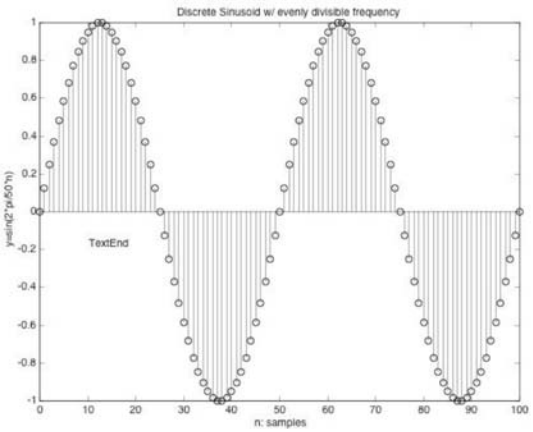
MAS.160 / MAS.510 / MAS.511 Signals, Systems and Information for Media Technology
Fall 2007

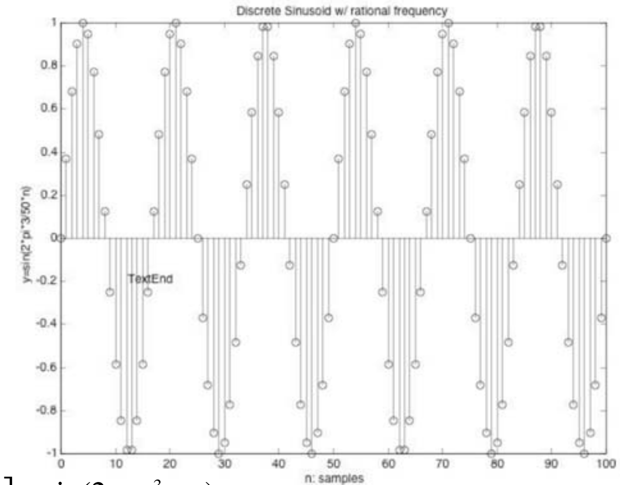
Period of a Discrete Sinusoid



$y[n] = \sin(2\pi \cdot \frac{1}{50} \cdot n)$ T=50 samples

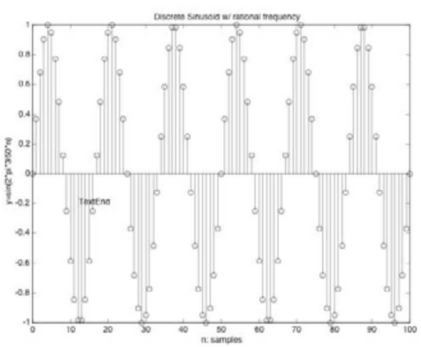
$y[n] = y[n+50]$

$\sin(0) = \sin(2\pi)$



$y[n] = \sin(2\pi \cdot \frac{3}{50} \cdot n)$

$y[n] = y[n+T]$ T=?? samples [integer] 50/3 ≠ integer



$y[n] = \sin(2\pi \cdot \frac{3}{50} \cdot n)$

$y[n] = y[n+T]$ T=?? samples

$\sin(0) = \sin(2\pi k)$

k=1,2…

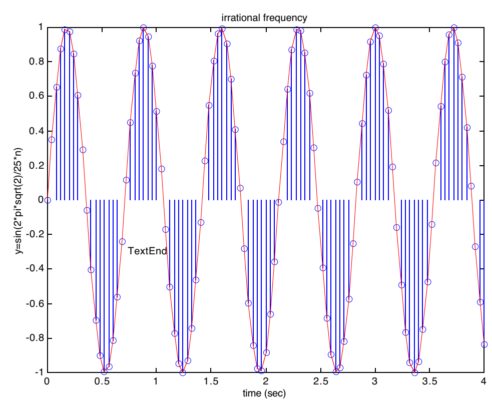
$2\pi \frac{3}{50} \cdot n = 2\pi k$

$\frac{n}{k} = \frac{50}{3} \frac{samples}{cycle}$ Ratio of integers

rational number

periodic

T=n=50 samples, k=3 cycles



$y(t) = \sin(2\pi \cdot \sqrt{2} \cdot t)$

$T = \frac{1}{\sqrt{2}}$ sec

continuous function periodic

Ts=1/25 sec

$y[n] = \sin(2\pi \cdot \frac{\sqrt{2}}{25} \cdot n)$

$y[n] = y[n+T]$ T=?? samples

$\sin(0) = \sin(2\pi k)$ k=1,2…

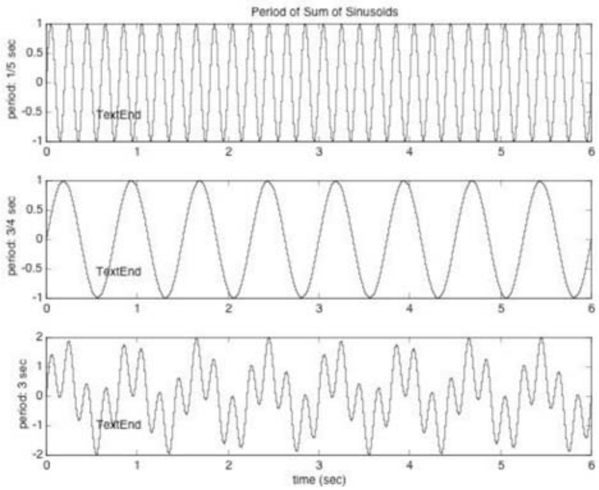
$2\pi \frac{\sqrt{2}}{25} \cdot n = 2\pi k$

$\frac{n}{k} = \frac{25\sqrt{2}}{2}$

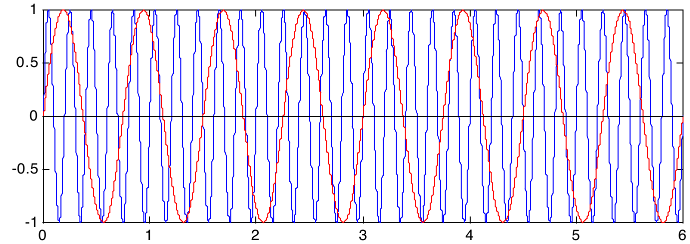
Equiv. discrete sinusoid not periodic

irrational number

