Analysis of large plastic deformation of elasto-plastic solids

- Friction involves large plastic deformation.
- There are different ways of solving the deformation of elasto-plastic solids.
- Also upper- and lower-bound solutions, which are approximate solutions, can be very useful in engineering.
- One of the methods used is the slip-line field method, which gives a physical feel for the deformation process. It is an exact analysis for deformation of rigid-perfectly plastic solids.

Classification of partial differential equations

- Three different types of pd equations:
  - Elliptic (elastic deformation)
  - Parabolic (heat transfer, mass transfer)
  - Hyperbolic equations (wave propagation, plastic deformation

Classification of partial differential equations

- Consider the following second-order partial differential equation:

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = e$$

Boundary conditions :

z, $\frac{\partial z}{\partial x}$, and $\frac{\partial z}{\partial y}$ are specified.

Classification of partial differential equations

Then the variation of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial x}$ may be expressed as

$$d\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} dx + \frac{\partial^2 z}{\partial x \partial y} dy$$

$$d\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} dx + \frac{\partial^2 z}{\partial y^2} dy$$

We have three equations and three unknowns, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, and $\frac{\partial^2 z}{\partial y^2}$.

Classification of partial differential equations

Solving for $\frac{\partial^2 z}{\partial x^2}$,

$$\frac{\partial^2 z}{\partial x^2} = \frac{|N|}{|D|}$$

Depending on whether $|D|$ equal to orgreater than 0,

the pde represents different physical phenomena.

$b^2 - 4ac < 0$, elliptic equation.

$b^2 - 4ac > 0$, hyperbolic equation.

$b^2 - 4ac = 0$, parabolic equation.

The Upper- and the Lower-Bound Solutions

- **We want to get approximation solutions for large deformation of rigid-perfectly plastic solid in plane strain.**

- **We have to satisfy**
  - **Equilibrium condition (F=ma)**
  - **Geometric compatibility**
  - **Stress-strain relationship (constitutive relationship)**
  - **Yield condition**
  - **Boundary conditions**

The Upper- and the Lower-Bound Solutions

- Lower bound solutions are obtained if we satisfy
  - Equilibrium condition
  - Yield condition
  - Boundary conditions on stress

