron) field.
The Hamiltonian of the system $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1(t) + \mathcal{H}_1'(t)$ is then:

$$\mathcal{H} = \frac{\omega_0}{2}\sigma_z + \frac{\omega_1}{2}\left[\cos(\omega t)\sigma_x + \sin(\omega t)\sigma_y\right] + \frac{\omega_1}{2}\left[\cos(\omega t)\sigma_x - \sin(\omega t)\sigma_y\right],$$

where we defined the rf frequency ω_1. We already know the eigenstates of $\mathcal{H}_0$ ($|0\rangle$ and $|1\rangle$). Thus we use the interaction picture to simplify the Hamiltonian, with $U_0 = e^{-i\omega\sigma_z/2}$ defining a frame rotating about the z-axis at a frequency ω: this is the so-called *rotating frame*. Remembering that $U_0 \sigma_z U_0^\dagger = \cos(\omega t)\sigma_x + \sin(\omega t)\sigma_y$, it's easy to see that the perturbation Hamiltonian in the interaction frame is $\mathcal{H}_{1I} = U_0^\dagger \mathcal{H}_1 U_0 = \frac{\omega_1}{2}\sigma_x$. We also have $\mathcal{H}_{1I}' = U_0^\dagger \mathcal{H}_1' U_0 = \frac{\omega_1}{2}(\cos(2\omega t)\sigma_x - \sin(2\omega t)\sigma_y)$. Under the assumptions that $\omega_1 \ll \omega$, this is a small, fast oscillating term, that quickly averages out during the evolution of the system and thus can be neglected. This approximation is called the *rotating wave approximation* (RWA). Under the RWA, the Hamiltonian in the rotating frame simplifies to

$$\mathcal{H}_I = \frac{\Delta\omega}{2}\sigma_z + \frac{\omega_1}{2}\sigma_x$$

where $\Delta\omega = \omega_0 - \omega$. Now we want to solve the same problem for a spin-1 system. The Hamiltonian of the system $\mathcal{H} = \mathcal{H}_0 + V(t)$ is :

$$\mathcal{H} = \Delta S_z^2 + \omega_0 S_z + 2\omega_1 \cos(\omega t) S_x,$$

a) Set $\omega_0 = 0$. Using the result in Problem 7:, make a transformation to the rotating frame $U_0 = e^{-i\omega t S_z^2}$ (consider the on resonance case for simplicity) and neglect any remaining time-dependent terms (rotating wave approximation).

Hint: you can express the oscillating field as a sum of "counter-rotating" fields $U_0^\dagger S_x U^0$ and $U_0 S_x U_0^\dagger$, of which you calculate in Problem 7: the explicit expression (although these do not describe real counter-rotating fields as in the spin-1/2 case, but are more of a mathematical trick here).

d) Assuming the spin is initially in the state $|0\rangle$, what are the probability of being in the state $|0\rangle$ and $|\pm 1\rangle$ as a function of time? What is the probability of being in the state $(|+1\rangle + |-1\rangle)/\sqrt{2}$? What about the state $(|+1\rangle - |-1\rangle)/\sqrt{2}$?

e) Now we set $\omega_0 \neq 0$ and we would like to only populate the states $|0\rangle$ and $|1\rangle$ (i.e. $p_{-1}(t) = 0$). To do so we set $\omega = \Delta + \omega_0$, and we go into the rotating frame as above. What is the Hamiltonian in the rotating frame?

f) Assuming that $\omega_0 \gg \omega_1$, show that $|-1\rangle$ is never populated (you can just give an intuitive explanation without doing any calculation)

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