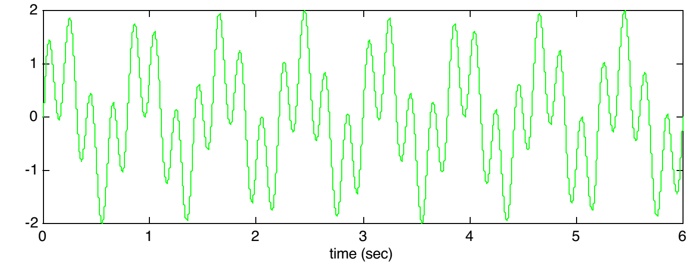
Period of Sum of Sinusoids



$$y(t) = y(t+T)$$



T1=0.2 seconds, T2=.75 seconds



Tsum=3 seconds

Least common multiple

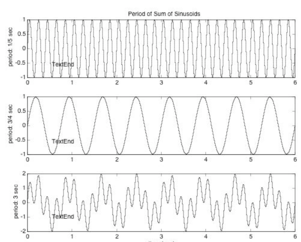
seconds to complete cycles

T1=1/5 seconds

1/5s, 2/5s, 3/5s …

4/20s, 8/20s, 12/20s, 16/20s, 20/20s, 24/20s, 28/20s, 32/20s, 36/20s, 40/20s, 44/20s, 48/20s, 52/20s, 56/20s, 60/20s

15 cycles



seconds to complete cycles

T2=3/4 seconds

3/4s, 6/4s, …

15/20s. 30/20s, 45/20s, 60/20s

4 cycles

1/5*k=3/4*l
k/l=15/4

rational number

Tsum=15*T1=15/5=3 seconds

Tsum=4*T2=3/4*4=3 seconds

Tsum=3 seconds

Complex Conversions

cartesian → polar		polar → cartesian	
s=a+jb	$s=\sqrt{a^2+b^2}e^{j\cdot a\tan(b/a)}$	s=re$^{j\theta}$	$s=r\cos\theta+jr\sin\theta$

Complex Arithmetic

Addition	cartesian	$(a_1+jb_1)+(a_2+jb_2)=(a_1+a_2)+j(b_1+b_2)$	
Subtraction	cartesian	$(a_1+jb_1)-(a_2+jb_2)=(a_1-a_2)+j(b_1-b_2)$	
Multiplication	polar	$r_1e^{j\theta_1}\cdot r_2e^{j\theta_2}=r_1r_2e^{j(\theta_1+\theta_2)}$	
Division	polar	$\frac{r_1e^{j\theta_1}}{r_2e^{j\theta_2}}=\frac{r_1}{r_2}e^{j(\theta_1-\theta_2)}$	
Powers	polar	$(re^{j\theta})^n=r^ne^{jn\theta}$	
Roots	polar	$z^n=s=re^{j\theta}$ $z=s^{1/n}=r^{1/n}e^{j(\theta/n+2\pi k/n)}$	$k=1,2\ldots n-1$