- Consider the punch indentation problem. The lower-bound can be obtained as

The Lower-Bound Solutions

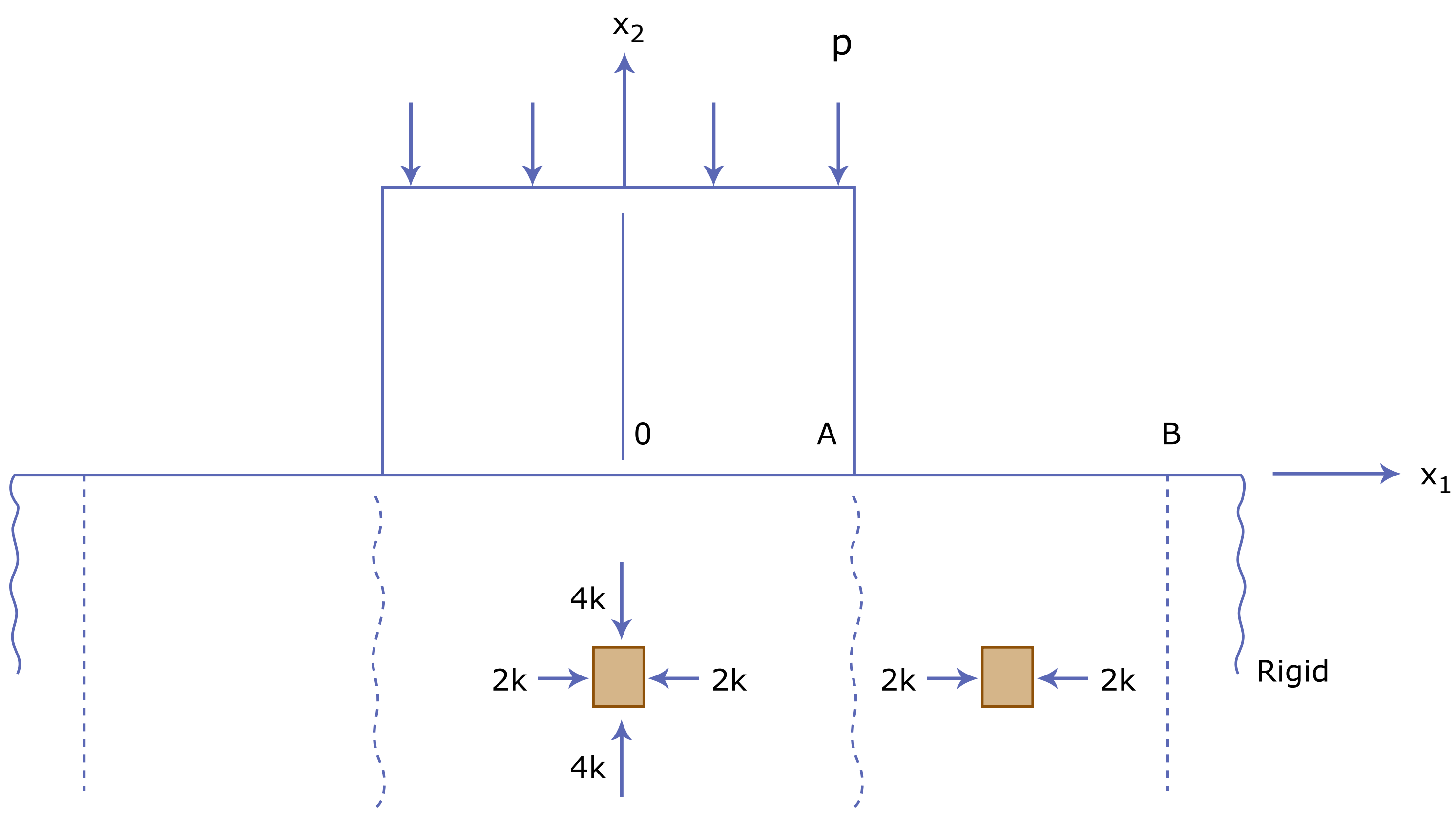

Figure by MIT OCW.

The Upper-Bound Solutions

The upper-bound-solutions are obtained by satisfying the following for an assumed displacement field:

1. **Incompressibility condition**
2. **Geometric compatibility**
3. **Velocity boundary conditions.**

The Upper-Bound Solutions

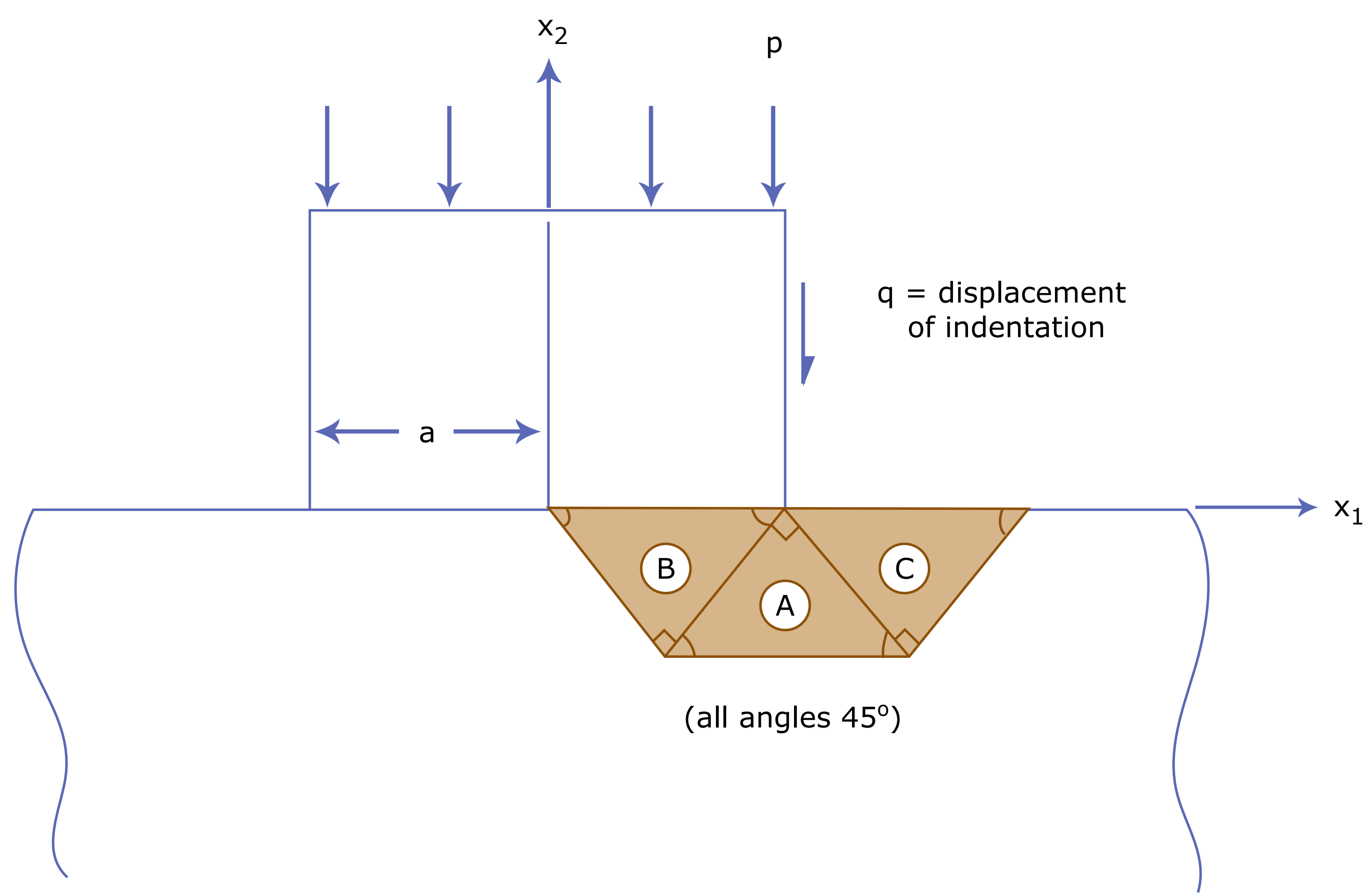

Figure by MIT OCW.

The Lower- and the Upper-Bound Solutions

The lower – bound solution

$$p \geq 4k□$$

The upper – bound solution

$$p \leq 6k□$$

The Slip-line Field Solutions

Equilibrium condition in plane strain:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial x} = 0$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial x} = 0$$

The Tresca yield condition:

$$\sigma_I - \sigma_{III} = 2k$$

The Slip-line Field Solutions

The stresses can be represented in terms of two invariants, p and k, as

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The Tresca yield condition:

$$\sigma_I - \sigma_{III} = 2k$$

The Slip-line Field Solutions

The stresses can be represented in terms of two invariants, p and k, as

$$\sigma_{xx} = -p - k\sin\phi$$
$$\sigma_{yy} = -p + k\sin\phi$$
$$\sigma_{xy} = k\cos\phi$$

where

$$p = -\frac{\sigma_{xx} + \sigma_{yy}}{2}$$

The Slip-line Field Solutions

Characteristic lines:

$$\frac{dy}{dx} = \tan\phi \qquad \alpha - \text{lines}$$

$$\frac{dy}{dx} = -\cot\phi \qquad \beta - \text{lines}$$

Characteristic equations:

$$p + 2k\phi = \text{constant} \qquad \alpha - \text{lines}$$

$$p - 2k\phi = \text{constant} \qquad \beta - \text{lines}$$

The Slip-line Field Solution for Asperity Deformation

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The Slip-line Field Solution for Asperity Deformation