Complex Conversions

cartesian → polar

$$3+j4=\sqrt{3^2+4^2}\,e^{j\cdot a\tan(4/3)}=5e^{j\cdot 0.927}$$

polar → cartesian

$$2e^{j\cdot\frac{\pi}{3}}=2\cos\tfrac{\pi}{3}+j2\sin\tfrac{\pi}{3}=1+j\sqrt{3}$$

Complex Arithmetic		
Addition	cartesian	$(1+j2)+(3+j4)=(4+j6)$
Subtraction	cartesian	$(1+j2)-(3+j4)=(-2-j2)$
Multiplication	polar	$5e^{j\cdot\frac{\pi}{3}}\cdot 6e^{j\cdot\frac{\pi}{4}}=5\cdot 6e^{j\cdot\left(\frac{\pi}{3}+\frac{\pi}{4}\right)}=30e^{j\cdot\frac{7\pi}{12}}$
Division	polar	$10e^{j\cdot\frac{\pi}{2}}\div 5e^{j\cdot\frac{\pi}{4}}=\left(\frac{10}{5}\right)e^{j\cdot\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}=2e^{j\cdot\frac{\pi}{4}}$
Powers	polar	$\left(3e^{j\cdot\frac{\pi}{4}}\right)^3=3^3\cdot e^{j\cdot\left(\frac{3\pi}{4}\right)}=27e^{j\cdot\left(\frac{3\pi}{4}\right)}$
Roots	polar	$z^3=64=64e^{j0}$ $z=64^{1/3}e^{j(0/3+2\pi k/3)}=4e^{j(2\pi k/3)}$ 4, $4e^{j(2\pi/3)}$, $4e^{j(4\pi/3)}$

Representations of Sinusoids

$$A\cos(2\pi k f_0 t+\phi_k)$$

$$\mathrm{Re}\left\{Ae^{j2\pi ft+\phi}\right\}=\mathrm{Re}\left\{Ae^{j\phi}e^{j2\pi ft}\right\}=\mathrm{Re}\left\{Xe^{j2\pi ft}\right\}$$

$$Ae^{j\phi}\cdot\left(\frac{e^{j2\pi ft}+e^{-j2\pi ft}}{2}\right)=X\cdot\left(\frac{e^{j2\pi ft}+e^{-j2\pi ft}}{2}\right)$$

Sum multiple cosines same frequency

$$\sum_{k=1}^{n}A_k\cos(2\pi ft+\phi_k)=\sum_{k=1}^{n}\mathrm{Re}\left\{A_ke^{2\pi ft+\phi_k}\right\}=\sum_{k=1}^{n}\mathrm{Re}\left\{A_ke^{\phi_k}e^{2\pi ft}\right\}$$

$$=\left(\sum_{k=1}^{n}\mathrm{Re}\left\{A_ke^{\phi_k}\right\}\right)e^{2\pi ft}$$

Ex.

$$3\cos\left(2\pi 40t+\tfrac{\pi}{2}\right)-1\cos\left(2\pi 40t-\tfrac{\pi}{6}\right)+2\cos\left(2\pi 40t+\tfrac{\pi}{3}\right)$$

$$\mathrm{Re}\left\{3e^{j\frac{\pi}{2}}e^{j2\pi 40t}-1e^{-j\frac{\pi}{6}}e^{j2\pi 40t}+2e^{\frac{\pi}{3}}e^{j2\pi 40t}\right\}$$

$$\mathrm{Re}\left\{\left(3e^{j\frac{\pi}{2}}-1e^{-j\frac{\pi}{6}}+2e^{\frac{\pi}{3}}\right)e^{j2\pi 40t}\right\}$$

$$\mathrm{Re}\left\{5.234e^{j1.545}e^{2\pi 40t}\right\}$$

$$5.234\cos(2\pi 40t+1.545)$$

multiply cosines of different frequency

$$A_1\cos(\omega_1 t)\cdot A_2\cos(\omega_2 t+\phi)$$

$$A_1\left(\frac{e^{j\omega_1 t}+e^{-j\omega_1 t}}{2}\right)A_2\left(\frac{e^{j(\omega_2 t+\phi)}+e^{-j(\omega_2 t+\phi)}}{2}\right)$$

$$\frac{A_1A_2}{4}\left(e^{j\omega_1 t}e^{j(\omega_2 t+\phi)}+e^{j\omega_1 t}e^{-j(\omega_2 t+\phi)}+e^{-j\omega_1 t}e^{j(\omega_2 t+\phi)}+e^{-j\omega_1 t}e^{-j(\omega_2 t+\phi)}\right)$$

$$\frac{A_1A_2}{4}\left(e^{j(\omega_1 t+\omega_2 t+\phi)}+e^{-j(\omega_2 t-\omega_1 t+\phi)}+e^{j(\omega_2 t-\omega_1 t+\phi)}+e^{-j(\omega_1 t+\omega_2 t+\phi)}\right)$$

$$\frac{A_1A_2}{2}\left(\cos((\omega_1+\omega_2)t+\phi)+\cos((\omega_2-\omega_1)t+\phi)\right)$$

Composite signals (waveform synthesis)

$$x(t)=A_0+\sum_{k=1}^{\infty}A_k\cos(2\pi kf_0t+\phi_k)=X_0+\mathrm{Re}\left\{\sum_{k=1}^{\infty}X_ke^{j2\pi kf_0t}\right\}$$

decompose a periodic signal x(t) into a sum of a series of sinusoids - the Fourier series.

Note: The sum of periodic functions is periodic.

ex.

$$X_k=\begin{cases}\dfrac{-8}{\pi^2k^2} & k\ \ odd\\ 0 & k\ \ even\end{cases}$$

$$f_0=25Hz$$

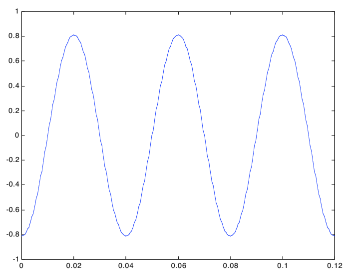
Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_k = \begin{cases} \dfrac{-8}{\pi^2 k^2} & k \quad odd \\ 0 & k \quad even \end{cases} \qquad X_k = \begin{cases} \dfrac{8}{\pi^2 k^2} e^{j\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

k=1

$$x(t) = 0.8105\cos(2\pi 25t + \pi)$$



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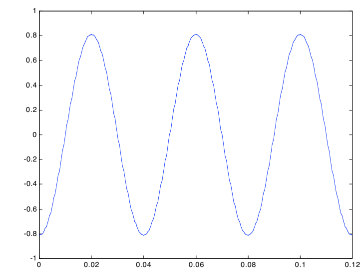


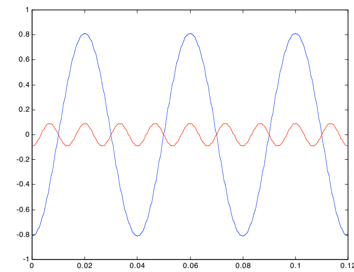
Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_k = \begin{cases} \dfrac{8}{\pi^2 k^2} e^{j\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

k=3

$$x(t) = 0.8105\cos(2\pi 25t + \pi) + 0.0901\cos(2\pi 75t + \pi)$$



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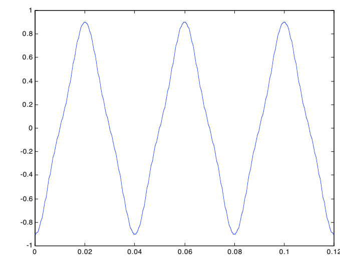


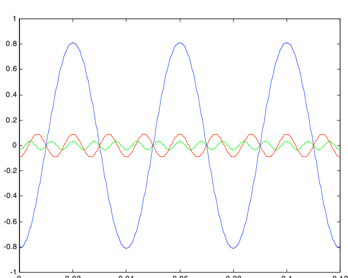
Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_k = \begin{cases} \dfrac{8}{\pi^2 k^2} e^{j\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

k=5

$$x(t) = 0.8105\cos(2\pi 25t + \pi) + 0.0901\cos(2\pi 75t + \pi) + 0.0324\cos(2\pi 125t + \pi)$$