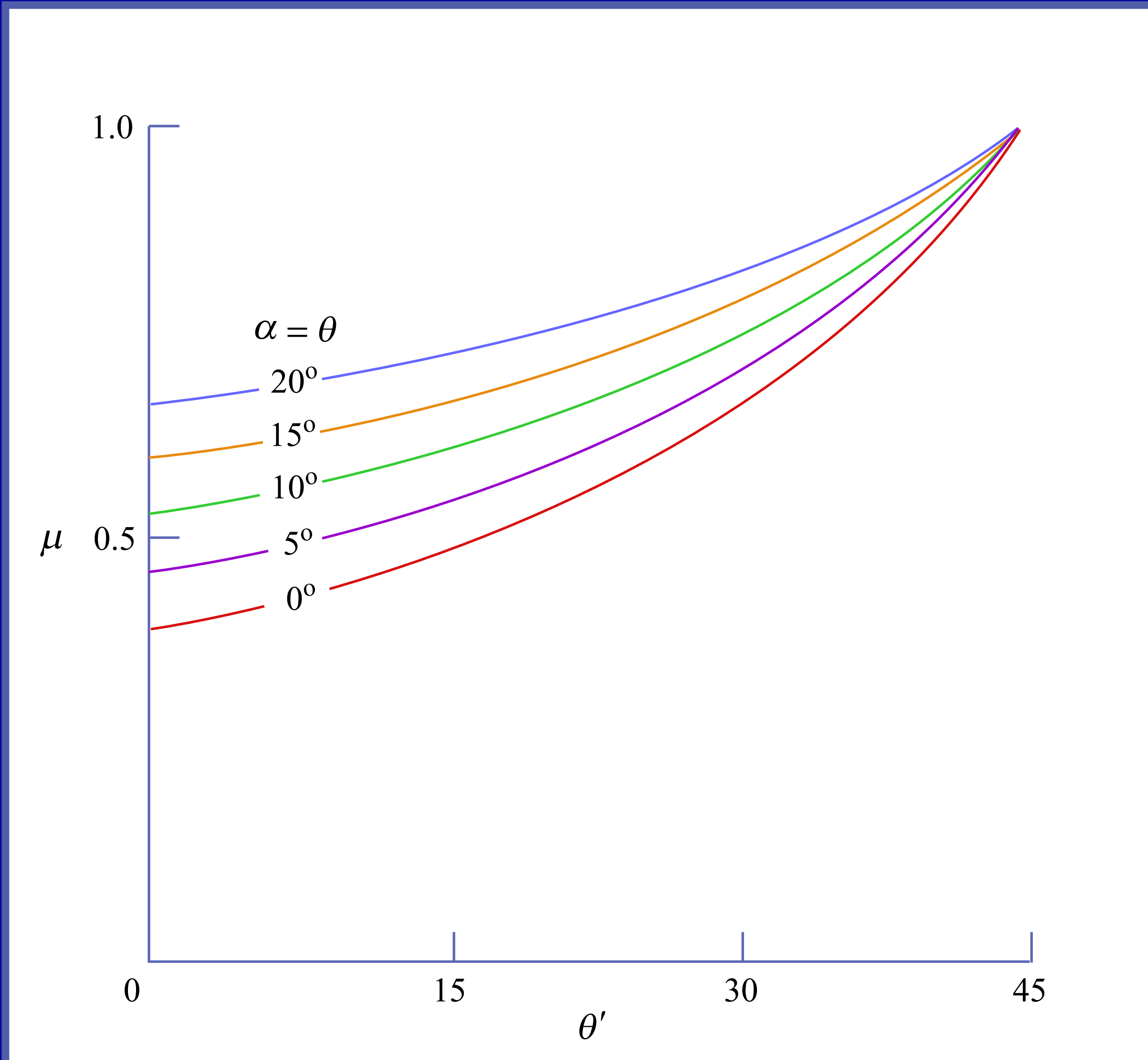

Figure by MIT OCW. After Suh, N. P., and H. C. Sin. "The Genesis of Friction." *Wear* 69 (1981): 91-114.