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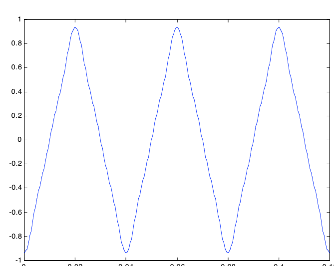
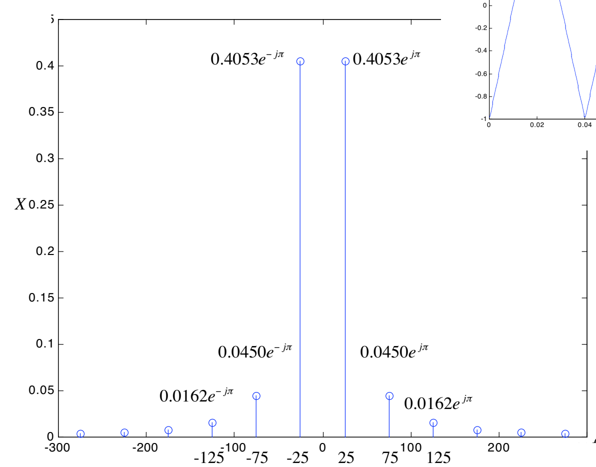
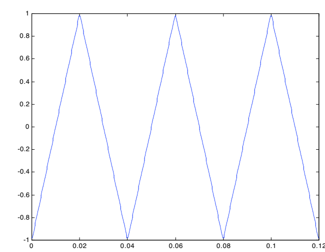


spectrum





$$x(t) = 0.8105\cos(2\pi 25t + \pi) + 0.0901\cos(2\pi 75t + \pi) + 0.0324\cos(2\pi 125t + \pi) + \ldots$$

Fourier Series

For a given signal, how do we find $X_k = A_k e^{j\phi_k}$ for each k ?

Fourier Analysis

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

where

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

$$X_k = \frac{2}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

f_0:fundamental frequency

$T_0 = 1/f_0$

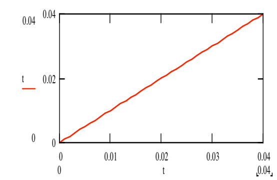
Fourier Series

$$x(t) = t \quad 0 \le t < T_0$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

$$= \frac{1}{T_0}\int_0^{T_0} tdt = \frac{1}{T_0}\left.\frac{t^2}{2}\right|_0^{T_0} = \frac{1}{T_0}\frac{T_0^{\,2}}{2} = \frac{T_0}{2}$$



Mathematica:
athena%add math
athena%math
In[1]:=1/T*Integrate[t,{t,0,T}]
Out[1]:=T/2

$$X_k = \frac{2}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

Fourier Series

$$x(t) = t \quad 0 \le t < T_0$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_0 = \frac{T_0}{2}$$

$$X_k = \frac{2}{T_0}\int_0^{T_0} te^{-j2\pi kt/T_0}dt$$

$$X_k = \frac{T_0}{2}\frac{(j2\pi k + 1)}{\pi^2 k^2}e^{-2j\pi k} - \frac{T_0}{2\pi^2 k^2}$$

$$X_k = \frac{T_0}{2}\frac{(j2\pi k + 1)}{\pi^2 k^2} - \frac{T_0}{2\pi^2 k^2}$$

$$X_k = j\frac{T_0}{2}\frac{2k\pi}{\pi^2 k^2} + \frac{T_0}{2\pi^2 k^2} - \frac{T_0}{2\pi^2 k^2}$$

$$X_k = j\frac{T_0}{\pi k} = \frac{T_0}{\pi k}e^{j\frac{\pi}{2}}$$

$$e^{-2j\pi k} = \left(e^{-j2\pi}\right)^k = 1^k = 1$$

Fourier Series

Mathematica:
In[2]:= 2/T*Integrate[t*Exp[-I*2*Pi*k*t/T],{t,0,T}]

$$X_k = \frac{2}{T_0}\int_0^{T_0} te^{-j2\pi kt/T_0}dt$$

```
                  (2 I) k Pi
        -((-1 + E           - (2 I) k Pi) T)
Out[2]= ------------------------------------
                 (2 I) k Pi  2   2
              2 E             k  Pi
```

In[3]:= Simplify[%,Element[k,Integers]]

$$e^{-2j\pi k} = 1$$

$$e^{-j\pi k} = -1^k$$

```
          I T
Out[3]= ----
         k Pi
```

$$X_k = j\frac{T_0}{\pi k} = \frac{T_0}{\pi k}e^{j\frac{\pi}{2}}$$

Fourier Series

$x(t) = t \quad 0 \le t < T_0$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_0 = \frac{T_0}{2}$$

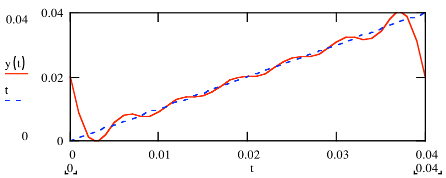
$$X_k = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

f_0:fundamental frequency

$T_0 = 1/f_0$

$$x(t) = \frac{T_0}{2} + \sum_{k=1}^{\infty} \frac{T_0}{\pi k} \cos\left(2\pi k f_0 t + \tfrac{\pi}{2}\right)$$

$$x(t) = \frac{T_0}{2} + \frac{T_0}{\pi} \cos\left(2\pi f_0 t + \tfrac{\pi}{2}\right) + \frac{T_0}{2\pi} \cos\left(2\pi 2 f_0 t + \tfrac{\pi}{2}\right) + \ldots$$



$f_0 = 25Hz$

$T_0 = 1/f_0 = 0.04$

7 terms

Fourier Series

$x(t) = t \quad 0 \le t < T_0$

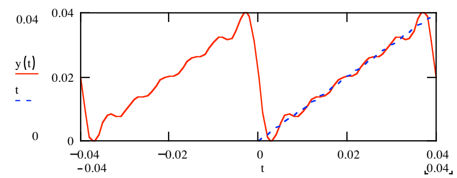
$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_0 = \frac{T_0}{2}$$

$$X_k = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

$$x(t) = \frac{T_0}{2} + \sum_{k=1}^{\infty} \frac{T_0}{\pi k} \cos\left(2\pi k f_0 t + \tfrac{\pi}{2}\right)$$

$$x(t) = \frac{T_0}{2} + \frac{T_0}{\pi} \cos\left(2\pi f_0 t + \tfrac{\pi}{2}\right) + \frac{T_0}{2\pi} \cos\left(2\pi 2 f_0 t + \tfrac{\pi}{2}\right) + \ldots$$

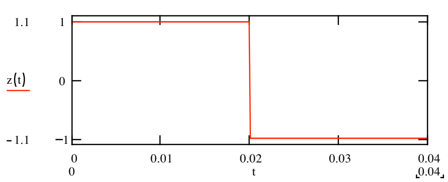


Defined between 0<t<0.04

Periodic with period 0.04

Fourier Series:Square Wave

$$x(t) = \begin{cases} 1 & 0 \le t < T_0/2 \\ -1 & T_0/2 \le t < T_0 \end{cases}$$



$$X_0 = \frac{1}{T_0}\int_0^{T_0/2} 1\,dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} -1\,dt$$

In[1]:=1/T*Integrate[1,{t,0,T/2}]+ 1/T*Integrate[-1,{t,T/2,T}]

Out[1]:=0

$X_0 = 0$

Fourier Series:Square Wave

$$X_k = \frac{2}{T_0}\int_0^{T_0/2} 1 e^{-j2\pi kt/T_0}\,dt + \frac{2}{T_0}\int_{T_0/2}^{T_0} -1 e^{-j2\pi kt/T_0}\,dt$$

```
In[2]:=2/T*Integrate[Exp[-I*2*Pi*k*t/T],{t,0,T/2}]+
        2/T*Integrate[- Exp[-I*2*Pi*k*t/T],{t,T/2,T}]

                 -I k Pi              I k Pi
        -I (1 - E        )    I (-1 + E       )
Out[2]= ----------------- + ----------------
             k Pi               (2 I) k Pi
                               E           k Pi

In[3]:= Simplify[%,Element[k,Integers]]

                         k 2
         -I (-1 + (-1) )
Out[7]= ----------------
              k Pi
```

$$X_k = \frac{-j\left(-1+(-1)^k\right)^2}{k\pi}$$

$$X_k = \begin{cases} -j\dfrac{4}{k\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$X_k = \frac{-j(-1-1)^2}{k\pi} = -\frac{j(-2)^2}{k\pi} = -\frac{j4}{k\pi}$$

$$X_k = \frac{-j(-1+1)^2}{k\pi}$$

$$x(t) = \begin{cases} 1 & 0 \le t < T_0/2 \\ -1 & T_0/2 \le t < T_0 \end{cases}$$

$$X_0 = 0$$

$$X_k = \begin{cases} -j\dfrac{4}{k\pi} & k \quad odd \\ 0 & k \quad even \end{cases} \longrightarrow X_k = \begin{cases} \dfrac{4}{k\pi} e^{-j\frac{\pi}{2}} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$x(t) = \tfrac{4}{\pi}\cos\left(2\pi f_0 t - \tfrac{\pi}{2}\right) + \tfrac{4}{3\pi}\cos\left(2\pi 3 f_0 t - \tfrac{\pi}{2}\right) + \ldots$$